THE SOLUTION OF SOME INTEGRAL EQUATIONS AND THEIR CONNECTION WITH DUAL INTEGRAL EQUATIONS AND SERIES

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1. The equations. We shall solve the equation

$$\int_0^1 \log\frac{x+t}{|x-t|}\,g(t)\,dt = \pi f(x) \qquad (0 < x < 1), \tag{1}$$

giving the solution in two forms, and give a new solution of

$$\int_0^1 \log|x-t|\,g(t)\,dt = \pi f(x) \qquad (0 < x < 1), \tag{2}$$

originally solved by Carlemann.

The latter will be extended to the case where the limits are a and b with $a < x < b$.

These equations will be found to be connected, respectively, with the pair of dual integral equations

$$\int_0^\infty \frac{\sin xt}{t}\,h(t)\,dt = f(x) \qquad (0 < x < 1), \tag{3}$$

$$\int_0^\infty \sin xt\,h(t)\,dt = 0 \qquad (x > 1) \tag{4}$$

and with the pair

$$\int_0^\infty \frac{\cos xt}{t}\,h(t)\,dt = f(x) \qquad (0 < x < 1), \tag{5}$$

$$\int_0^\infty \cos xt\,h(t)\,dt = 0 \qquad (x > 1), \tag{6}$$

and indeed they have in a sense the same solution in certain cases.

The equations can also be used to solve certain problems involving dual trigonometric series, and we give an example. Incidentally we point out some limitations in Tranter's solutions of the dual equations and series.

We also use the solution of (2) to obtain an alternative solution of a singular integral equation of Carlemann.

A modified form of (2) has arisen in a problem in slender wing theory; this indeed was the original motive for studying these equations.

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2. The solution of equation (2). Carlemann [**1**] gives the solution as

$$g(y)=\frac{1}{\pi\{y(1-y)\}^{\frac{1}{2}}}\int_0^1\frac{\{x(1-x)\}^{\frac{1}{2}}}{x-y}f'(x)\,dx-\frac{1}{2\pi\log 2\{y(1-y)\}^{\frac{1}{2}}}\int_0^1\frac{f(x)\,dx}{\{x(1-x)\}^{\frac{1}{2}}},\tag{7}$$

where the first integral is to have its principal value.

Now we can show that an alternative solution is

$$g(y)=\frac{1}{\pi y^{\frac{1}{2}}}\frac{d}{dy}\int_y^1\frac{S(k)\,dk}{(k-y)^{\frac{1}{2}}}-\frac{1}{2\pi\log 2\{y(1-y)\}^{\frac{1}{2}}}\int_0^1\frac{f(x)\,dx}{\{x(1-x)\}^{\frac{1}{2}}},\tag{8}$$

where

$$S(k)=k\frac{d}{dk}\int_0^k\frac{f(x)\,dx}{\{x(k-x)\}^{\frac{1}{2}}}=\int_0^k\frac{x^{\frac{1}{2}}f'(x)\,dx}{(k-x)^{\frac{1}{2}}}.\tag{9}$$

This result can be established by showing that

$$\frac{1}{\pi y^{\frac{1}{2}}}\frac{d}{dy}\int_y^1\frac{dk}{(k-y)^{\frac{1}{2}}}\int_0^k\frac{x^{\frac{1}{2}}f'(x)\,dx}{(k-x)^{\frac{1}{2}}}=\frac{1}{\pi\{y(1-y)\}^{\frac{1}{2}}}\int_0^1\frac{\{x(1-x)\}^{\frac{1}{2}}f'(x)\,dx}{x-y},\tag{10}$$

and a proof of this will be found in Appendix 1.

