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NEAR OPTIMAL THRESHOLDS FOR EXISTENCE OF DILATED CONFIGURATIONS IN $\mathbb{F}_q^d$

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Abstract

Let $\mathbb{F}_q^d$ denote the d-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. Define $\|\alpha\| := \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$. Let $k \in \mathbb{N}$, A be a nonempty subset of $\{(i, j) : 1 \le i < j \le k+1\}$ and $r \in (\mathbb{F}_q)^2 \setminus 0$, where $(\mathbb{F}_q)^2 = \{a^2 : a \in \mathbb{F}_q\}$. If $E \subset \mathbb{F}_q^d$, our main result demonstrates that when the size of the set E satisfies $|E| \ge C_k q^{d/2}$, where C_k is a constant depending solely on k, it is possible to find two $(k+1)$-tuples in E such that one of them is dilated by r with respect to the other, but only along $|A|$ edges. To be more precise, we establish the existence of $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that, for $(i, j) \in A$, we have $\|y_i - y_j\| = r\|x_i - x_j\|$, with the conditions that $x_i \ne x_j$ and $y_i \ne y_j$ for $1 \le i < j \le k+1$, provided that $|E| \ge C_k q^{d/2}$ and $r \in (\mathbb{F}_q)^2 \setminus \{0\}$. We provide two distinct proofs of this result. The first uses the technique of group actions, a powerful method for addressing such problems, while the second is based on elementary combinatorial reasoning. Additionally, we establish that in dimension 2, the threshold $d/2$ is sharp when $q \equiv 3 \pmod 4$. As a corollary of the main result, by varying the underlying set A, we determine thresholds for the existence of dilated k-cycles, k-paths and k-stars (where $k \ge 3$) with a dilation ratio of $r \in (\mathbb{F}_q)^2 \setminus \{0\}$. These results improve and generalise the findings of Xie and Ge ['Some results on similar configurations in subsets of $\mathbb{F}_q^d$', *Finite Fields Appl.* **91** (2023), Article no. 102252, 20 pages].

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1. Introduction

Let $\mathbb{F}_q^d$, where $d \ge 2$, be the d-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. We assume that q is a power of an odd prime p.

Given a set E in $\mathbb{F}_q^d$, the distance set $\Delta(E)$ is defined by

$$\Delta(E) := \{\|x - y\| \in \mathbb{F}_q : x, y \in E\},$$

where $\|\alpha\| = \sum_{i=1}^d \alpha_i^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$.

In the finite field setting, the Falconer distance problem asks for the smallest exponent $\alpha > 0$ such that, for any $E \subset \mathbb{F}_q^d$ with $|E| \ge Cq^\alpha$, we have $|\Delta(E)| \ge cq$, where

$C > 1$ represents a sufficiently large constant and $0 < c \leq 1$ represents a constant independent of q and $|E|$. Here, $|E|$ denotes the cardinality of the set E.

This problem was initially proposed by Iosevich and Rudnev in [7] as a finite field counterpart of the Falconer distance problem in Euclidean space. The formulation of the finite field Falconer problem was also inspired by the Erdős distinct distances problem over finite fields, as introduced by Bourgain *et al.* in [2]. As a result, this problem is often referred to as the Erdős–Falconer distance problem.

One can consider a stronger version of the Erdős–Falconer distance problem known as the Mattila–Sjölin distance problem over finite fields. This problem seeks the smallest threshold $\beta > 0$ such that, for any $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\beta}$, we have $\Delta(E) = \mathbb{F}_q$. Iosevich and Rudnev, using Fourier analytic techniques and Kloosterman sum estimates, successfully determined the threshold to be $(d+1)/2$ for all dimensions $d \geq 2$.

THEOREM 1.1 (Iosevich and Rudnev, [7]). *If $E \subset \mathbb{F}_q^d$ $(d \geq 2)$ and $|E| > 2q^{(d+1)/2}$, then $\Delta(E) = \mathbb{F}_q$.*

The threshold $(d+1)/2$ in Theorem 1.1 is currently the best-known result for the Mattila–Sjölin distance problem over finite fields in all dimensions $d \geq 2$. It is considered as a challenging problem to improve the $(d+1)/2$ threshold. In odd dimensions, this threshold is proven to be optimal, as demonstrated in [4].

However, in even dimensions, there is a belief that the exponent $(d+1)/2$ can potentially be improved, but as of now, there is no reasonable evidence or conjecture stated in the literature. In dimension 2, Murphy and Petridis in [8] have shown that the threshold cannot be lower than $4/3$ for the Mattila–Sjölin distance problem over finite fields.

