CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Strategic behavior, social optimization, and revenue maximization in queues with infinite servers

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Motivated by the impact of emerging technologies on (toll) parks, this paper studies a problem of customers' strategic behavior, social optimization, and revenue maximization for infinite-server queues. More specifically, we assume that a customer's utility consists of a positive reward for receiving service minus a cost caused by the other customers in the system. In the observable setting, we show the existence, uniqueness, and expressions of the individual equilibrium threshold, the socially optimal threshold, and the optimal revenue threshold, respectively. Then, we prove that the optimal revenue threshold is smaller than the socially optimal threshold, which is smaller than the individual one. Furthermore, we also extend the cost functions to any finite polynomial function with nonnegative coefficients. In the unobservable setting, we derive the joining probabilities of individual equilibrium and optimal revenue. Finally, using numerical experiments, we complement our results and compare the social welfare and the revenue under these two information levels.

1. Introduction

In the past few decades, studying queueing systems from an economic perspective has become increasingly prominent. More specifically, a specific reward-cost structure is imposed on the queueing system to reflect customers' desire to be served and unwillingness to wait. Arriving customers are permitted to make decisions about whether to join the queue. All customers would like to maximize their profit, taking into consideration that all the other customers also have the same goal. When any customer can choose to join, the system should intuitively be modeled by an infinite-server queue. In the real world, many service systems can be approximated as infinite-server queues, for example, large resorts, (toll) parks in urban areas. Although the interior of these systems could be divided into several different queueing models or networks, it is still a reasonable approximation to view them as an infinite-server queue as a whole. Therefore, the literature on infinite-server queues is very extensive, see, for instance, [1–3, 5, 9, 22, 24, 26] and the extensive references therein. However, to the best of our knowledge, such queues have not been studied from an economic perspective.

In the following, let's first briefly review the development of studying the queueing problems from the economic analysis. In a seminal paper, Naor [23] introduced the reward and the linear delay cost of customers into the $M/M/1$ queueing model. If the queue length can be accurately observed by the customers, Naor [23] gave threshold strategies of the individual equilibrium, the socially optimal welfare, and the optimal revenue and proposed the idea of levying fees to induce the social optimal strategy. Edelson and Hilderbrand [8] complemented Naor's research from an unobservable case. Their conclusion shows that the social welfare and the revenue are equal when the queue length is not observed by customers. Therefore, they proposed a method of levying observation fees to make the social welfare

and the revenue still coincide when customers' queueing strategy has a threshold type. However, in the case of non-homogeneous costs, the aforementioned results do not always hold. Hassin [12] compared Naor's observable model with Edelson and Hilderbrand's unobservable model. The conclusion shows that providing real-time information is not always beneficial to profit maximization of the manager and the social welfare under the profit maximizing admission fee also has the similar results. Since then, because the strategic queueing models have been widely used in various service industries, more and more scholars have paid attention to the problem of strategic queue and numerous excellent papers have been published, such as vacation queues in the transportation industry (Guo and Hassin [11]), retrial queueing systems with applications in networks (Wang and Zhang [30], Cui, Su, and Veeraraghavan [6]), double-ended queues in the passenger-taxi service system (Shi and Lian [27]), priority queues with discriminatory pricing (Hassin and Haviv [14], Wang, Cui, and Wang [29]), queues with uncertain/different information (Cui and Veeraraghavan [7], Hassin, Haviv, and Oz [16], Chen and Hasenbein [4], Liu [19]), etc. The basic knowledge of strategic queues is summarized in Hassin and Haviv [15]. Recently, the book Hassin [13] lists most of the relevant literature. Interested readers can refer to it and the extensive references therein.

Models with infinite servers to approximatively characterize and analyze real problems arise in various situations in practice. An example of infinite-server queues may be illustrated by the decision making of tourists in modern parks. In modern parks, congestion problems occur from time to time due to the centralized travel of people. For example, in Fantawild (Disneyland, Universal studios) of China, it is always reported that there are too many people staying in the park during the holidays, and in urban (toll) parks located in densely populated metropolitan areas, we also usually see a large number of people traveling on weekends. It is no difficult to find that whether tourists are willing to enter the park has a lot to do with the number of people staying in the park. Intuitively, the more the people stay in the park, the more reluctant tourists are to join it. The reason is that according to the empirical (expected) information or the real-time information provided by the park on the mobile platform (the bulletin board), rational tourists will judge whether it is worth entering the park and their individual utilities are negatively correlated with the number of people in the park. Based on this phenomenon, we could model these parks as infinite-server queues, regard tourists as customers, quantify tourists' behavior by using the game theory, and analyze tourists' equilibrium strategic behavior, social optimization, and revenue maximization under different information levels, so as to provide some valuable advice to the public. In the following sections, we use the terms of queueing theory to explain the conclusions. These explanations can also correspond to the practical examples given above.

In the traditional literature, we usually see that some basic hypotheses of the queueing model have the following salient characteristics: the customer's reward is assumed to be $R > 0$ and customers' own cost is positively correlated with their sojourn time. In the context of infinite-server queues, customers are assumed to receive a reward that depends negatively on the number of customers in the system. An interesting practical explanation is that in the park example, this assumption is able to reflect the impact of the park population on tourists satisfaction in modern parks. Under this structure, there are several contributions in the present paper. First, according to whether to announce the real-time number of people, we divide the problem into the observable model and the unobservable model. For these two cases, we analyze the individual equilibrium, the optimal social welfare, and the optimal revenue of the infinite-server queue and gives computable expressions for these optimal policies. Furthermore, we theoretically show the relationship of these optimal strategies and make some monotonic analyses. Finally, we numerically compare the social welfare and the revenue with different thresholds and information levels, and some valuable suggestions for the system administrator (SA) are also presented.

The rest of this article is arranged as follows. In Section 2, we give a detailed description of the model and the reward-cost structure. Sections 3 and 4 are devoted to the observable and unobservable cases of the model. Section 5 shows numerical analyses including a mini example which gives a simple operation procedure for calculating each quantity. The proofs of the main results are postponed to Section 6. The paper ends presenting some conclusions and potential research directions in Section 7.

2. Formulation and preliminaries

Following the background described in Section 1, here we consider an infinite-server queue. We assume that customers are homogeneous and arrive at the system according to a Poisson process with potential arrival rate Λ. The sojourn times of all customers in the system are independent and follow a common general distribution function $B(x)$ with a mean of $1/\mu$. A customer's utility is assumed to consist of a reward for receiving service minus a cost caused by the other customers in the system. More specifically, if the SA announces the real-time number of customers in the system, customers receive a state dependent reward $R - C_1 N - C_2 N^2$ upon successful completion of service, where $R > 0$ and N is the number of customers in the system. There are many practical explanations when C_1 and C_2 take different values.

1. If $C_1 > 0$ and $C_2 = 0$, we have $C_1 N + C_2 N^2 = C_1 N$, which means that the cost is a linear function of the current number of customers in the system. Thus, this corresponds to the risk-neutral customers.
2. If $C_1 = 0$ and $C_2 > 0$, we have $C_1 N + C_2 N^2 = C_2 N^2$, which implies that the cost is a quadratic function with respect to the real-time number of customers in the system. This represents the risk-averse customers.

If the system does not announce the real-time number of customers, we assume that customers use the expected information to estimate the number of customers in the system. Therefore, customers receive a state dependent reward $R - C_1 \mathbb{E}(L) - C_2 \mathbb{E}(L)^2$ after the service is completed, where $\mathbb{E}(L)$ is the average number of customers in the system. Similarly, we could get corresponding interpretations when customers use the cost structure $C_1 \mathbb{E}[L] + C_2 \mathbb{E}[L]^2$. Moreover, if $C_1 = C(1 + \mu A''(1) \int_0^\infty [1 - B(x)]^2 dy)$ and $C_2 = C$, we also have $C_1 \mathbb{E}[L] + C_2 \mathbb{E}[L]^2 = C \mathbb{E}[L^2]$, where $A(z) = \mathbb{E}[z^X]$ (see Holman, Chaudhry, and Kashyap [17]). This expression indicates that customers' costs are linear to the second moment of queue length. In the following sections, we only require $C_1 C_2 \geq 0$ and $\max\{C_1, C_2\} > 0$. Therefore, the above statements are only a special case. We also investigate that the cost structures are any finite polynomial function with nonnegative coefficients, but for brevity, if the extended conclusions can be obtained in the same way, we will only state them in remarks. Besides the individual utility, the additive social utility composed of the sum of individual utilities and the revenue composed of long-term gains from monopolist pricing are also analyzed in the following sections.