3. Extension to limits a and b. If in equation (2) we write

$$x=\frac{X-a}{b-a},\qquad y=\frac{Y-a}{b-a},$$

$$g\left(\frac{Y-a}{b-a}\right)=G(Y),\qquad f\left(\frac{X-a}{b-a}\right)=F(X),$$

then we find that the solution of

$$\int_a^b G(Y)\{\log|X-Y|-\log(b-a)\}\,dY=\pi F(X)\qquad(a<X<b)\tag{11}$$

is

$$G(Y)=\frac{1}{\pi\{(Y-a)(b-Y)\}^{\frac{1}{2}}}\int_a^b\frac{\{(X-a)(b-X)\}^{\frac{1}{2}}}{X-Y}F'(X)\,dX-\frac{1}{2\pi\log 2}\frac{I}{\{(Y-a)(b-Y)\}^{\frac{1}{2}}},\tag{12}$$

where

$$I=\int_a^b\frac{F(X)\,dX}{\{(X-a)(b-X)\}^{\frac{1}{2}}}.\tag{13}$$

There is a slight difficulty here in that (11) is not the form we require to solve, owing to the term $\log(b-a)$ in the integrand. We deal with this in Appendix 2 and so, replacing

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capitals by lower case letters, we find that the solution of

$$\int_a^b \log|x-y|\,g(y)\,dy = \pi f(x) \qquad (a < x < b) \tag{14}$$

is

$$g(y) = \frac{1}{\pi\{(y-a)(b-y)\}^{\frac{1}{2}}}\int_a^b \frac{\{(x-a)(b-x)\}^{\frac{1}{2}} f'(x)\,dx}{x-y}$$

$$+\frac{1}{\pi\{(y-a)(b-y)\}^{\frac{1}{2}}}\,\frac{1}{\log(b-a)-2\log 2}\int_a^b \frac{f(x)\,dx}{\{(x-a)(b-x)\}^{\frac{1}{2}}}. \tag{15}$$

The other form of solution is

$$g(y) = \frac{1}{\pi(y-a)^{\frac{1}{2}}}\frac{d}{dy}\int_y^b \frac{S(k)\,dk}{(k-y)^{\frac{1}{2}}} + T, \tag{16}$$

where

$$S(k) = (k-a)\frac{d}{dk}\int_a^k \frac{f(x)\,dx}{\{(x-a)(k-x)\}^{\frac{1}{2}}} = \int_a^k \left(\frac{x-a}{k-x}\right)^{\frac{1}{2}} f'(x)\,dx \tag{17}$$

and T is the same as the last term on the right of (15).

An exception is when $b-a = 4$. In such a case the last term on the right of (15) or (16) is now written

$$\frac{C}{\{(y-a)(b-y)\}^{\frac{1}{2}}}, \tag{18}$$

where C is an arbitrary constant. It can be verified that this expression is a null solution of (14) when $b-a = 4$.

Often we encounter the equation

$$\int_a^b g(y)\log|x^2-y^2|\,dy = \pi f(x).$$

(Note: We do not here allow a to be negative.) The solution of this can be obtained from that already given by writing

$$x = X^2,\ y = Y^2,\ k = K^2,\ 2Yg(Y^2) = G(Y),\ f(X^2) = F(X),\ a = A^2,\ b = B^2$$

in the equations and then replacing capitals by lower case letters. We shall not write down the solutions but they can easily be obtained by the above means.

An integral equation given by Durran and Lord [2], connected with a problem in slender wing theory, is

$$\int_0^1 g(y)\log\frac{|x^2-y^2|}{y^2}\,dy = \pi f(x).$$

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In this case it can be shown that $C(1-y^2)^{-\frac{1}{2}}$ is a null solution. Thus the solution can be obtained from (7) or (8) and (9) but with the last term in (7) and (8) replaced by $C(1-y^2)^{-\frac{1}{2}}$, where C is an arbitrary constant.

We find, therefore, that the solution of Durran and Lord's equation may be written

$$g(y)=\frac{2}{\pi(1-y^2)^{\frac{1}{2}}}\int_0^1\frac{x(1-x^2)^{\frac{1}{2}}f'(x)\,dx}{x^2-y^2}+\frac{C}{(1-y^2)^{\frac{1}{2}}},$$

or

$$g(y)=\frac{2}{\pi y}\frac{d}{dy}\int_y^1\frac{kS(k)\,dk}{(k^2-y^2)^{\frac{1}{2}}}+\frac{C}{(1-y^2)^{\frac{1}{2}}},$$

where

$$S(k)=k\frac{d}{dk}\int_0^k\frac{f(x)\,dx}{(k^2-x^2)^{\frac{1}{2}}}=\int_0^k\frac{xf'(x)\,dx}{(k^2-x^2)^{\frac{1}{2}}}.$$

If $f(x)$ is a constant the equation has no solution.