However, Iosevich and Rudnev in [7] conjectured that the threshold $(d+1)/2$ can be lowered to $d/2$ for the Erdős–Falconer distance problem in even dimensions. In dimension 2, the threshold $4/3$ was proven in [3] (see also [1]). Additionally, if q is a prime number, then the exponent $4/3$ was improved to $5/4$ by Murphy *et al.* in [9]. The threshold $(d+1)/2$ cannot be improved for the Erdős–Falconer distance problem in odd dimensions (see also [4]).

The distance problems over finite fields have been extended in various directions. Although numerous variants of the distance problems have been extensively studied, the threshold $d/2$ for the set E in $\mathbb{F}_q^d$ had not been addressed for any distance-type problems until Iosevich *et al.* in [5] studied the Mattila–Sjölin problem for the quotient set of the distance set over finite fields.

1.1. The Mattila–Sjölin problem for the quotient set of the distance set. If E is a subset of $\mathbb{F}_q^d$ $(d \geq 2)$, then the quotient set of the distance set, denoted by $\Delta(E)/\Delta(E)$, is defined as follows:

$$\frac{\Delta(E)}{\Delta(E)} := \left\{ \frac{||x-y||}{||z-w||} : x, y, z, w \in E,\ ||z-w|| \neq 0 \right\}.$$

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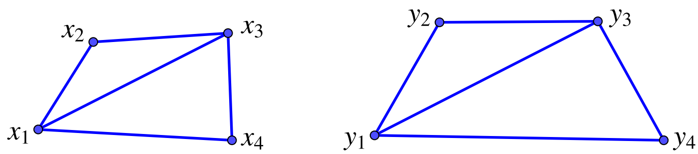

FIGURE 1. Pair of quadrilaterals with a dilation ratio of $r \in \mathbb{F}_q^*$.

One can ask the following question: determine the smallest exponent $\gamma > 0$ such that for any set $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^\gamma$, we have

$$\frac{\Delta(E)}{\Delta(E)} = \mathbb{F}_q.$$

In [5], the authors obtained the threshold $d/2$ in even dimensions for the Mattila–Sjölin problem for the quotient set of the distance set over finite fields. More precisely, they proved the following result. Here, $(\mathbb{F}_q)^2 := \{a^2 : a \in \mathbb{F}_q\}$.

THEOREM 1.2 [5, Theorems 1.1 and 1.2]. *If $E \subset \mathbb{F}_q^d$ $(d \geq 2)$, then the following statements hold:*

(i) *if $d \geq 2$ is even and $|E| \geq 9q^{d/2}$, then $\Delta(E)/\Delta(E) = \mathbb{F}_q$;*
(ii) *if $d \geq 3$ is odd and $|E| \geq 6q^{d/2}$, then $\Delta(E)/\Delta(E) \supseteq (\mathbb{F}_q)^2$.*

Theorem 1.2 has been extended and generalised with improved constants to general nondegenerate quadratic distances by Iosevich, Koh and the second author (see [6]).

For convenience, we will use the notation $\mathbb{F}_q^*$ to represent the set of all nonzero elements in $\mathbb{F}_q$.

The equality $\Delta(E)/\Delta(E) = \mathbb{F}_q$ implies that for each $r \in \mathbb{F}_q^*$, there exist $(x, y) \in E^2$ and $(x', y') \in E^2$ such that $||x - y|| \neq 0$ and $||x' - y'|| = r||x - y||$. In other words, if $r \in \mathbb{F}_q^*$ and $E \subseteq \mathbb{F}_q^d$ with $|E| \geq 9q^{d/2}$, then one can find two pairs $(x, y) \in E^2$ and $(x', y') \in E^2$ such that the length of one of them is r times that of the other.

The following natural question arises: is it possible to generalise this result to other point configurations in $E \subseteq \mathbb{F}_q^d$? For example, consider $r \in \mathbb{F}_q^*$ and $E \subset \mathbb{F}_q^2$. We aim to find the smallest exponent $\gamma > 0$ such that, for any $E \subset \mathbb{F}_q^2$ with $|E| \geq Cq^\gamma$, there exist 4-tuples $(x_1, x_2, x_3, x_4) \in E^4$ and $(y_1, y_2, y_3, y_4) \in E^4$ satisfying the conditions:

(1) $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq 4$;
(2) $||y_i - y_j|| = r||x_i - x_j||$ if $(i, j) \in A$, where $A = \{(1, 2), (2, 3), (3, 4), (1, 4), (1, 3)\}$ (refer to Figure 1).