3. The observable model

3.1. Individual equilibrium

In the observable setting, we assume that the real-time number of customers in the system are always posted on the bulletin board and all rational customers can clearly know this information before deciding whether to join the system. According to the reward-cost structure, an arriving customer who finds n customers in the system joins the queue with the individual utility $R - (C_1 n + C_2 n^2)$ and balks with the individual utility 0. It follows from $R > 0$ that rational customers will join the system if and only if the individual utility is nonnegative. Therefore, the maximum integer $n_e - 1$ that the customers decide to join the system will satisfy the following two inequalities

$$R - [C_1(n_e - 1) + C_2(n_e - 1)^2] \geq 0, \tag{1}$$

$$R - [C_1 n_e + C_2 n_e^2] < 0.$$

Solving the above two inequalities, we have the following theorem.

Theorem 1. *In the observable infinite-server queue, there exists a unique equilibrium strategy $n_e = \max\left\{N : N \leq \frac{2C_2 - C_1 + \sqrt{(C_1)^2 + 4C_2 R}}{2C_2}\right\}$ such that customers join the system if and only if $n < n_e$.*

3.2. Social optimality

Now, we could consider the problem of maximizing the expected total net gain of all customers per time unit, that is, the socially optimal welfare. Since the actual (long-run) joining rate is an important index, we must proceed in a different mode from the individual equilibrium.

Denote $\rho = \frac{\Lambda}{\mu}$ and let $\mathbb{P}(N = j)$ ($j = 1, 2, \ldots, n$) be the stationary distribution of the $M/G/n/n$ queue. Note that when all customers consistently use balking strategies n (join the system if and only if the number of customers is less than n), we can regard this process as an $M/G/n/n$ queue. It follows from the results of $M/G/n/n$ queue (see Fakinos [9] or Shortle, Thompson, Gross, and Harris [28]) that

$$\mathbb{P}(N = j) = \frac{(\frac{\Lambda}{\mu})^j \frac{1}{j!}}{\sum_{k=0}^{n} (\frac{\Lambda}{\mu})^k \frac{1}{k!}} = \frac{\frac{\rho^j}{j!}}{\sum_{k=0}^{n} \frac{\rho^k}{k!}}, \tag{2}$$

$$\mathbb{E}(L_n) = \frac{\sum_{k=0}^{n} k \frac{\rho^k}{k!}}{\sum_{k=0}^{n} \frac{\rho^k}{k!}} = \frac{\rho \sum_{k=0}^{n-1} \frac{\rho^k}{k!}}{\sum_{k=0}^{n} \frac{\rho^k}{k!}}, \tag{3}$$

where $\mathbb{E}(L_n)$ is the expected queue length of the $M/G/n/n$ queue. Then, if all customers consistently use the balking strategy n, the actual joining rate of customers is

$$\Lambda \mathbb{P}(N < n) = \Lambda \frac{\sum_{k=0}^{n-1} \frac{\rho^k}{k!}}{\sum_{k=0}^{n} \frac{\rho^k}{k!}} = \mu \mathbb{E}(L_n).$$

Let $S^r(n)$ be the expected total net gain per time unit under balking strategy n. Using the PASTA property (see Wolff [31]), we arrive at the following expressions of $S^r(n)$:

$$S^r(n) = \Lambda \sum_{m=0}^{n-1} \mathbb{P}(N = m) \left[R - (C_1 m + C_2 m^2)\right] \tag{4}$$

$$= \Lambda \left[R \mathbb{P}(N < n) - \left(C_1 \sum_{m=0}^{n-1} m \mathbb{P}(N = m) + C_2 \sum_{m=0}^{n-1} m^2 \mathbb{P}(N = m)\right)\right]$$

$$= \mu R \mathbb{E}(L_n) - \mu\rho \left[C_1 \frac{\sum_{m=0}^{n-1} m \frac{\rho^m}{m!}}{\sum_{m=0}^{n} \frac{\rho^m}{m!}} + C_2 \frac{\sum_{m=0}^{n-1} m^2 \frac{\rho^m}{m!}}{\sum_{m=0}^{n} \frac{\rho^m}{m!}}\right]. \tag{5}$$

According to the expression of (5), we could get the following key results about the socially optimal welfare. The proof can be found in Section 6.

Theorem 2. *In the observable infinite-server queue, there exists a unique socially optimal threshold strategy*

$$n_s = \max\left\{N : \frac{\rho \sum_{i=1}^{2} C_i \left[\frac{\sum_{m=0}^{N-1} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{N} \frac{\rho^m}{m!}} - \frac{\sum_{m=0}^{N-2} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{N-1} \frac{\rho^m}{m!}}\right]}{\mathbb{E}(L_N) - \mathbb{E}(L_{N-1})} \leq R\right\} \tag{6}$$

such that $S^r(n)$ is strictly increasing in n when $n \leq n_s$ and strictly decreasing in n when $n > n_s$. Furthermore, n_s is decreasing in ρ.

Remark 1. There exists an intuitive explanation for the relationship between n_s and ρ. As ρ increases, if n_s remains the same, the number of customers in the system will be *stochastically increasing* in ρ. This, together with the individual utility $R - (C_1 n + C_2 n^2)$, implies that the (long-run) average marginal net utility for each arriving customer will become very small. At this point, the optimal threshold n_s should be reduced to increase the average net utility of each customer in the system and further increase the additive social utility. This interpretation is consistent with the monotonicity of n_s with respect to ρ. On the other hand, the smaller ρ is, the greater the probability that arriving customers will see a small number of customers in the system. Simple calculations yield

$$\lim_{\rho \to 0} \frac{\rho \left[\frac{\sum_{m=0}^{n} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\rho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{n} \frac{\rho^m}{m!}} \right]}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)} = n^i.$$

This combining with (1) and (6) implies that when ρ tends to 0, n_s and n_e are getting closer and closer and therefore allowing customers with the positive individual utility value to enter the system will increase the social welfare.

Remark 2.

(a) Letting $\Lambda' > \Lambda$, we consider a new strategy such that if the number of customers in the system is less than n_s, the new system with parameter Λ' (all the other parameters are assumed to be the same) allows customers to enter with probability $\frac{\Lambda}{\Lambda'}$. According to the decomposability of the Poisson flow and the expression (5), we see that under this new strategy, the social welfare of this new system is also equal to $S^r(n_s)$. Note that we can regard this social welfare problem of the infinite-server queue as a particular (long-run) average reward model in the theory of the Markov decision processes (MDPs). Thus, according to the results of the MDPs (see Chapter 11 in Puterman [25] or Feinberg and Yang [10]), the deterministic stationary optimal strategy (or optimal pure strategy) always exists, which means that $S^r(n_s)$ is increasing in Λ.

(b) For fixed $n < n_e$, let $\mathbb{P}[N(\rho) = j], j = 1, 2, \ldots, n$, be the stationary distribution of the $M/G/n/n$ queue with parameter ρ. Using the method of the sample path comparison, we easily have $N(\rho) \leq_{st} N(\rho')$ when $\rho < \rho'$, where $\leq_{st}$ is the usual stochastic order (see Müller and Stoyan [21] or Keilson and Kester [18]). Therefore, for the decreasing sequence $R - (C_1 n + C_2 n^2)$ with respect to n, we have

$$\sum_{m=0}^{n-1} \mathbb{P}(N(\rho) = m) \left[R - (C_1 m + C_2 m^2) \right] > \sum_{m=0}^{n-1} \mathbb{P}(N(\rho') = m) \left[R - (C_1 m + C_2 m^2) \right],$$

which, together with (4), implies that $\frac{S^r(n)}{\Lambda}$ is strictly decreasing in ρ. This means that $S^r(n_s)$ is strictly increasing in μ.

3.3. The system's revenue

In this subsection, we introduce a price P_o to study the system's revenue maximizing problem. Because customers respond to P_o, we model the interaction between the SA and the customers as a Stackelberg game, where the SA is the leader whose action is to set the price, while customers are the followers whose roles are determine their utilities and strategies based on the price set by the SA. The goal of the SA is to maximize its revenue while anticipating customers equilibrium strategies. Similar to the

traditional analysis (see Section 2.4 of Hassin and Haviv [15]), under a balking strategy n, the best price is $P_o = R - [C_1(n-1) + C_2(n-1)^2]$ and thus, the expected total net revenue, $S_m^r(n)$, can be expressed as follows:

$$\begin{aligned} S_m^r(n) &= \Lambda \mathbb{P}(N < n) P_o \\ &= \mu \mathbb{E}(L_n)[R - (C_1(n-1) + C_2(n-1)^2)]. \end{aligned} \tag{7}$$

According to this expression, we are able to obtain the following theorem about the optimal revenue. The proof can be found in Section 6.

Theorem 3. *In the observable infinite-server queue, there exists a unique optimal threshold strategy*

$$n_m = \max\left\{N : \frac{\sum_{i=1}^2 C_i(\mathbb{E}(L_N)(N-1)^i - \mathbb{E}(L_{N-1})(N-2)^i)]}{\mathbb{E}(L_N) - \mathbb{E}(L_{N-1})} \leq R\right\} \tag{8}$$

such that $S_m^r(n)$ is strictly increasing in n when $n \leq n_m$ and strictly decreasing in n when $n > n_m$. Moreover, n_m is increasing in ρ and the optimal price for the SA is $\widetilde{P}_o = R - [C_1(n_m-1) + C_2(n_m-1)^2]$.