4. The solution of equation (1). In this case we first obtain a solution reminiscent of (8) and (9) by a method analogous to that of Copson [3], and then transform this solution into one which we may call a Carlemann type.

It can be shown (see Appendix 3) that

$$I=\tfrac{1}{2}\log\frac{x+t}{|x-t|}=\int_0^{\min(x,t)}\frac{\alpha\,d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}}. \tag{20}$$

Substitute in equation (1) and we have

$$\pi f(x)=2\int_0^1 Ig(t)\,dt=2\int_0^x g(t)\,dt\int_0^t\frac{\alpha\,d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}}+2\int_x^1 g(t)\,dt\int_0^x\frac{\alpha\,d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}}. \tag{21}$$

Inverting the order of integration we find that

$$\pi f(x)=2\int_0^x\frac{\alpha\,d\alpha}{(x^2-\alpha^2)^{\frac{1}{2}}}\int_\alpha^1\frac{g(t)\,dt}{(t^2-\alpha^2)^{\frac{1}{2}}}=2\int_0^x\frac{\alpha S(\alpha)\,d\alpha}{(x^2-\alpha^2)^{\frac{1}{2}}}, \tag{22}$$

where

$$S(\alpha)=\int_\alpha^1\frac{g(t)\,dt}{(t^2-\alpha^2)^{\frac{1}{2}}}. \tag{23}$$

(23) is a form of Abel's integral equation and its solution is

$$g(t)=-\frac{2}{\pi}\frac{d}{dt}\int_t^1\frac{\alpha S(\alpha)\,d\alpha}{(\alpha^2-t^2)^{\frac{1}{2}}}. \tag{24}$$

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Again, the solution of (22) is

$$S(\alpha)=\frac{1}{\alpha}\frac{d}{d\alpha}\int_0^\alpha \frac{xf(x)\,dx}{(\alpha^2-x^2)^{\frac{1}{2}}}=\frac{f(0)}{\alpha}+\int_0^\alpha \frac{f'(k)\,dk}{(\alpha^2-k^2)^{\frac{1}{2}}}, \tag{25}$$

where $f(0)$ is taken to mean $f(0+)$. Thus the solution of (1) is effected.

The last form shows that we may write this alternatively

$$g(t)=-\frac{2}{\pi}\frac{d}{dt}\int_t^1 \frac{\alpha S(\alpha)\,d\alpha}{(\alpha^2-t^2)^{\frac{1}{2}}}+\frac{2}{\pi}\frac{f(0)}{t(1-t^2)^{\frac{1}{2}}}, \tag{26}$$

where

$$S(\alpha)=\int_0^\alpha \frac{f'(k)\,dk}{(\alpha^2-k^2)^{\frac{1}{2}}}, \tag{27}$$

which is often more convenient.

An alternative solution of (1) can now be found by substituting the value of $S(\alpha)$ from (27) into (26) and inverting the order of integration. The analysis is closely allied to that of Appendix 1 and will not be given in detail. It leads to

$$g(t)=-\frac{2}{\pi}\frac{t}{(1-t^2)^{\frac{1}{2}}}\int_0^1 \frac{(1-x^2)^{\frac{1}{2}}f'(x)\,dx}{x^2-t^2}+\frac{2}{\pi}\frac{f(0)}{t(1-t^2)^{\frac{1}{2}}}, \tag{28}$$

which is what we have called the Carlemann form of the solution.

5. Alternative solution of Carlemann's singular integral equation. We now consider the equation

$$\int_a^b \frac{g(y)\,dy}{x-y}=\pi F(x) \qquad (a<x<b), \tag{29}$$

whose solution is well-known [4]. It can be written in several different forms. We give one such, namely

$$g(y)=\frac{1}{\pi\{(y-a)(b-y)\}^{\frac{1}{2}}}\int_a^b \frac{\{(x-a)(b-y)\}^{\frac{1}{2}}}{x-y}F(x)\,dx+\frac{D}{\{(y-a)(b-y)\}^{\frac{1}{2}}}, \tag{30}$$

where D is an arbitrary constant.