We proceed to formulate the main question. Consider $r \in \mathbb{F}_q^*$ and $E \subset \mathbb{F}_q^d$, and assume that A is a nonempty subset of $\{(i, j) : 1 \leq i < j \leq k + 1\}$, where $k \geq 1$. How large must E be to ensure the existence of two sets of points, $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$, satisfying the conditions:

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(1) $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$;
(2) $\|y_i - y_j\| = r\|x_i - x_j\|$ for $(i,j) \in A$?

We address this question and provide two proofs. The first proof uses the machinery of group actions and revolves around the investigation of L^k and L^{k+1} norms of the counting function:

$$\lambda_{r,\theta}(z) := |\{(u,v) \in E^2 : u - \sqrt{r}\theta v = z\}|.$$

Here, $r \in (\mathbb{F}_q)^2 \setminus \{0\}$, $\theta \in \mathrm{O}_d(\mathbb{F}_q)$ and $z \in \mathbb{F}_q^d$, where $\mathrm{O}_d(\mathbb{F}_q)$ represents the group of $d \times d$ orthogonal matrices with entries in $\mathbb{F}_p$. We note that $\sqrt{r}$ is an element of $\mathbb{F}_q$ such that its square is r. For $r \in (\mathbb{F}_q)^2 \setminus \{0\}$, there are two square roots. However, it does not affect our result, as in the end, we square the element, resulting in the restoration of the original element r. The second proof is elementary and is based on fairly straightforward combinatorial reasoning.

Before delving into the details, let us present the main result of this paper.

THEOREM 1.3. *Suppose $r \in (\mathbb{F}_q)^2 \setminus \{0\}$ and $\emptyset \neq A \subset \{(i,j) : 1 \leq i < j \leq k+1\}$, where $k \geq 1$. If $E \subset \mathbb{F}_q^d$ with $|E| \geq 2kq^{d/2}$, then there exist $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that $\|y_i - y_j\| = r\|x_i - x_j\|$ for $(i,j) \in A$, and $x_i \neq x_j$, $y_i \neq y_j$ for $1 \leq i < j \leq k+1$.*

We remark that the second proof gives a stronger result. In particular, we show in Section 4 that the result holds if $|E| \geq \sqrt{k+1}|E|^{d/2}$.

As a straightforward corollary, by varying the underlying set A, we can establish thresholds for the existence of dilated k-cycles, k-paths and k-stars (for $k \geq 3$) with a dilation ratio of $r \in (\mathbb{F}_q)^2 \setminus \{0\}$ (see more details in Section 5).

COROLLARY 1.4 (k-paths with a dilation ratio of $r \in (\mathbb{F}_q)^2$). *Let $k \in \mathbb{N}$, $r \in (\mathbb{F}_q)^2 \setminus \{0\}$ and $E \subset \mathbb{F}_q^d$ such that $|E| \geq 2kq^{d/2}$. Then, there exist two $(k+1)$-point configurations $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that the following conditions hold:*

(1) *$\|y_{i+1} - y_i\| = r\|x_{i+1} - x_i\|$ for $i \in \{1, \ldots, k\}$;*
(2) *$x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$.*

Corollary 1.4 easily follows from Theorem 1.3 by setting $A = \{(1,2), \ldots, (k, k+1)\}$. Corollary 1.4 generalises [10, Theorem 1.5]. Xie and Ge [11, Theorem 1.9] proved similar results, but only for paths of length 4 with exponents $d/2$ or $(2d+1)/3$ depending on d and q. We obtain the exponent $d/2$, which is exponentially better, and we also improve the constants.

COROLLARY 1.5 (k-stars with a dilation ratio of $r \in (\mathbb{F}_q)^2$). *Let $k \geq 2$, $r \in (\mathbb{F}_q)^2 \setminus \{0\}$ and $E \subset \mathbb{F}_q^d$ such that $|E| \geq 2kq^{d/2}$. Then, there exist two $(k+1)$-point configurations $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that the following conditions hold:*

(1) *$\|y_i - y_1\| = r\|x_i - x_1\|$ for $i \in \{2, \ldots, k+1\}$;*
(2) *$x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$.*

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Corollary 1.5 easily follows from Theorem 1.3 by setting $A = \{(1,2), \ldots, (1, k+1)\}$. Corollary 1.5 improves the constants of [11, Theorem 1.8]. There, the constants depend on k quadratically, whereas in our result, the dependence on k is linear.

The rest of this paper will focus on proving Theorem 1.3.

2. Notation

We recall some basic notation which we will use throughout the paper.

Let $\mathrm{O}_d(\mathbb{F}_q)$ denote the group of orthogonal $d \times d$ matrices with entries in $\mathbb{F}_q$.