Remark 3. The relationship between n_m and ρ has an interesting interpretation. As ρ increases, increasing n_m will allow more customers to be served, thereby increasing the (long-run) revenue of the system. This reflects the small profit but quick turnover strategy often used in economics.

Having obtained the optimal threshold strategies of individual equilibrium, social welfare, and revenue, now, we could compare the relationship of size between them. The following results show that for the observable case, if the customers' costs are positively correlated with the real-time number of customers in the system, the three thresholds are generally different and the unequal relationship is consistent with the traditional conclusion found by Naor [23].

Theorem 4. *In the observable infinite-server queue, we have $n_m \leq n_s \leq n_e$.*

Proof. It follows from (1) and (4) that $n_s \leq n_e$ is straightforward, therefore we just need to show $n_m \leq n_s$. Since

$$\lim_{\rho \to \infty} \frac{\rho\left[\frac{\sum_{m=0}^n m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\rho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^n \frac{\rho^m}{m!}}\right]}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)} = n^i(n+1) - n(n-1)^i$$

and

$$\lim_{\rho \to \infty} \frac{\mathbb{E}(L_{n+1})n^i - \mathbb{E}(L_n)(n-1)^i}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)} = n^i(n+1) - n(n-1)^i,$$

by the definition of (6) and (8), we have that $n_m = n_s$ when $\rho \to \infty$. Using the results of Theorems 2 and 3, we know that n_s is decreasing in ρ while n_m is increasing in ρ, which immediately indicates $n_m \leq n_s$ for $\rho < \infty$.   □

Remark 4. The first inequality of Theorem 4 shows that a revenue-maximizing SA sets a higher entrance price than that maximizes the social welfare, potentially making the system less available to the public. It follows from the proof of Theorem 4 that the capacity gap between the socially optimal strategy and the revenue-maximizing strategy gradually decreases as ρ increases. An interesting

explanation is that as ρ increases, there will be more customers left in the system and the profits of all new arrivals are closer to the system's pricing, which brings the socially optimal welfare closer to the optimal revenue. The second inequality indicates us that under the fully free condition, the system is generally not able to achieve the social optimum. Thus, appropriate tolls are still a good way to reach the socially optimal threshold.

4. The unobservable model

In the unobservable model, we assume that customers can not obtain the real-time number of customers in the system upon arrival. Customers make decisions based on the information of the system including Λ, μ, R, and the cost structure $C_1 \mathbb{E}(L) + C_2 \mathbb{E}(L)^2$, where $\mathbb{E}(L)$ is the average number of people in the system. To consider a symmetric equilibrium, we suppose that customers join the system with probability q $(0 \leq q \leq 1)$ upon arrival. In this section, we first derive the equilibrium strategy with no price set. When the SA chooses a desired threshold n and sets the maximum price P_u to ensure this threshold, like in Section 3.3, the results of this Stackelberg game is $P_u = R - [C_1 \mathbb{E}(L) + C_2 \mathbb{E}(L)^2]$. Since the system is unobservable, the social welfare and the revenue have the same expressions, thus we don't need to distinguish them in the following study.

4.1. Equilibrium

When $R - [C_1 \rho + C_2 \rho^2] \leq 0$, suppose that the equilibrium strategy of a customer to join the system is q_e, $0 \leq q_e \leq 1$, then q_e should satisfy

$$
\begin{aligned}
0 &= R - (C_1 \mathbb{E}(L) + C_2 \mathbb{E}(L)^2) \\
&= R - [C_1(\rho q_e) + C_2(\rho q_e)^2],
\end{aligned}
$$

where $\mathbb{E}[L]$ is the expected queue length of the system, which equals to that of an $M/G/\infty$ queue with arrival Λq_e and the service rate μ. Solving the above equation, we see that $q_e = \frac{-C_1 + \sqrt{C_1^2 + 4RC_2}}{2C_2 \rho}$ when $R \leq C_1 \rho + C_2 \rho^2$, which yields the following theorem immediately.

Theorem 5. *In the unobservable infinite-server queue, the unique equilibrium strategy for customers is given as follows:*

(a) *If $R - (C_1 \rho + C_2 \rho^2) \geq 0$, $q_e = 1$.*
(b) *If $R - (C_1 \rho + C_2 \rho^2) < 0$, $q_e = \frac{-C_1 + \sqrt{C_1^2 + 4RC_2}}{2C_2 \rho}$.*

4.2. Revenue (social) optimality

Like in Section 3.3, when the SA sets an entrance fee P_u, the reward of customers drops from R to $R - P_u$, which will change the equilibrium probability of joining the system. Let $q(P_u)$ denote the joining probability associated with a given fee P_u and without confusion, we use q to represent. Then, we have the expression of revenue

$$
S(q) = q\Lambda[R - (C_1(\rho q) + C_2(\rho q)^2)], \tag{9}
$$

where $P_u = R - [C_1 \rho q + C_2(\rho q)^2]$. When $R \leq (2\rho C_1 + 3\rho^2 C_2)$, by differentiating $S(q)$ with respect to q and finding the nonnegative root, we have

$$
q = \frac{-C_1 + \sqrt{C_1^2 + 3RC_2}}{3C_2 \rho}.
$$

Summarizing the above discussions, we could naturally develop the following theorem.

Theorem 6. *In the unobservable infinite-server queue, let $\widetilde{q}$ be the optimal joining probability of revenue and $\widetilde{P}_u$ be an optimal entrance price, then the following statements hold.*

(a) *If $R \geq (2\rho C_1 + 3\rho^2 C_2)$, we have*
    1. $\widetilde{q} = 1$;
    2. $S(\widetilde{q}) = \mu(R\rho - C_1\rho^2 - C_2\rho^3)$;
    3. $\widetilde{P}_u = R - C_1\rho - C_2\rho^2$.
(b) *If $R < (2\rho C_1 + 3\rho^2 C_2)$, we have*
    1. $\widetilde{q} = \dfrac{-C_1 + \sqrt{C_1^2 + 3RC_2}}{3C_2\rho}$;
    2. $S(\widetilde{q}) = \dfrac{-2C_1^3 - 9RC_1C_2 + (2C_1^2 + 6RC_2)\sqrt{C_1^2 + 3RC_2}}{27C_2^2}\mu$;
    3. $\widetilde{P}_u = \dfrac{6RC_2 + C_1^2 - C_1\sqrt{C_1^2 + 3RC_2}}{9C_2}$.

Remark 5.

(a) It follows from Theorems 5 and 6 that $\widetilde{q} \leq q_e$ and $S(q_e) = 0$, which means that for the unobservable case, the system is still not able to achieve the social optimum under the fully free condition. In fact, a customer who decides to join the system would impose negative externalities on future arrivals. Thus, appropriate tolls can reach the socially optimal threshold.

(b) It follows from Theorem 6(b) that if $R < (2\rho C_1 + 3\rho^2 C_2)$, $\widetilde{P}_u$ and $S(\widetilde{q})$ are both constants with respect to Λ while $S(\widetilde{q})$ is increasing in R and μ. This implies that the change in the arrival flow of customers does not affect the revenue and there is no need to adjust the entrance price. If $R \geq (2\rho C_1 + 3\rho^2 C_2)$, $\widetilde{P}_u$ is decreasing in ρ but $S(\widetilde{q})$ is increasing in ρ and R. This shows that when $R \geq (2\rho C_1 + 3\rho^2 C_2)$ and ρ increases, the revenue-maximizing SA should reduce the price to increase the effective arrival rate. Like the observable case, increasing individual reward (R) will always increase the optimal social welfare.

Remark 6.

(a) The relationship between $S_m^r(n_m)$ and $S(\widetilde{q})$ can not be completely determined. In fact, if $\mathbb{E}(L_{n_m}) \leq (n_m - 1)$, it follows from (7) and (9) that $S_m^r(n_m) \leq S(\widetilde{q})$. However, as $\rho \to \infty$, for $R = 20, \mu = 1, C_1 = 1, C_2 = 0$, we have $S(\widetilde{q}) = \frac{\mu R^2}{4C_1} = 100, S_m^r(n_m) = \mu n_m[R - (C_1(n_m - 1))] > 10[20 - (10 - 1)] = 110 > S(\widetilde{q})$. Thus, whether a profit-maximizing SA publishes the real-time number of customers depends on system parameters. Similarly, the relationship of size between $S_m^r(n_s)$ and $S(\widetilde{q})$ cannot be completely determined.