We can now find an alternative solution in the following way. Differentiate (2) with respect to x and it becomes

$$\int_a^b \frac{g(y)\,dy}{x-y}=\pi f'(x), \tag{31}$$

which is (29) with $f'(x)$ in place of $F(x)$. The solution of (2) is given in (15) or (16) and (17).

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The solution (15) simply gives (30), whilst (16) and (17) give

$$g(y)=\frac{1}{\pi(y-a)^{\frac{1}{2}}}\frac{d}{dy}\int_y^b\frac{S(k)\,dk}{(k-y)^{\frac{1}{2}}}+\frac{C}{\{(y-a)(b-y)\}^{\frac{1}{2}}},\tag{32}$$

where

$$S(k)=\int_a^k\left(\frac{x-a}{k-x}\right)^{\frac{1}{2}}F(x)\,dx.\tag{33}$$

We do not need to investigate C any further, since it is known to be arbitrary.

We may note that Lundgren and Chiang [5] have given a solution which (with our extension to limits a and b) has the same $S(k)$ as in (33), but in which the first term in (32) is

$$\frac{1}{\pi}\frac{d}{dy}(y-a)^{\frac{1}{2}}\int_y^b\frac{S(k)\,dk}{(k-a)(k-y)^{\frac{1}{2}}}.$$

Lundgren and Chiang checked this by a method closely similar to that of Appendix 1. With a different value of the constant C, the term concerned may also be written

$$\frac{1}{\pi}\frac{1}{(y-a)^{\frac{1}{2}}}\int_y^b\frac{S'(k)\,dk}{(k-y)^{\frac{1}{2}}}.$$

6. Connection with dual integral equations.

6.1. The sine pair (3) and (4).

We consider first the pair (3) and (4) and suppose that

$$\int_0^\infty h(t)\sin xt\,dt=g(x)\qquad(0<x<1).\tag{34}$$

It is often more important in applications to find $g(x)$ than $h(t)$. By Fourier's Integral Theorem we have

$$h(t)=\frac{2}{\pi}\int_0^1 g(k)\sin kt\,dk.\tag{35}$$

Substituting in (3), we have

$$\frac{2}{\pi}\int_0^\infty\frac{\sin xt}{t}\,dt\int_0^1 g(k)\sin kt\,dk=f(x).$$

Inverting the order of integration, we obtain

$$\frac{1}{\pi}\int_0^1 g(k)\,dk\int_0^\infty\frac{\cos(x-k)t-\cos(x+k)t}{t}\,dt=f(x).\tag{36}$$

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The inner integral converges and satisfies the condition for Frullani's integrals, namely that

$$\int^{\infty} \cos x\, dx$$

oscillates finitely, and so (36) reduces to

$$\int_0^1 g(k) \log \frac{x+k}{|x-k|}\, dk = \pi f(x). \tag{37}$$

Hence the solutions already given for this equation will satisfy the dual integral equations with $h(t)$ found from (35). It is necessary, however, to impose a restriction on $f(x)$ which is not necessary in solving (1). This is that $f(0+) = 0$. Suppose, for instance, that $f(x) = C$; then the solution of (37) is

$$g(k) = -\frac{2C}{\pi} \frac{1}{k(1-k^2)^{\frac{1}{2}}}.$$

By (35), this would give

$$h(t) = -\frac{4C}{\pi^2} \int_0^1 \frac{\sin kt\, dk}{k(1-k^2)^{\frac{1}{2}}} = \frac{2C}{\pi} \int_0^t J_0(u)\, du,$$

where J_0 is a Bessel function, but in fact this leads to a non-convergent integral when substituted in (4), since $h(\infty)$ is finite. It does, however, satisfy (3) and would indeed satisfy (4) if this integral were interpreted to be

$$\lim_{z \to 0+} \int_0^{\infty} e^{-zt} \sin xt\, h(t)\, dt, \tag{38}$$

which it often is in practical applications.