Let $\mathbb{F}_q^*$ denote the set of nonzero elements in $\mathbb{F}_q$, that is, $\mathbb{F}_q^* = \{a \in \mathbb{F}_q : a \neq 0\}$.

Let $(\mathbb{F}_q)^2$ denote the set of quadratic residues in $\mathbb{F}_q$, that is, $(\mathbb{F}_q)^2 = \{a^2 : a \in \mathbb{F}_q\}$.

Define the map $\|\cdot\| : \mathbb{F}_q^d \to \mathbb{F}_q$ by $\|\alpha\| := \alpha_1^2 + \cdots + \alpha_d^2$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$. This map is not a norm, as we do not impose a metric structure on $\mathbb{F}_q^d$. However, it shares an important property with the Euclidean norm: it is invariant under orthogonal transformations.

If X is a finite set, let $|X|$ denote its cardinality.

3. First proof of Theorem 1.3

Let $k \in \mathbb{N}$ and $\emptyset \neq A \subset \{(i,j) : 1 \leq i < j \leq k+1\}$. For $r \in (\mathbb{F}_q)^2 \setminus \{0\}$, $z \in \mathbb{F}_q^d$ and $\theta \in \mathrm{O}_d(\mathbb{F}_q)$, define the counting function:

$$\lambda_{r,\theta}(z) := |\{(u,v) \in E^2 : u - \sqrt{r}\theta v = z\}|. \tag{3.1}$$

For $p \geq 1$, we define the L^p-norm of $\lambda_{r,\theta}(z)$ as follows:

$$\|\lambda_{r,\theta}(z)\|_p^p := \sum_{\theta \in \mathrm{O}_d(\mathbb{F}_q)} \sum_{z \in \mathbb{F}_q^d} \lambda_{r,\theta}(z)^p.$$

From (3.1),

$$\lambda_{r,\theta}(z)^{k+1} = |\{(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1}) \in E^{2k+2} : u_i - \sqrt{r}\theta v_i = z,\ i \in [k+1]\}|.$$

Then, we obtain

$$\|\lambda_{r,\theta}(z)\|_{k+1}^{k+1} = \sum_{\theta,z} |\{(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1}) \in E^{2k+2} : u_i - \sqrt{r}\theta v_i = z,\ i \in [k+1]\}|$$

$$= \sum_{\theta} |\{(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1}) \in E^{2k+2} : u_i - u_j = \sqrt{r}\theta(v_i - v_j),\ 1 \leq i < j \leq k+1\}|.$$

Let us introduce the notation:

$$\Lambda_\theta(r) := \{(u_1, \ldots, u_{k+1}, v_1, \ldots, v_{k+1}) \in E^{2k+2} : u_i - u_j = \sqrt{r}\theta(v_i - v_j),\ 1 \leq i < j \leq k+1\}.$$

With this notation, one can express the L^{k+1}-norm of the function $\lambda_{r,\theta}(z)$ in terms of $|\Lambda_\theta(r)|$:

$$\|\lambda_{r,\theta}(z)\|_{k+1}^{k+1} = \sum_{\theta \in \mathrm{O}_d(\mathbb{F}_q)} |\Lambda_\theta(r)|.$$

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Let us consider the auxiliary sets:

$$\mathcal{N}_{A,\theta}(r) := \left\{(u_1,\dots,u_{k+1},v_1,\dots,v_{k+1}) \in E^{2k+2} : \begin{array}{l} u_i - u_j = \sqrt{r}\theta(v_i - v_j),\ (i,j) \in A, \\ v_i \neq v_j,\ 1 \le i < j \le k+1 \end{array}\right\},$$

$$\mathcal{N}_{\theta}(r) := \left\{(u_1,\dots,u_{k+1},v_1,\dots,v_{k+1}) \in E^{2k+2} : \begin{array}{l} u_i - u_j = \sqrt{r}\theta(v_i - v_j), \\ v_i \neq v_j,\ 1 \le i < j \le k+1 \end{array}\right\}.$$

To prove Theorem 1.3, we need to show that $|\mathcal{S}_A| > 0$, where

$$\mathcal{S}_A := \left\{(x_1,\dots,x_{k+1},y_1,\dots,y_{k+1}) \in E^{2k+2} : \begin{array}{l} \|y_i - y_j\| = r\|x_i - x_j\|,\ (i,j) \in A, \\ x_i \neq x_j,\ y_i \neq y_j,\ 1 \le i < j \le k+1 \end{array}\right\}.$$