(b) For any $n \leq n_e$, we have

$$\Lambda \sum_{m=0}^{n-1} \mathbb{P}(N = m)\left[R - (C_1 m + C_2 m^2)\right] \geq \Lambda \sum_{m=0}^{n-1} \mathbb{P}(N = m)\left[R - (C_1(n-1) + C_2(n-1)^2)\right]$$

$$= \Lambda \mathbb{P}(N < n)[R - (C_1(n-1) + C_2(n-1)^2)]$$

$$= \mu\mathbb{E}(L_n)[R - (C_1(n-1) + C_2(n-1)^2)],$$

which, combining with (4) and (7), implies that $S^r(n_m) > S_m^r(n_m)$. This shows that if the revenue-maximizing SA chooses to release the real-time number of customers, that is, $S_m^r(n_m) \geq S(\widetilde{q})$, this

behavior for the public should also be actively advocated because $S^r(n_m) \geq S^r_m(n_m) \geq S(\widetilde{q})$ implies $S^r(n_m) \geq S(\widetilde{q})$.

5. Numerical comparisons

In this section, we present some numerical results for both observable and unobservable models. We mainly focus on comparing the social welfare and revenue under these two models to gain insight into some valuable results that have been or have not been proven. Finally, we also give a simple example to calculate each quantity.

In Figure 1, we compare the social welfare with different Λ (ρ with $\mu = 1$). From the figure, we could observe the following facts.

1. $S^r(n_s)$ and $S^r(n_m)$ are increasing in Λ, respectively. $S^r(n_s)$ is always bigger than $S(\widetilde{q})$. In fact, when $C_2 = 0$, we have $\rho q = \sum_{m=0}^{\infty} m \mathbb{P}(N = m)$ in (9). This, combining with expression of (4), shows that we can think of the unobservable social welfare problems as an observable (long-run) average reward model in the theory of MDPs with the stochastic Markov strategy, see Chapter 11 in Puterman [25]. Note that n_s is the deterministic stationary optimal strategy in this average reward model, thus we have $S(\widetilde{q}) \leq S^r(n_s)$.

2. $S^r(n_e)$ increases first and then decreases with respect to Λ. When Λ is relatively large, from the figure, a reasonable toll is a better choice to achieve the social optimum, which coincides with the actual strategy adopted. When Λ is relatively small, even if there is no charge, $S^r(n_e)$ is closer to the social optimal welfare. The reasons for this phenomenon have been analyzed in Remark 1.

3. When Λ gradually increases, $S^r(n_s)$ and $S^r(n_m)$ get closer and closer until they coincide. In fact, it follows from the proof of Theorem 4 that this is caused by the gradual approach of the two thresholds. Therefore, when Λ is large, the optimal strategy of the SA is gradually in line with the goal of social maximization.

4. When Λ is relatively small, we can see that $S^r(n_m)$ is less than $S(\widetilde{q})$. At this time, under the revenue-maximizing admission fee, not providing the number of customers is good for social welfare. When Λ is relatively large, $S^r(n_m) > S(\widetilde{q})$. At this time, providing the real-time number of customers is beneficial to the social welfare. In short, whether to publish the real-time number of customers needs depend on the choice of real parameters.

In Figure 2, we provide the revenue with different Λ. Observing the figure, we have the following statements.

1. $S^r_m(n_m)$ and $S(\widetilde{q})$ are increasing in Λ, respectively and after a simple judgment, $S(\widetilde{q})$ is a constant when $\Lambda \geq 7.5$. From the figure, we can also see that $S^r_m(n_s)$ is increasing in Λ although we can not prove it theoretically. An intuitive reason is that n_s and n_m gradually approach as Λ increases.

2. When Λ is relatively small, $S^r_m(n_m) < S(\widetilde{q})$, which means that under the revenue-maximizing admission fee, the SA has no incentive to publish the real-time number of customers. Meanwhile, we also see that $S^r_m(n_s) < S(\widetilde{q})$, so under the social welfare maximization threshold, the unobservable case will also have greater revenue than the observable case. That is to say, it is beneficial for the SA not to publish the real-time information in this circumstance.

3. When Λ gradually increases, $S^r_m(n_m)$ and $S^r_m(n_s)$ will get closer and closer and will exceed $S(\widetilde{q})$. Thus, for sufficiently large Λ, the SA is willing to publish real-time information under various optimal thresholds. In fact, we can see that there exists Λ such that $S^r_m(n_m) > S(\widetilde{q}) > S^r_m(n_s)$. Under this condition, the decision of the SA is the opposite to the decision with the social welfare-maximizing objective. Therefore, in order to maximize the optimal social welfare, it is necessary to induce the SA to publish the real-time information.
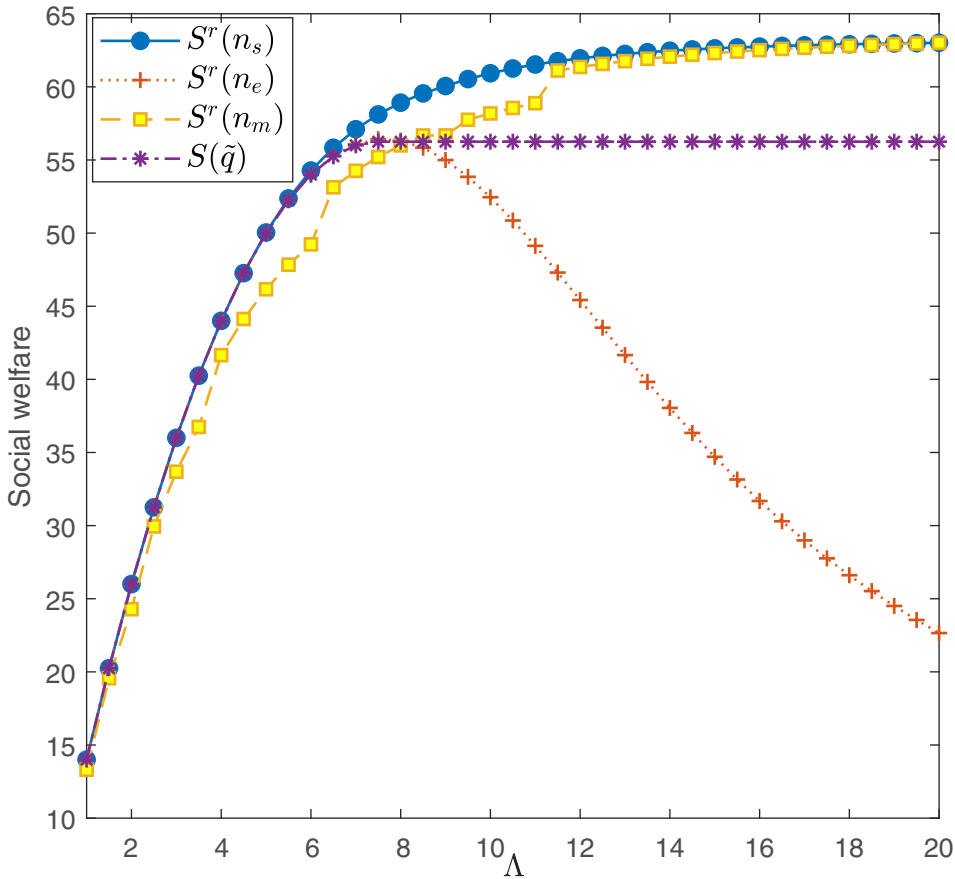
Figure 1. *Comparison of the social welfare per time unit vs. ρ for $\mu = 1, R = 15, C_1 = 1, C_2 = 0$.*

Finally, we also complement a concrete example to give a numerical solution of various quantities.

Example 1. Suppose that potential customers arrive at a system 20 persons per minute, the average sojourn time is 60 minutes, the service reward $R = 400$ and the cost is $C_1 = 0, C_2 = 0.01$. Then, $\rho = 1,200$ and using the results of Sections 3 and 4, we have $n_e = 201$, $n_s = 116$, $n_m = 116$, $q_e = \frac{1}{6}$, $q_s = \frac{\sqrt{3}}{18}$, $S^r(n_s) = 517.64$, $S^r(n_e) = 2.66$, $\widetilde{P}_o = 265.44$, $S_m^r(n_m) = 512.71$, $\widetilde{P}_u = 266.67$, $S(q_e) = 0, S(\widetilde{q}) = 513.20$.

In this example, the social optimal threshold and the revenue optimal threshold are the same. Comparing $S^r(n_s)$ and $S^r(n_e)$, whether it is a free system that controls the number of people or a toll system that controls the threshold through charging, the social welfare will be significantly improved.