Tranter [6] gave a solution of the pair (3) and (4), namely

$$h(t) = \frac{2t}{\pi} \int_0^1 pJ_0(pt)\, dp \int_0^p \frac{f'(x)\, dx}{(p^2-x^2)^{\frac{1}{2}}}, \tag{39}$$

but if this is substituted in (3) and suitable inversions of the order of integration are made, it leads to

$$\int_0^{\infty} \frac{h(t) \sin xt}{t}\, dt = f(x) - f(0), \tag{40}$$

so that Tranter's solution only applies if $f(0) = 0$. If this is not so, then the right-hand side of (3) must be written

$$\{f(x) - f(0)\} + f(0).$$

Tranter's solution will then apply to the part in curly brackets, and the term

$$\frac{2f(0)}{\pi}\int_0^t J_0(u)\,du$$

must be added to the right-hand side of (39) and non-convergent integrals must be interpreted as in (38).

We can go further and replace the zero in the right-hand side of (4) by $m(x)$. This adds a term

$$-\int_1^\infty m(k)\log\frac{k+x}{k-x}\,dk$$

to the right-hand side of (37), and the solution is (26) and (27) with

$$-\pi\int_1^\infty \frac{m(k)\,dk}{(k^2-\alpha^2)^{\frac{1}{2}}}$$

added to the right-hand side of (27); alternatively, we may leave (27) as it is and add a term

$$-\frac{2t}{(1-t^2)^{\frac{1}{2}}}\int_1^\infty \frac{m(k)(k^2-1)^{\frac{1}{2}}}{k^2-t^2}\,dk$$

to the right-hand side of (26). This term should also be added to the right-hand side of the alternative version (28). These solutions for the sine pair with the extra term $m(x)$ on the right-hand side of (4) seem to be simpler than any so far given in the literature.

6.2. The cosine pair (5) and (6).

The analysis is not so satisfactory here and we omit it. It does, however, suggest that the corresponding $g(k)$ for this pair is the solution of

$$\int_0^1 g(k)\log|x^2-k^2|\,dk = -\pi f(x),$$

but only if

$$\int_0^1 \frac{f(x)\,dx}{(1-x^2)^{\frac{1}{2}}} = 0. \tag{41}$$

We have not found a satisfactory proof of this, but we have verified it in a few special cases.

Tranter [6] gave the solution of this pair as

$$h(t) = \frac{2t^2}{\pi}\int_0^1 pJ_0(pt)\,dp\int_0^p \frac{f(x)\,dx}{(p^2-x^2)^{\frac{1}{2}}}. \tag{42}$$

However, it can be shown that this leads to a divergent integral unless equation (41) holds, but that the result is correct if divergent integrals are interpreted as in (38). For instance, if

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$f(x) = C$, then (41) is not satisfied and (42) gives $h(t) = CtJ_1(t)$. This satisfies (5) but makes the integral in (6) divergent. However this integral is zero when it is interpreted as in (38).

7. Application to dual trigonometric series. We give one example, solved by Tranter [7], [8], namely

$$\sum_1^\infty \frac{a_n \sin nx}{n} = f(x) \qquad (0 < x < c), \tag{43}$$

$$\sum_1^n a_n \sin nx = 0 \qquad (c < x < \pi). \tag{44}$$

Suppose that, for $0 < x < c$,

$$\sum a_n \sin nx = g(x). \tag{45}$$

Then we have

$$a_n = \frac{2}{\pi}\int_0^c g(t)\sin nt\,dt. \tag{46}$$

Substitute in (43) and interchange summation and integration and we have

$$\frac{2}{\pi}\int_0^c g(t)\sum_1^\infty \frac{\sin nt \sin nx}{n}\,dt = f(x).$$

When the series is summed, this gives

$$\int_0^c g(t)\log\frac{\sin\frac{1}{2}(t+x)}{\sin\frac{1}{2}|t-x|}\,dt = \pi f(x) \tag{47}$$

or

$$\int_0^c g(t)\log\frac{\tan\frac{1}{2}t+\tan\frac{1}{2}x}{|\tan\frac{1}{2}t-\tan\frac{1}{2}x|}\,dt = \pi f(x). \tag{48}$$