It is easy to verify that $\mathcal{N}_{A,\theta}(r) \subset \mathcal{S}_A$ for each $\theta \in \mathrm{O}_d(\mathbb{F}_q)$ and hence

$$|\mathcal{S}_A| \ge \frac{1}{|\mathrm{O}_d(\mathbb{F}_q)|} \sum_{\theta \in \mathrm{O}_d(\mathbb{F}_q)} |\mathcal{N}_{A,\theta}(r)|. \tag{3.2}$$

Since $\mathcal{N}_{\theta}(r) \subset \mathcal{N}_{A,\theta}(r)$ for each $\theta \in \mathrm{O}_d(\mathbb{F}_q)$, (3.2) yields

$$|\mathcal{S}_A| \ge \frac{1}{|\mathrm{O}_d(\mathbb{F}_q)|} \sum_{\theta \in \mathrm{O}_d(\mathbb{F}_q)} |\mathcal{N}_{\theta}(r)|. \tag{3.3}$$

For each pair (α,β) such that $1 \le \alpha < \beta \le k+1$, we define

$$A_{\alpha,\beta} := \left\{(u_1,\dots,u_{k+1},v_1,\dots,v_{k+1}) \in E^{2k+2} : \begin{array}{l} u_i - u_j = \sqrt{r}\theta(v_i - v_j), \\ 1 \le i < j \le k+1,\ v_\alpha = v_\beta \end{array}\right\}.$$

From the set equality

$$\Lambda_\theta(r) \setminus \bigcup_{1 \le \alpha < \beta \le k+1} A_{\alpha,\beta} = \mathcal{N}_\theta(r),$$

we obtain

$$|\mathcal{N}_\theta(r)| \ge |\Lambda_\theta(r)| - \sum_{1 \le \alpha < \beta \le k+1} |A_{\alpha,\beta}|. \tag{3.4}$$

One can verify that, for (α,β) with $1 \le \alpha < \beta \le k+1$,

$$|A_{\alpha,\beta}| = \sum_{z \in \mathbb{F}_q^d} \lambda_{r,\theta}(z)^k. \tag{3.5}$$

Substituting (3.5) into (3.4}, we obtain

$$|\mathcal{N}_\theta(r)| \ge |\Lambda_\theta(r)| - \binom{k+1}{2} \sum_{z \in \mathbb{F}_q^d} \lambda_{r,\theta}(z)^k. \tag{3.6}$$

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Summing (3.6) over all $\theta \in \mathrm{O}_d(\mathbb{F}_q)$ yields

$$\sum_{\theta} |\mathcal{N}_\theta(r)| \geq \sum_{\theta} |\Lambda_\theta(r)| - \binom{k+1}{2} \sum_{\theta,z} \lambda_{r,\theta}(z)^k. \tag{3.7}$$

Combining (3.7) and (3.3),

$$|\mathcal{S}_A| \geq \frac{1}{|\mathrm{O}_d(\mathbb{F}_q)|}\Big(\|\lambda_{r,\theta}(z)\|_{k+1}^{k+1} - \binom{k+1}{2} \|\lambda_{r,\theta}(z)\|_k^k \Big). \tag{3.8}$$

Note that the lower bound for the size of $\mathcal{S}_A$ is independent of A. This is a key aspect of the result being proven.

Now, we will demonstrate that if $|E| \geq C_k q^{d/2}$, then $|\mathcal{S}_A| > 0$. By applying Hölder's inequality, we can establish a lower bound for the L^{k+1}-norm of $\lambda_{r,\theta}(z)$, namely

$$\sum_{\theta,z} \lambda_{r,\theta}(z)^{k+1} \geq |\mathrm{O}_d(\mathbb{F}_q)| \cdot \frac{|E|^{2k+2}}{q^{dk}}. \tag{3.9}$$

The second term in (3.8) can be divided into two separate sums, where the terms are categorised as either 'small' or 'large'. More precisely,

$$\|\lambda_{r,\theta}(z)\|_k^k = \sum_{\theta,z} \lambda_{r,\theta}(z)^k = \sum_{\lambda \geq c} \lambda_{r,\theta}(z)^k + \sum_{\lambda < c} \lambda_{r,\theta}(z)^k, \tag{3.10}$$

where the parameter $c \in \mathbb{R}_{\geq 0}$ will be chosen later. Combining (3.10) with (3.8), we obtain the lower bound:

$$|\mathcal{S}_A| \geq \frac{1}{|\mathrm{O}_d(\mathbb{F}_q)|}(\mathcal{S}_1 + \mathcal{S}_2), \tag{3.11}$$

where

$$\mathcal{S}_1 = \frac{1}{2}\|\lambda_{r,\theta}(z)\|_{k+1}^{k+1} - \binom{k+1}{2} \sum_{\lambda \geq c} \lambda_{r,\theta}(z)^k,$$

$$\mathcal{S}_2 = \frac{1}{2}\|\lambda_{r,\theta}(z)\|_{k+1}^{k+1} - \binom{k+1}{2} \sum_{\lambda < c} \lambda_{r,\theta}(z)^k.$$