6. Proofs of the main results

This part is devoted to the proofs of Theorems 2 and 3. To begin with, we need to state and prove some lemmas and propositions.
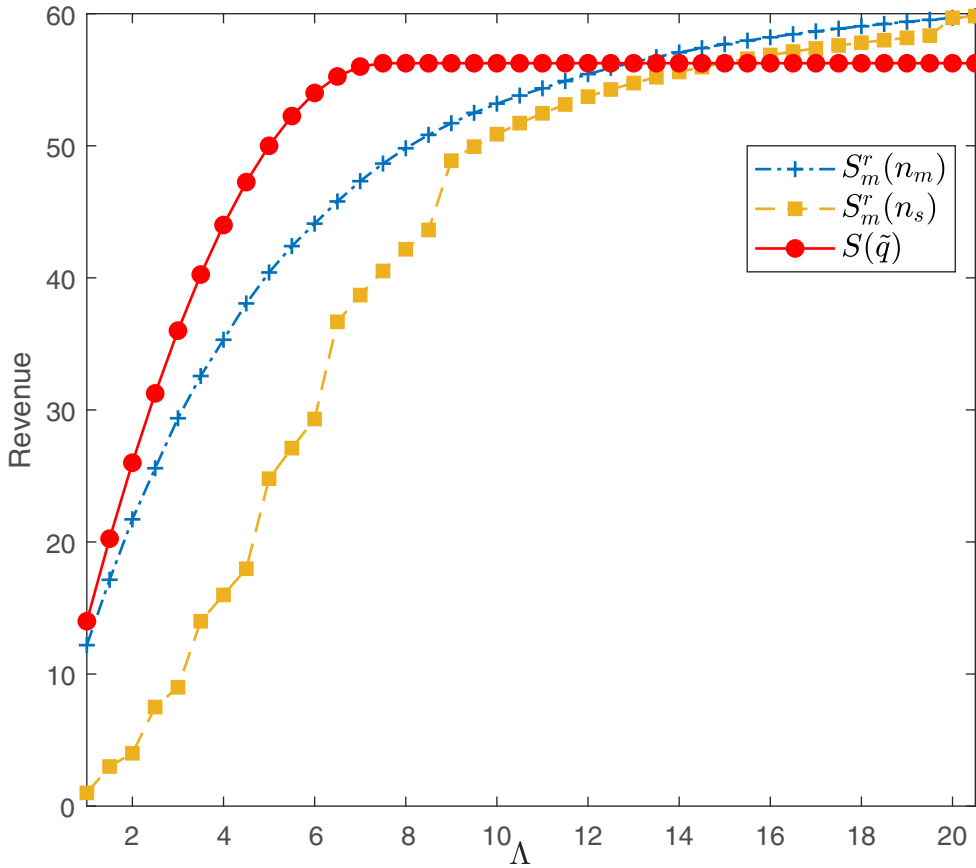
Figure 2. *Comparison of optimal revenue per time unit vs. ρ for $\mu = 1$, $R = 15$, $C_1 = 1$, $C_2 = 0$.*

Lemma 1. *Let function $f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ $(b_i > 0)$ and $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ $(a_i > 0)$. Then, if $\frac{b_n}{a_n} \geq \frac{b_{n-1}}{a_{n-1}} \geq \cdots \geq \frac{b_0}{a_0}$, $\frac{f(x)}{g(x)}$ is increasing in x. In particular, when one of the above inequalities is strict, $\frac{f(x)}{g(x)}$ is strictly increasing in x.*

Proof. Through differentiation with respect to x, we have

$$
\mathrm{d}\left(\frac{f(x)}{g(x)}\right) / \mathrm{d}x = \frac{\left(\sum_{k=1}^n k b_k x^{k-1}\right)\left(\sum_{k=0}^n a_k x^k\right) - \left(\sum_{k=0}^n b_k x^k\right)\left(\sum_{k=1}^n k a_k x^{k-1}\right)}{g(x)^2}
$$

$$
= \frac{\sum_{m=0}^{2n-1}\left[\sum_{\substack{i+j=m \\ 0 \leq i \leq n-1 \\ 0 \leq j \leq n}} (i+1) b_{i+1} a_j x^m - \sum_{\substack{i+j=m \\ 0 \leq i \leq n \\ 0 \leq j \leq n-1}} (j+1) b_i a_{j+1} x^m\right]}{g(x)^2}
$$

$$
= \frac{\sum_{m=0}^{2n-1}\left[\sum_{i=\max\{m-n,0\}}^{\min\{m,n-1\}} (i+1) b_{i+1} a_{m-i} x^m - \sum_{i=\max\{m-n,0\}}^{\min\{m,n-1\}} (i+1) b_{m-i} a_{i+1} x^m\right]}{g(x)^2}. \quad (10)
$$

Since $\frac{b_{i+1}}{a_{i+1}} \geq \frac{b_{m-i}}{a_{m-i}}$ for $i+1 > m-i$, we have $[(i+1)b_{i+1}a_{m-i} + (m-i)b_{m-i}a_{i+1}] - [(i+1)b_{m-i}a_{i+1} + (m-i)b_{i+1}a_{m-i}] = (2i+1-m)b_{i+1}a_{m-i} - (2i+1-m)b_{m-i}a_{i+1} \geq 0$. This immediately implies that

$$\sum_{i=\max\{m-n,0\}}^{\min\{m,n-1\}} (i+1)b_{i+1}a_{m-i} - (i+1)b_{m-i}a_{i+1} \geq 0. \tag{11}$$

Thus, $\frac{f(x)}{g(x)}$ is increasing in x. Obviously, when one of the inequalities is strict, the monotonicity is also strict from the (10) and (11). $\qquad\square$

Proposition 1. *Let* $f_n(\rho) = \sum_{m=0}^{n} \frac{\rho^m}{m!}$ *for* $n \geq 0$ *and* $f_{-1}(\rho) = 0$. *Then, the following statements hold.*

(a) $\frac{f_{n-t}(\rho)f_n(\rho) - f_{n-1-t}(\rho)f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho)f_{n+1}(\rho)}$ *is strictly increasing in* n *for* $t = 1, 2, \ldots, n$.

(b) $\frac{\rho^t f_{n-t}(\rho)f_n(\rho) - \rho^t f_{n-1-t}(\rho)f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho)f_{n+1}(\rho)}$ *is strictly increasing in* ρ *for* $t = 1, 2, \ldots, n$.

(c) $\frac{\rho\left[\frac{\sum_{m=0}^{n} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\rho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\rho^m}{m!}}{\sum_{m=0}^{n} \frac{\rho^m}{m!}}\right]}{\mathbb{E}[L_{n+1}] - \mathbb{E}[L_n]}$ *is strictly increasing in* n *and* ρ, *respectively, for* $i = 1, 2, 3, \ldots$.

Proof.

(a) We just need to show $\frac{f_{n-t}(\rho)f_n(\rho) - f_{n-1-t}(\rho)f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho)f_{n+1}(\rho)} > \frac{f_{n-1-t}(\rho)f_{n-1}(\rho) - f_{n-2-t}(\rho)f_n(\rho)}{f_{n-1}(\rho)^2 - f_{n-2}(\rho)f_n(\rho)}$, which holds if and only if for $n \geq t+1$,

$$\begin{aligned}
&f_{n-t}(\rho)f_{n-1}(\rho)^2 + f_{n-2-t}(\rho)f_n(\rho)^2 + f_{n-1-t}(\rho)f_{n-2}(\rho)f_{n+1}(\rho) \\
&> f_{n-1-t}(\rho)f_{n-1}(\rho)f_n(\rho) + f_{n-t}(\rho)f_{n-2}(\rho)f_n(\rho) + f_{n-2-t}(\rho)f_{n-1}(\rho)f_{n+1}(\rho).
\end{aligned} \tag{12}$$

Denoted the coefficients of the mth power on both sides of inequality (12) by

$$G(m) \triangleq \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n-1}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-2 \\ 0 \leq k \leq n+1}} \frac{1}{i!j!k!}, \tag{13}$$

$$H(m) \triangleq \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-2 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n+1}} \frac{1}{i!j!k!}. \tag{14}$$

Then, it is clear to see that (12) holds if $G(m) \geq H(m)$ for $0 \leq m \leq 3n - 2 - t$ and at least one inequality sign is strictly established. Next, we show that this condition is always true.