To solve this equation, write

$$\tan\tfrac{1}{2}t = T\tan\tfrac{1}{2}c, \qquad \tan\tfrac{1}{2}x = X\tan\tfrac{1}{2}c,$$

when it becomes of the form (1). Writing down the solution and then transforming back, we find that

$$g(t) = -\frac{1}{\pi}\frac{\tan\frac{1}{2}t\sec^2\frac{1}{2}t}{(\tan^2\frac{1}{2}c-\tan^2\frac{1}{2}t)^{\frac{1}{2}}}\int_0^c \frac{(\tan^2\frac{1}{2}c-\tan^2\frac{1}{2}x)^{\frac{1}{2}}f'(x)}{\tan^2\frac{1}{2}x-\tan^2\frac{1}{2}t}\,dx+\frac{2}{\pi}f(0)\frac{\tan\frac{1}{2}c}{\sin t(\tan^2\frac{1}{2}c-\tan^2\frac{1}{2}t)^{\frac{1}{2}}}. \tag{49}$$

The alternative solution (26) and (27) leads to

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^c \frac{S(\beta)\tan\frac{1}{2}\beta\sec^2\frac{1}{2}\beta\,d\beta}{(\tan^2\frac{1}{2}\beta-\tan^2\frac{1}{2}t)^{\frac{1}{2}}}+\frac{2}{\pi}f(0)\frac{\tan\frac{1}{2}c}{\sin t(\tan^2\frac{1}{2}c-\tan^2\frac{1}{2}t)^{\frac{1}{2}}}, \tag{50}$$

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where

$$S(\beta)=\int_0^\beta \frac{f'(k)\,dk}{(\tan^2\frac{1}{2}\beta-\tan^2\frac{1}{2}k)^{\frac{1}{2}}}. \tag{51}$$

If (44) has the term $m(x)$ on the right-hand side instead of zero, then there is an additional term

$$-\pi\int_c^\pi \frac{m(k)\,dk}{(\tan^2\frac{1}{2}k-\tan^2\frac{1}{2}\beta)^{\frac{1}{2}}}$$

on the right-hand side of (51); alternatively we may leave (51) as it is and add a term

$$-\frac{\tan\frac{1}{2}t\sec^2\frac{1}{2}t}{(\tan^2\frac{1}{2}c-\tan^2\frac{1}{2}t)^{\frac{1}{2}}}\int_c^\pi \frac{m(\alpha)(\tan^2\frac{1}{2}\alpha-\tan^2\frac{1}{2}c)^{\frac{1}{2}}\,d\alpha}{\tan^2\frac{1}{2}\alpha-\tan^2\frac{1}{2}t}$$

to the right-hand side of (50). This term should also be added to the alternative version (49).

APPENDIX 1

Consider

$$I=\int_y^1 \frac{dk}{(k-y)^{\frac{1}{2}}}\int_0^k \frac{x^{\frac{1}{2}}f'(x)\,dx}{(k-x)^{\frac{1}{2}}}.$$

Inverting the order of integration, we have

$$I=\int_0^y x^{\frac{1}{2}}f'(x)\,dx\int_y^1 (k-y)^{-\frac{1}{2}}(k-x)^{-\frac{1}{2}}\,dk+\int_y^1 x^{\frac{1}{2}}f'(x)\,dx\int_x^1 (k-y)^{-\frac{1}{2}}(k-x)^{-\frac{1}{2}}\,dk.$$

The first inner integral may be integrated to become

$$2\sinh^{-1}\left(\frac{1-y}{y-x}\right)^{\frac{1}{2}},$$

and the second inner integral is the same as this with x and y interchanged. Hence we have

$$I=2\int_0^y x^{\frac{1}{2}}f'(x)\sinh^{-1}\left(\frac{1-y}{y-x}\right)^{\frac{1}{2}}dx+2\int_y^1 x^{\frac{1}{2}}f'(x)\sinh^{-1}\left(\frac{1-x}{x-y}\right)^{\frac{1}{2}}dx.$$

We can write this as the limit as $\varepsilon\to 0$ of

$$\int_0^{y-\varepsilon}\dots dx+\int_{y+\varepsilon}^1\dots dx$$

and we have

$$\frac{d}{dy}\sinh^{-1}\left(\frac{1-y}{y-x}\right)^{\frac{1}{2}}=-\frac{d}{dy}\sinh^{-1}\left(\frac{1-x}{x-y}\right)^{\frac{1}{2}}=-\frac{1}{2}\left(\frac{1-x}{1-y}\right)^{\frac{1}{2}}\frac{1}{y-x}.$$