By using the definition of the L^{k+1}-norm of $\lambda_{r,\theta}(z)$, we can establish the following lower bound for $\mathcal{S}_1$:

$$\mathcal{S}_1 \geq \frac{1}{2}\Big(\sum_{\lambda \geq c} \lambda_{r,\theta}(z)^{k+1} - k(k+1) \sum_{\lambda \geq c} \lambda_{r,\theta}(z)^k \Big) = \frac{1}{2}\Big(\sum_{\lambda \geq c} \lambda_{r,\theta}(z)^k (\lambda_{r,\theta}(z) - k(k+1)) \Big).$$

Choosing $c = k(k+1)$ shows that $\mathcal{S}_1 \geq 0$.

Next, we estimate $\mathcal{S}_2$: for the first term, we use (3.9), and for the second term, we provide a straightforward estimate. Thus,

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$$\begin{aligned}\mathcal{S}_2 &\geq \frac{|E|^{2k+2}}{2q^{dk}}\cdot|\mathrm{O}_d(\mathbb{F}_q)| - \binom{k+1}{2}\sum_{\lambda<c}\lambda_{r,\theta}(z)^k\\ &\geq \frac{|E|^{2k+2}}{2q^{dk}}\cdot|\mathrm{O}_d(\mathbb{F}_q)| - c^k\binom{k+1}{2}\sum_{\theta,z}1\\ &\geq |\mathrm{O}_d(\mathbb{F}_q)|\Big(\frac{|E|^{2k+2}}{2q^{dk}} - \frac{(k^2+k)^{k+1}q^d}{2}\Big).\end{aligned} \tag{3.12}$$

We note that $|\mathrm{O}_d(\mathbb{F}_q)| > 0$. By combining this observation with the inequalities $\mathcal{S}_1 \geq 0$ as well as (3.12) and (3.11), we obtain

$$|\mathcal{S}_A| \geq \frac{|E|^{2k+2}}{2q^{dk}} - \frac{(k^2+k)^{k+1}q^d}{2}.$$

It is not difficult to verify that if $|E| \geq 2kq^{d/2}$, then $|\mathcal{S}_A| > 0$, which completes the proof of Theorem 1.3.

4. Second proof of Theorem 1.3

In this section, we will prove Theorem 1.3 using elementary combinatorial arguments.

Suppose $E \subset \mathbb{F}_q^d$ and $r \in (\mathbb{F}_q)^2 \setminus \{0\}$, so $r = t^2$ where $t \neq 0$. For such t, define tE as $tE = \{tv : v \in E\}$.

Now, consider the set:

$$H = \{(x,a) : x \in tE \cap (E+a),\ a \in \mathbb{F}_q^d\}.$$

We will use double counting to determine the cardinality of H. First,

$$|H| = \sum_{x\in tE}\sum_{\substack{a\in\mathbb{F}_q^d\\ x\in E+a}} 1 = \sum_{x\in tE}\sum_{\substack{a\in\mathbb{F}_q^d\\ a\in x-E}} 1 = \sum_{x\in tE}|x-E| = \sum_{x\in tE}|E| = |tE||E| = |E|^2. \tag{4.1}$$

However, by changing the order of variables,

$$|H| = \sum_{a\in\mathbb{F}_q^d}\sum_{\substack{x\in tE\\ x\in E+a}} 1 = \sum_{a\in\mathbb{F}_q^d}|tE \cap (E+a)|. \tag{4.2}$$

Comparing (4.1) with (4.2) yields

$$\sum_{a\in\mathbb{F}_q^d}|tE \cap (E+a)| = |E|^2.$$

Therefore,

$$\max_{a\in\mathbb{F}_q^d}|tE \cap (E+a)| \geq \frac{|E|^2}{q^d}.$$

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We observe that if $|E| \geq \sqrt{k+1}q^{d/2}$, then $\max_{a\in\mathbb{F}_q^d}|tE \cap (E+a)| \geq k+1$. Thus, there exists an element $a \in \mathbb{F}_q^d$ such that $|tE \cap (E+a)| \geq k+1$. Consequently, we can establish the existence of a sequence $\{z_1, \ldots, z_{k+1}\}$ such that $\{z_1, \ldots, z_{k+1}\} \subset tE \cap (E+a)$. This implies the existence of sequences $\{x_1, \ldots, x_{k+1}\} \subset E$ and $\{y_1, \ldots, y_{k+1}\} \subset E$, such that $z_i = tx_i$ and $z_i = y_i + a$ for $1 \leq i \leq k+1$.