We give a concrete proof for $2 \leq t \leq n-2$ and for $t = 1$ or $t = n-1$, we can also prove it by a similar method. If $m < n - t$, by (13) and (14), we have

$$
\begin{aligned}
G(m) - H(m) \quad = \quad & \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n-t}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n-t}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n-t}} \frac{1}{i!j!k!} \\
& - \left[\sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n-t}} \frac{1}{i!j!k!} + \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n-t \\ 0 \leq k \leq n-t}} \frac{1}{i!j!k!} \right] = 0.
\end{aligned}
$$

If $m \geq n - t$, taking differences, we have

$$
\begin{aligned}
\sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n-1}} \frac{1}{i!j!k!} - \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-t \\ 0 \leq j \leq n-2 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} \quad = \quad & \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n-1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-2 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} \\
= \quad & \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0 \leq k \leq n-1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-2 \\ k=n}} \frac{1}{i!j!k!}. \quad (15)
\end{aligned}
$$

Similarly, we arrive at the following two expressions.

$$
\sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-2 \\ 0 \leq k \leq n+1}} \frac{1}{i!j!k!} - \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-1-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} = \sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-2 \\ k=n+1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!}, \quad (16)
$$

$$
\sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} - \sum_{\substack{i+j+k=m \\ 0 \leq i \leq n-2-t \\ 0 \leq j \leq n-1 \\ 0 \leq k \leq n+1}} \frac{1}{i!j!k!} = \sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ j=n \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-1 \\ k=n+1}} \frac{1}{i!j!k!}. \quad (17)
$$

This, combining with (13) and (14), implies that

$$
\begin{aligned}
G(m) - H(m) = & \left[\sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0 \leq k \leq n-1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-2 \\ k=n}} \frac{1}{i!j!k!} \right] \\
& + \left[\sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-2 \\ k=n+1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} \right] \\
& + \left[\sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ j=n \\ 0 \leq k \leq n}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ 0 \leq j \leq n-1 \\ k=n+1}} \frac{1}{i!j!k!} \right]. \quad (18)
\end{aligned}
$$

By taking differences, we also have that

$$
\sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0\le k\le n-1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ j=n-1 \\ 0\le k\le n}} \frac{1}{i!j!k!}
$$

$$
= \frac{1}{(n-t)!(n-1)!(m-2n+t+1)!} - \frac{1}{(m-2n+1)!(n-1)!n!}\mathbf{1}_{\{m\ge 2n-1\}}, \tag{19}
$$

$$
\sum_{i=0}^{n-1-t} \sum_{\substack{i+j+k=m \\ 0\le j\le n-2 \\ k=n+1}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ 0\le j\le n-1 \\ k=n+1}} \frac{1}{i!j!k!}
$$

$$
= \frac{1}{(n-1-t)!(m-2n+t)!(n+1)!} - \frac{1}{(m-2n)!(n-1)!(n+1)!}\mathbf{1}_{\{m\ge 2n\}}, \tag{20}
$$

and

$$
\sum_{i=0}^{n-2-t} \sum_{\substack{i+j+k=m \\ j=n \\ 0\le k\le n}} \frac{1}{i!j!k!} - \sum_{i=0}^{n-t} \sum_{\substack{i+j+k=m \\ 0\le j\le n-2 \\ k=n}} \frac{1}{i!j!k!}
$$

$$
= -\frac{1}{(n-t)!(m-2n+t)!n!} - \frac{1}{(n-1-t)!(m-2n+t+1)!n!}
$$

$$
+\frac{1}{(m-2n)!n!n!}\mathbf{1}_{\{m\ge 2n\}} + \frac{1}{(m-2n+1)!(n-1)!n!}\mathbf{1}_{\{m\ge 2n-1\}}, \tag{21}
$$

where $\mathbf{1}_B$ is the indicator function of B taking value 1 if the event B is true and 0 otherwise and (21) also holds formally for $m = 3n - 2 - t$. We substitute (19), (20), and (21) into (18), and simple calculations show that

$$
G(m) - H(m) = \frac{1}{(n-t)!(n-1)!(m-2n+t+1)!} + \frac{1}{(n-1-t)!(m-2n+t)!(n+1)!}
$$

$$
- \frac{1}{(m-2n)!(n-1)!(n+1)!}\mathbf{1}_{\{m\ge 2n\}} - \frac{1}{(n-t)!(m-2n+t)!n!}
$$

$$
- \frac{1}{(n-1-t)!(m-2n+t+1)!(n)!} + \frac{1}{(m-2n)!n!n!}\mathbf{1}_{\{m\ge 2n\}}
$$

$$
= \frac{3n-t-1-m}{(n-t)!n!(m-2n+t+1)!} + \frac{m-3n+t}{(n-1-t)!(m-2n+t+1)!(n+1)!}
$$

$$
+ \frac{1}{(m-2n)!n!(n+1)!}\mathbf{1}_{\{m\ge 2n\}}
$$

$$
= \frac{(3n-t-m)(t+1)-(n+1)}{(n-t)!(n+1)!(m-2n+t+1)!} + \frac{1}{(m-2n)!n!(n+1)!}\mathbf{1}_{\{m\ge 2n\}}.
$$

When $n-t\le m<2n$, we have $(3n-t-m)(t+1)-(n+1) > (n-t)(t+1)-(n+1) = (n-t-1)t-1>0$. When $m \ge 2n$, it is clear to see that

$$
\frac{(3n-t-m)(t+1)-(n+1)}{(n-t)!(n+1)!(m-2n+t+1)!} + \frac{1}{(m-2n)!n!(n+1)!}\mathbf{1}_{\{m\ge 2n\}} > 0 \tag{22}
$$

if and only if

$$\frac{(m - 2n + t + 1)!(n - t)!}{(m - 2n)!n!} - (m - 2n)(t + 1) > (n + 1) - (n - t)(t + 1). \tag{23}$$

Since

$$\frac{(z + t + 1)(z + t) \cdots (z + 1)}{n(n - 1) \cdots (n - t + 1)} - z(t + 1)$$

is decreasing in z for $0 \le z \le n - 2 - t$ and for $m = 3n - 2 - t$,

$$\frac{(3n - t - m)(t + 1) - (n + 1)}{(n - t)!(n + 1)!(m - 2n + t + 1)!} + \frac{1}{(m - 2n)!(n)!(n + 1)!}\mathbf{1}_{\{m \ge 2n\}}$$

$$= \frac{t^2 + t}{(n - t)!(n + 1)!n!} > 0,$$

then, (23) holds for $m \ge 2n$. This yields that $G(m) > H(m)$ when $m \ge n - t$. Therefore, (12) always holds for any $\rho > 0$, which completes the proof.

(b) First, by rearranging terms, we arrive at the following more compact representation:

$$\rho^t f_{n-t}(\rho) f_n(\rho) - \rho^t f_{n-1-t}(\rho) f_{n+1}(\rho)$$

$$= \rho^t \sum_{m=0}^{2n-t} \left[\sum_{i=\max\{m-n,0\}}^{\min\{m,n-t\}} \frac{\rho^m}{i!(m-i)!} - \sum_{i=\max\{m-n-1,0\}}^{\min\{m,n-1-t\}} \frac{\rho^m}{i!(m-i)!} \right]$$

$$= \rho^t \sum_{m=0}^{2n-t} \left[\sum_{i=\max\{m-n,0\}}^{\min\{m,n-t\}} \frac{1}{i!(m-i)!} - \sum_{i=\max\{m-n-1,0\}}^{\min\{m,n-1-t\}} \frac{1}{i!(m-i)!} \right] \rho^m. \tag{24}$$

In order to analyze the relationship between the coefficients of $\frac{\rho^t f_{n-t}(\rho) f_n(\rho) - \rho^t f_{n-1-t}(\rho) f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho) f_{n+1}(\rho)}$ and use the results of Lemma 1, we first define $F(m, t) = \sum_{i=\max\{m-n,0\}}^{\min\{m,n-t\}} \frac{1}{i!(m-i)!} - \sum_{i=\max\{m-n-1,0\}}^{\min\{m,n-1-t\}} \frac{1}{i!(m-i)!}$. Then, we have that for $t \ge 0$,

1. if $m \le n - 1 - t$, $F(m, t) = 0$;
2. if $n - t \le m \le n$, $F(m, t) = \frac{1}{(n-t)!(m-n+t)!} > 0$;
3. if $n + 1 \le m \le 2n - t$, $F(m, t) = \frac{1}{(n-t)!(m-n+t)!} - \frac{1}{(m-n-1)!(n+1)!} > 0$.

Thus, $\sum_{m=0}^{2n-t} F(m, t) > 0$ and after simple calculations we also have that

1. if $m \le n - 1 - t$ $(m + t \le n - 1)$, $F(m, t) = 0$ and $F(m + t, 0) = 0$;
2. if $m = n - t$ $(m + t = n)$, $F(m, t) = \frac{1}{(n-t)!(m-n+t)!} > 0$ and $F(m + t, 0) = \frac{1}{n!(m-n+t)!} > 0$;
3. if $n - t + 1 \le m \le n$ $(n + 1 \le m + t \le n + t)$, $F(m, t) = \frac{1}{(n-t)!(m-n+t)!}$ and $F(m + t, 0) = \frac{1}{n!(m-n+t)!} - \frac{1}{(m+t-n-1)!(n+1)!}$;
4. if $n + 1 \le m \le 2n - t$ $(n + 1 + t \le m + t \le 2n)$, $F(m, t) = \frac{1}{(n-t)!(m-n+t)!} - \frac{1}{(m-n-1)!(n+1)!}$ and $F(m + t, 0) = \frac{1}{n!(m-n+t)!} - \frac{1}{(m+t-n-1)!(n+1)!}$.