Hence

$$\frac{dI}{dy}=\frac{1}{(1-y)^{\frac{1}{2}}}\int_0^1 \frac{\{x(1-x)\}^{\frac{1}{2}}f'(x)}{x-y}\,dx,$$

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where the principal value is to be taken, together with a term due to differentiation of the limits. This has the limit zero as $\varepsilon \to 0$. Hence, finally, we have

$$\frac{1}{\pi y^{\frac{1}{2}}}\frac{d}{dy}\int_y^1 \frac{dk}{(k-y)^{\frac{1}{2}}}\int_0^k \frac{x^{\frac{1}{2}}f'(x)\,dx}{(k-x)^{\frac{1}{2}}} = \frac{1}{\pi\{y(1-y)\}^{\frac{1}{2}}}\int_0^1 \frac{\{x(1-x)\}^{\frac{1}{2}}f'(x)\,dx}{x-y}.$$

APPENDIX 2

It can be shown that

$$\int_a^b \frac{\log|y-x|\,dy}{\{(y-a)(b-y)\}^{\frac{1}{2}}} = \pi\{\log(b-a)-2\log 2\} \qquad (a<x<b) \tag{52}$$

and we know that the solution of (11), namely

$$\int_a^b \{\log(X-Y)-\log(b-a)\}G(Y)\,dY = \pi F(X), \tag{53}$$

is equation (12), which we write

$$G(Y) = K(Y) - \frac{1}{2\pi\log 2}\,\frac{I}{\{(Y-a)(b-Y)\}^{\frac{1}{2}}}. \tag{54}$$

Now, if we multiply (53) by $\{(X-a)(b-X)\}^{-\frac{1}{2}}$ and integrate between the limits a and b, we find, using (52), that

$$\int_a^b G(Y)\{\pi\log(b-a)-2\pi\log 2\}\,dY - \pi\log(b-a)\int_a^b G(Y)\,dY = \pi I,$$

and so

$$\int_a^b G(Y)\,dY = -\frac{I}{2\log 2}.$$

Hence

$$\int_a^b \log|X-Y|\,G(Y)\,dY + \frac{\log(b-a)}{2\log 2}I = \pi F(X),$$

and using (52) again we have

$$\int_a^b \log|Y-X|\left\{G(Y)+\frac{1}{\{(Y-a)(b-Y)\}^{\frac{1}{2}}}\,\frac{\log(b-a)}{2\log 2}\,\frac{I}{\pi\{\log(b-a)-2\log 2\}}\right\}dY = \pi F(X),$$

and so the solution of

$$\int_a^b \log|y-x|\,g(y)\,dy = \pi f(x)$$

is

$$\begin{aligned} g(y) &= K(y) + \frac{I}{\{(y-a)(b-y)\}^{\frac{1}{2}}}\left[-\frac{1}{2\pi\log 2}+\frac{\log(b-a)}{2\pi\log 2\{\log(b-a)-2\log 2\}}\right] \\ &= K(y) + \frac{I}{\pi\{(y-a)(b-y)\}^{\frac{1}{2}}}\,\frac{1}{\log(b-a)-2\log 2}. \end{aligned}$$

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APPENDIX 3

Consider

$$I=\int_0^x \frac{\alpha\, d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}},$$

with $0<x<t<1$.

Write $\alpha = x\sin\theta$, and we have

$$I=\int_0^{\frac{1}{2}\pi} \frac{x\sin\theta\, d\theta}{(t^2-x^2+x^2\cos^2\theta)^{\frac{1}{2}}} = \sinh^{-1}\left(\frac{x^2}{t^2-x^2}\right)^{\frac{1}{2}} = \log\frac{x+t}{(t^2-x^2)^{\frac{1}{2}}} = \tfrac{1}{2}\log\frac{x+t}{t-x}.$$

Also, if $0<t<x<1$, we can show in the same way that

$$\int_0^t \frac{\alpha\, d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}} = \tfrac{1}{2}\log\frac{x+t}{x-t}.$$

Hence

$$\int_0^{\min(x,t)} \frac{\alpha\, d\alpha}{\{(x^2-\alpha^2)(t^2-\alpha^2)\}^{\frac{1}{2}}} = \tfrac{1}{2}\log\frac{x+t}{|x-t|}.$$

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