In summary, we have demonstrated the existence of $(k+1)$-tuples $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ satisfying the conditions:

(1) $x_i \neq x_j, y_i \neq y_j$ for $1 \leq i < j \leq k+1$;
(2) $y_i + a = tx_i$ for $i \in \{1, \ldots, k+1\}$.

Therefore, for $1 \leq i < j \leq k+1$,

$$\|y_i - y_j\| = \|(tx_i - a) - (tx_j - a)\| = \|tx_i - tx_j\| = t^2\|x_i - x_j\| = r\|x_i - x_j\|.$$

Since A is a nonempty subset of $\{(i,j) : 1 \leq i < j \leq k+1\}$, we have demonstrated the existence of two $(k+1)$-point configurations: $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ satisfying:

(1) $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$;
(2) $\|y_i - y_j\| = r\|x_i - x_j\|$ for $(i,j) \in A$.

This completes the proof of Theorem 1.3.

5. Corollaries

In this section, we will explore some interesting corollaries that follow from Theorem 1.3. For simplicity, we assume a two-dimensional scenario, that is, $d = 2$.

5.1. Paths of length *k*. Let $k \geq 2$ and fix any element $r \in (\mathbb{F}_q)^2 \setminus \{0\}$. Taking $A = \{(1,2), \ldots, (k, k+1)\}$, we can see that if $E \subset \mathbb{F}_q^2$ with $|E| \geq 2kq$, then we can find two k-paths in E, that is, $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$, such that $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$, and $\|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|$ for $i \in \{1, \ldots, k\}$ (see Figure 2).

5.2. Cycles of length *k*. Assuming that $k \geq 3$, for k-cycles, one needs to take $A = \{(1,2), \ldots, (k-1, k), (k, 1)\}$. Then, we can see that for $E \subset \mathbb{F}_q^2$ with $|E| \geq 2kq$, there exist k-cycles in E. That is, there exist $(x_1, \ldots, x_k) \in E^k$ and $(y_1, \ldots, y_k) \in E^k$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k$, and $\|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|$ for $i \in \{1, \ldots, k-1\}$, and $\|y_k - y_1\| = r\|x_k - x_1\|$ (see Figure 3).

5.3. Stars with *k* edges. Assuming that $k \geq 1$, for k-stars, one needs to take $A = \{(1,2), \ldots, (1, k+1)\}$. Then, for $E \subset \mathbb{F}_q^2$ with $|E| \geq 2kq$, there exist k-stars in E. That is, there exist $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k+1$, and $\|y_1 - y_i\| = r\|x_1 - x_i\|$ for $i \in \{2, \ldots, k+1\}$ (see Figure 4).

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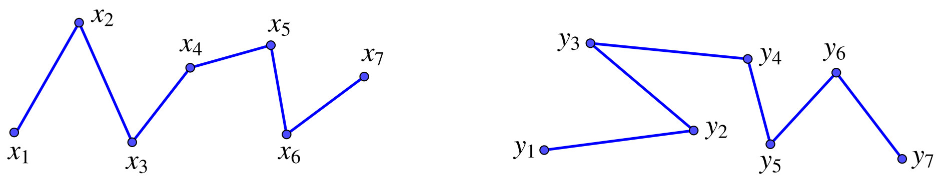

FIGURE 2. 6-paths with a dilation ratio of $r \in (\mathbb{F}_q)^2 \setminus \{0\}$.

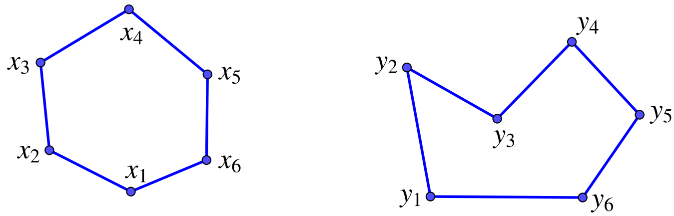

FIGURE 3. 6-cycles with a dilation ratio of $r \in (\mathbb{F}_q)^2 \setminus \{0\}$.