This immediately implies that for

$$\left[(n + 1) \cdots (n - t + 1) - (m + 1 - n + t) \cdots (m + 1 - n)\mathbf{1}_{\{m+1 \ge n+1\}} \right] \left[(n + 1) \right.$$

$$- (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}\big] - \big[(n + 1) \cdots (n - t + 1) - (m - n + t) \cdots (m - n)\mathbf{1}_{\{m \geq n+1\}}\big]$$
$$\big[(n + 1) - (m + 1 - n + t)\mathbf{1}_{\{m+1 \geq n-t+1\}}\big]$$
$$= (n + 1) \cdots (n - t + 1)\big[(m + 1 - n + t)\mathbf{1}_{\{m+1 \geq n-t+1\}} - (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}\big]$$
$$+ \big[(m - n + t) \cdots (m - n)\mathbf{1}_{\{m \geq n+1\}}\big]\big[(n + 1) - (m + 1 - n + t)\mathbf{1}_{\{m+1 \geq n-t+1\}}\big]$$
$$- \big[(m + 1 - n + t) \cdots (m + 1 - n)\mathbf{1}_{\{m+1 \geq n+1\}}\big]\big[(n + 1) - (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}\big], \qquad (25)$$

we have the following case:

1. if $m = n - t$, $(25) = (n + 1) \cdots (n - t + 1) > 0$;
2. if $n - t + 1 \leq m \leq n - 1$, $(25) = (n + 1) \cdots (n - t + 1) > 0$ (If $t = 1$, there is no such item);
3. if $m = n$, $(25) = (n + 1) \cdots (n - t + 1) - [(t + 1)!\mathbf{1}_{\{m+1 \geq n+1\}}][(n + 1) - (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}] > 0$;
4. if $m \geq n + 1$, $(25) = (n + 1) \cdots (n - t + 1) - ((n + 1) - t(2n - t - m))(m - n + t) \cdots (m - n + 1) > 0$.

Note that for $m \geq n - t$,

$$\frac{F(m, t)}{F(m + t, 0)} < \frac{F(m + 1, t)}{F(m + 1 + t, 0)}$$

$$\Leftrightarrow \frac{(n + 1) \cdots (n - t + 1) - (m - n + t) \cdots (m - n)\mathbf{1}_{\{m \geq n+1\}}}{(n + 1) - (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}} <$$
$$\frac{(n + 1) \cdots (n - t + 1) - (m + 1 - n + t) \cdots (m + 1 - n)\mathbf{1}_{\{m+1 \geq n+1\}}}{(n + 1) - (m + 1 - n + t)\mathbf{1}_{\{m+1 \geq n-t+1\}}}$$

$$\Leftrightarrow [(n + 1) \cdots (n - t + 1) - (m + 1 - n + t) \cdots (m + 1 - n)\mathbf{1}_{\{m+1 \geq n+1\}}][(n + 1)$$
$$- (m - n + t)\mathbf{1}_{\{m \geq n-t+1\}}]$$
$$> [(n + 1) \cdots (n - t + 1) - (m - n + t) \cdots (m - n)\mathbf{1}_{\{m \geq n+1\}}][(n + 1)$$
$$- (m + 1 - n + t)\mathbf{1}_{\{m+1 \geq n-t+1\}}],$$

which is always true and have been proved in (25). Thus,

$$\frac{\rho^t F(m, t)\rho^m}{F(m + t, 0)\rho^{m+t}}$$

is strictly increasing in m for $m \geq n - t$. Because $F(m, t)$ and $F(m + t, 0)$ are the coefficients of $m + t$ power of $\rho^t f_{n-t}(\rho)f_n(\rho) - \rho^t f_{n-1-t}(\rho)f_{n+1}(\rho)$ and $f_n(\rho)^2 - f_{n-1}(\rho)f_{n+1}(\rho)$ (when $t = 0$), respectively, it immediately follows from Lemma 1 that $\frac{\rho^t f_{n-t}(\rho)f_n(\rho) - \rho^t f_{n-1-t}(\rho)f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho)f_{n+1}(\rho)}$ is strictly increasing in ρ for $n \geq t \geq 1$.

(c) For any $i = 1, 2, 3, \ldots$, we have

$$\sum_{m=0}^{n} m^i \frac{\rho^m}{m!} = \rho \sum_{m=0}^{n-1} (m + 1)^{i-1} \frac{\rho^m}{m!} = \rho \left[\sum_{m=0}^{n-1}\left(\sum_{k=0}^{i-1} \binom{i-1}{k} m^k \frac{\rho^m}{m!}\right)\right]$$
$$= \rho \sum_{k=1}^{i-1}\left[\sum_{m=0}^{n-1} \binom{i-1}{k} m^k \frac{\rho^m}{m!}\right] + \rho \sum_{m=0}^{n-1} \frac{\rho^m}{m!}.$$

Then, repeating the above arguments, we could derive the following expansion:

$$\sum_{m=0}^{n} m^i \frac{\rho^m}{m!} = \sum_{j=1}^{\min\{n,i\}} a_j \rho^j \sum_{m=0}^{n-j} \frac{\rho^m}{m!},$$

where a_j is a positive constant. This means that

$$\frac{\rho\left[\frac{\sum_{m=0}^{n} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\varrho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n} \frac{\varrho^m}{m!}}\right]}{\mathbb{E}[L_{n+1}] - \mathbb{E}[L_n]}$$

$$= \frac{\frac{\sum_{j=1}^{\min\{n,i\}} a_j \rho^j \sum_{m=0}^{n-j} \frac{\varrho^m}{m!}}{f_{n+1}(\rho)} - \frac{\sum_{j=1}^{\min\{n-1,i\}} a_j \rho^j \sum_{m=0}^{n-1-j} \frac{\varrho^m}{m!}}{f_n(\rho)}}{\frac{f_n(\rho)}{f_{n+1}(\rho)} - \frac{f_{n-1}(\rho)}{f_n(\rho)}}$$

$$= \sum_{j=1}^{\min\{n-1,i\}} a_j \frac{\rho^j f_{n-j}(\rho) f_n(\rho) - \rho^j f_{n-1-j}(\rho) f_{n+1}(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho) f_{n+1}(\rho)} + a_n \mathbf{1}_{\{n \geq i\}} \frac{\rho^n f_0(\rho) f_n(\rho)}{f_n(\rho)^2 - f_{n-1}(\rho) f_{n+1}(\rho)}. \quad (26)$$

Using the linearity of summation, by (a) and (b), the result holds immediately.

$\square$

Proof of Theorem 2 By the sample path comparison, it is easy to have $n_s \leq n_e$, which ensures that n_s is finite. According to the definition of n_s, we have $S^r(n_s) \geq S^r(n_s - 1)$ and $S^r(n_s) > S^r(n_s + 1)$. Using algebraic manipulations analogous to those in Section 2.4 in Hassin and Haviv [15], it follows from (5) that these relations can also be rewritten as

$$\frac{\rho \sum_{i=1}^{2} C_i \left[\frac{\sum_{m=0}^{n_s} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n_s+1} \frac{\varrho^m}{m!}} - \frac{\sum_{m=0}^{n_s-1} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n_s} \frac{\varrho^m}{m!}}\right]}{\mathbb{E}(L_{n_s+1}) - \mathbb{E}(L_{n_s})} > R \geq \frac{\rho \sum_{i=1}^{2} C_i \left[\frac{\sum_{m=0}^{n_s-1} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n_s} \frac{\varrho^m}{m!}} - \frac{\sum_{m=0}^{n_s-2} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n_s-1} \frac{\varrho^m}{m!}}\right]}{\mathbb{E}(L_{n_s}) - \mathbb{E}(L_{n_s-1})}.$$

By the results of Proposition 1(c), $\dfrac{\rho \sum_{i=1}^{2} C_i \left[\frac{\sum_{m=0}^{n} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\varrho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n} \frac{\varrho^m}{m!}}\right]}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)}$ is strictly increasing in n, so n_s is unique.

Since $\dfrac{\rho \sum_{i=1}^{2} C_i \left[\frac{\sum_{m=0}^{n} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n+1} \frac{\varrho^m}{m!}} - \frac{\sum_{m=0}^{n-1} m^i \frac{\varrho^m}{m!}}{\sum_{m=0}^{n} \frac{\varrho^m}{m!}}\right]}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)}$ is strictly increasing in ρ, n_s is decreasing in ρ. $\square$

Proposition 2.

(a) $\mathbb{E}(L_n)$ is strictly increasing in n.
(b) $\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n) > \mathbb{E}(L_{n+2}) - \mathbb{E}(L_{n+1})$.
(c) $\mathbb{E}(L_n)^2 > \mathbb{E}(L_{n+1})\mathbb{E}(L_{n-1})$, that is, $\frac{\mathbb{E}(L_n)}{\mathbb{E}(L_{n-1})} > \frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)}$.
(d) $\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)}$ is increasing in ρ.

Proof.

(a) We use a coupling method here although we may also directly prove it by taking differences. Noting that $M/G/n/n$ and $M/M/n/n$ have the same expression of $\mathbb{E}(L_n)$, we only need to consider the Markovian case. Suppose that Process 1($P1$) and Process 2($P2$) is an $M/M/n/n$ queueing process and an $M/M/(n+1)/(n+1)$ queueing process, respectively, and once customers enter the system, they will not leave until the service is finished. Follow the sample paths of two processes defined on the same probability space and starting in the same state $s < n$. Both processes remain coupled and thus see the same arrivals, services for each customer when the number of customers shall not exceed n. This implies that both systems have the same customers. Consider two processes enter the state n for the first time and a new customer arrives. $P1$ rejects the customer but $P2$ accepts this customer. At this point, the queue length of the $P2$ is greater than the length queue of the $P1$.

Then, if $P1$ rejects an arriving customer at state n, $P2$ also rejects the customer at state $n+1$. If a service is the next event for $P1$, $P2$ also completes a service with probability 1. If $P1$ accepts an arriving customer, $P2$ must accept this customer and vice versa. If only $P2$ completes a service for the "$n+1$" customer, both processes remain coupled until the next time in state n with an arriving customer.

   Thus, the queue length of $P1$ is always less than the queue length of $P2$. Since the state n is positive recurrent for $P2$, the unequal relationship of expected queue length must be strict, that is, $\mathbb{E}(L_n) < \mathbb{E}(L_{n+1})$, which completes the proof.

(b) We still use the coupling method and consider four systems, an $M/M/n/n$ system P1, two $M/M/(n+1)/(n+1)$ systems $P2$ and $P2'$, and an $M/M/(n+2)/(n+2)$ system $P3$. All four systems share the same arrival stream. The service times of an arrival to the four systems are not necessarily the same but are coupled in the following way.

   (i) If the customer enters service in all systems, the service times are all assigned by the same random variable. We also say that the customer in system $P1$ is coupled with that in system $P2$, and the customer in $P3$ is coupled with that in $P2'$.

   (ii) If the customer enters service in system $P2$ and $P3$, the service times in the two system are assigned by the same random variable. We say that the customer in $P3$ is coupled with that in $P2$.

   (iii) If the customer enters service in system $P2'$ and $P3$, the service times in the two system are assigned by the same random variable. We say that the customer in $P3$ is coupled with that in $P2'$.

   (iv) If the customer enters service in system $P2$, $P2'$, and $P3$, the service times in system $P2$ and $P3$ are assigned by the same random variable, while that in system $P2'$ is assigned with a new independent random variable. We say that the customer in $P3$ is coupled with that in $P2$, while the customer in $P2'$ is uncoupled.

   (v) If the customer enters service in system $P3$ only, and the customer finds an uncoupled customer in $P2'$, the service time of the customer in $P3$ is assigned by the remaining service time of the uncoupled customer in $P2'$, which is exponentially distributed according to the memoryless property. We say that the uncoupled customer is now coupled with the new arrival in $P3$.

   (vi) The customer does not enters any system and there is nothing to consider.

   One should argue that any arrival will see one case from (i) to (vi) using induction. In particular, any arrival that enters $P1$ must enter the other three systems, while any arrival to $P3$ only must find an uncoupled customer in $P2'$. The induction should also imply that each customer that enters service in $P1$ or $P3$ has a coupled customer who enters service in $P2$ or $P2'$ either earlier or at the same time, such that the services between coupled customers complete at the same time. Therefore, one must have $L_1(t) + L_3(t) \le L_2(t) + L_{2'}(t)$ for any t, in which $L_i(t)$ is the head count in system Pi ($i = 1, 2, 2', 3$), implying Proposition 2(b) by sending $t \to \infty$.

(c) It follows from (b) that

$$[2\mathbb{E}(L_n)]^2 > [\mathbb{E}(L_{n-1}) + \mathbb{E}(L_{n+1})]^2 \ge [\mathbb{E}(L_{n-1}) - \mathbb{E}(L_{n+1})]^2 + 4\mathbb{E}(L_{n-1})\mathbb{E}(L_{n+1}).$$

   Thus, $\mathbb{E}(L_n)^2 > \mathbb{E}(L_{n+1})\mathbb{E}(L_{n-1})$.

(d) It follows from (3) that $\frac{d\mathbb{E}(L_n)}{d\rho} = \rho^{-1}\mathbb{E}(L_n)[1 - \mathbb{E}(L_n) + \mathbb{E}(L_{n-1})]$. This, together with (b), shows that

$$\frac{d(\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)})}{d\rho} = \frac{\rho^{-1}\mathbb{E}(L_{n+1})[1 - \mathbb{E}(L_{n+1}) + \mathbb{E}(L_n)]\mathbb{E}(L_n) - \mathbb{E}(L_{n+1})\rho^{-1}\mathbb{E}(L_n)[1 - \mathbb{E}(L_n) + \mathbb{E}(L_{n-1})]}{\mathbb{E}(L_n)^2}$$

$$= \frac{\mathbb{E}(L_{n+1})\mathbb{E}(L_n)[2\mathbb{E}(L_n) - \mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)]}{\rho\mathbb{E}(L_n)^2} > 0.$$

Thus, $\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)}$ is strictly increasing in ρ. The proof is complete.

$\square$

Proof of Theorem 3 By the definition of n_m, we have $S(n_m) \geq S(n_m - 1)$ and $S(n_m) > S(n_m + 1)$. Applying a similar argument to that in the proof of Theorem 2, these relations can also be rewritten as

$$\frac{\sum_{i=1}^{2} C_i [\mathbb{E}(L_{n_m+1})(n_m)^i - \mathbb{E}(L_{n_m})(n_m-1)^i]}{\mathbb{E}(L_{n_m+1}) - \mathbb{E}(L_{n_m})} > R \geq \frac{\sum_{i=1}^{2} C_i [\mathbb{E}(L_{n_m})(n_m-1)^i - \mathbb{E}(L_{n_m-1})(n_m-2)^i]}{\mathbb{E}(L_{n_m}) - \mathbb{E}(L_{n_m-1})}.$$

Note that

$$\begin{aligned}
\frac{\mathbb{E}(L_{n+1})n^i - \mathbb{E}(L_n)(n-1)^i}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)} &= n^i + \frac{\mathbb{E}(L_n)(n^i - (n-1)^i)}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)} \\
&= (n+1)^i + \frac{n^i - (n-1)^i}{\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)} - 1}.
\end{aligned}$$

By Lemma 2(c), $\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)}$ is strictly decreasing in n and $n^i - (n-1)^i$ is increasing in n for $n \geq 1$, which means that $\frac{\mathbb{E}(L_{n+1})n^i - \mathbb{E}(L_n)(n-1)^i}{\mathbb{E}(L_{n+1}) - \mathbb{E}(L_n)}$ is strictly increasing in n. Thus, n_m is unique. By Lemma 2(d), we have that $\frac{n^i - (n-1)^i}{\frac{\mathbb{E}(L_{n+1})}{\mathbb{E}(L_n)} - 1}$ is decreasing in ρ, which immediately shows that n_m is increasing in ρ. The proof is complete. $\square$

Remark 7. Although the cost function in Theorems 2 and 3 is a combination of a linear function and a quadratic function, the methods in these two proofs are very general and suitable for any $i \geq 1$. Thus, the results of Theorems 2 and 3 can be generalized to the case in which the cost is any finite polynomial function with nonnegative coefficients.

7. Conclusions and extensions

In this paper, we consider the equilibrium, social welfare, and revenue of an infinite-server queue in both observable and unobservable contexts and get the existence, uniqueness, and computable expressions of optimal strategies for these goals. We also numerically compare the social welfare and the revenue with different thresholds and information levels, and insight into some useful information under different conditions.

On this topic, there is no denying that our hypothesis is somewhat rough compared to the actual background. Because of this, many expansion questions are worth studying. We make some comments on potential problems in the following.

1. In the actual environment, the arrival of customers is affected by many aspects, such as weather or holidays in the park examples. Therefore, analogous to Chen and Hasenbein [4], it is interesting and practical to study the model with uncertain arrival rates. For unobservable case, we could investigate it by similar methods. However, for the observable case, affected by expectations, we still encounter some monotonic proofs that need to be solved urgently, although a large number of numerical results show that they are correct. We look forward to proving it in the future.

2. In the notices posted in the system, we often see that customers are non-homogeneous and the system has price discrimination. For example, ticket prices of (toll) parks are related to age groups, regions or other requirements. Therefore, it is a meaningful direction to research and design (pricing) the infinite-server queue with multiple types of customers. There is a lot of literature focusing

on such queueing problems, such as Feinberg and Yang [10], Zhou, Chao, and Gong [32], Liu and Hasenbein [20], etc., so we believe that a similar method can be used to solve the infinite-server queue. Furthermore, considering the model of customers arriving in batches is also a more practical problem.

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