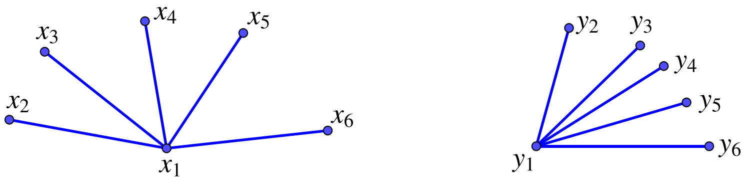

FIGURE 4. 5-stars with a dilation ratio of $r \in (\mathbb{F}_q)^2 \setminus \{0\}$.

6. Sharp examples

In this section, we will demonstrate that in dimension 2, the exponent $d/2$ in Theorem 1.3 is optimal under certain mild conditions.

6.1. The case when $|A| \geq 1$. In this setting, we will demonstrate that if $q = p^{2\ell}$ with $p \equiv 3 \pmod 4$ and $\ell \equiv 1 \pmod 2$, the exponent $d/2$ in Theorem 1.3 is optimal.

The ambient space is $\mathbb{F}_{p^{2\ell}}^2$, a 2-dimensional vector space over the finite field $\mathbb{F}_{p^{2\ell}}$ with $p^{2\ell}$ elements. The field $\mathbb{F}_{p^{2\ell}}$ contains the subfield $\mathbb{F}_{p^\ell}$ with p^ℓ elements. Consider the subset $E := \mathbb{F}_{p^\ell} \times \mathbb{F}_{p^\ell}$ of $\mathbb{F}_{p^{2\ell}}^2$ and observe that $|E| = q$.

We know that $|(\mathbb{F}_{p^{2\ell}})^2 \setminus \{0\}| = \frac{1}{2}(p^{2\ell} - 1) > p^\ell = |\mathbb{F}_{p^\ell}|$, and one can always choose $r \in (\mathbb{F}_{p^{2\ell}})^2$ such that $r \notin \mathbb{F}_{p^\ell}$. Then, for any $x_i, x_j, y_i, y_j \in E$ such that $x_i \neq x_j$ and $y_i \neq y_j$, we have

$$\frac{||y_i - y_j||}{||x_i - x_j||} \in \mathbb{F}_{p^\ell}.$$

Here, we have used the fact that $||x|| = 0$ if and only if $x = (0, 0)$, which is true since $p \equiv 3 \pmod 4$ and $\ell \equiv 1 \pmod 2$. However, r was chosen such that $r \notin \mathbb{F}_{p^\ell}$. This observation demonstrates that the exponent $d/2$ is optimal in this setting.

6.2. The case when A contains $(a, b), (a, c), (b, c)$. In this scenario, we will establish that if $q \equiv 3 \pmod 4$, then the exponent $d/2$ is the optimal exponent in Theorem 1.3.

Without loss of generality, assume that $(1, 2), (1, 3), (2, 3) \in A$, and let $r \in (\mathbb{F}_q)^2$ be such that $r \neq 1$. Define E as the set:

$$E := \{(u, v) \in \mathbb{F}_q^2 : u^2 + v^2 = 1\},$$

which represents a sphere of radius 1 in $\mathbb{F}_q^2$. It can be shown that $|E| = q + 1$. Now, suppose there exist two sets of points $(x_1, \ldots, x_{k+1}) \in E^{k+1}$ and $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ such that $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i < j \leq k + 1$, and $\|y_i - y_j\| = r\|x_i - x_j\|$ for $(i, j) \in A$. In particular, this implies that $\|y_i - y_j\| = r\|x_i - x_j\|$ for $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$.

For each $i \in \{1, 2, 3\}$, define $z_i := \sqrt{r} x_i$, which implies $z_i - z_j = \sqrt{r}(x_i - x_j)$. Therefore, $\|y_i - y_j\| = \|z_i - z_j\|$ for $1 \leq i < j \leq 3$. This implies the existence of a transformation $T \in \mathrm{ISO}(2)$ such that $T(y_i) = z_i$ for $i \in \{1, 2, 3\}$, where $\mathrm{ISO}(2)$ is the group of rigid motions.

Since $z_i = \sqrt{r} x_i$, it follows that $\|z_i\| = r\|x_i\| = r$. Also, we observe that $\|z_i - T(0)\| = \|y_i\| = 1$. Therefore, we have demonstrated that $\{z_1, z_2, z_3\} \subset S(T(0); 1) \cap S(0; r)$, where $S(a; R)$ denotes the sphere of radius R centred at a. However, the spheres $S(T(0); 1)$ and $S(0; r)$ are distinct since $r \neq 1$. This assumption leads to a contradiction since two distinct spheres in $\mathbb{F}_q^2$ have at most two points in their intersection. Thus, we have established that the exponent $d/2$ is indeed sharp in this setting.

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