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POSITIVE DEPENDENCE OF SIGNALS

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Abstract

In this paper we further investigate the problem considered by Mizuno (2006) in the special case of identically distributed signals. Specifically, we first propose an alternative sufficient condition of crossing type for the convex order to hold between the conditional expectations given signal. Then, we prove that the bivariate (2,1)-increasing convex order ensures that the conditional expectations are ordered in the convex sense. Finally, the L^2 distance between the quantity of interest and its conditional expectation given signal (or expected conditional variance) is shown to decrease when the strength of the dependence increases (as measured by the (2,1)-increasing convex order).

Keywords: Concordance order; supermodularity; convex order; conditional expectation; bivariate order of increasing-convex type

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1. Introduction and motivation

There are numerous situations in which we cannot observe directly a variable X of interest but we only have at our disposal a signal S for X. The signal S is correlated to X so that observing S brings some information about the hidden X: the more S is correlated to X, the more information it contains. Hence, we prefer signals S as perfectly correlated to X as possible. Considering two identically distributed signals S_1 and S_2 for X, the strength of the dependence in the pairs (X, S_1) and (X, S_2) can be compared with appropriate bivariate stochastic order relations, such as the concordance order for instance.

We can also compare signals based on the property that more informative signals lead to greater variability of the conditional expectation $\mathrm{E}[X \mid S]$. If the signal S is independent of X then the variance of $\mathrm{E}[X \mid S] = \mathrm{E}[X]$ is 0 so that S does not contain any information about X. On the other hand, if the signal is perfect, that is, if $S = X$, then the variance of $\mathrm{E}[X \mid S]$ is maximum, being equal to the variance of X. The convex order is often used in applied probability to compare the variability inherent to probability distributions beyond standard deviations. Recall that a random variable Y is said to be smaller than another random variable Z in the convex order, henceforth denoted as $Y \preceq_{\mathrm{cx}} Z$, if $\mathrm{E}[Y] = \mathrm{E}[Z]$ and $\mathrm{E}[(Y - t)_+] \leq \mathrm{E}[(Z - t)_+]$ for all $t \in \mathbb{R}$. The name convex order comes from the fact that $Y \preceq_{\mathrm{cx}} Z$ if and only if $\mathrm{E}[g(Y)] \leq \mathrm{E}[g(Z)]$ for all the convex functions g for which the expectations exist. For more details, we refer the reader to, e.g. Shaked and Shanthikumar (2007) or to Denuit *et al.* (2005). Here, we consider that a signal S_2 is more informative than another signal S_1 if $\mathrm{E}[X \mid S_1] \preceq_{\mathrm{cx}} \mathrm{E}[X \mid S_2]$. The literature about auction theory says that S_2 is more integral precise than S_1 in such a case. See Ganuza and Penalva (2010).

In this paper we show that these two approaches for comparing signals S_1 and S_2 are essentially equivalent: if the pair (X, S_2) is more positively dependent than the pair (X, S_1) then

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there is more information in S_2 about X compared to S_1. We restrict our analysis to identically distributed signals so that we concentrate on the dependence structure of (X, S_1) and (X, S_2), avoiding any marginal effect.

Following Mizuno (2006), let us consider the trivariate nonnegative random vector (X, S_1, S_2), where S_1 and S_2 are interpreted as noisy signals of the unobservable random variable X. Let F denote the distribution function of X, i.e. $F(x) = \Pr[X \leq x]$, and let G denote the common distribution function of S_1 and S_2, i.e. $G(s) = \Pr[S_1 \leq s] = \Pr[S_2 \leq s]$. Furthermore, $G_i(\cdot \mid x)$ is the conditional distribution function of S_i given $X = x$, $i = 1, 2$, and $m_i(s) = \mathrm{E}[X \mid S_i = s]$ is the conditional expectation of X given $S_i = s$, $i = 1, 2$. Throughout the paper, we assume, as in Mizuno (2006), that both m_1 and m_2 are nondecreasing. Mizuno (2006) proved that if the function $x \mapsto G_1(s_1 \mid x) - G_2(s_2 \mid x)$ changes sign at most once from negative to positive as x increases, then $m_1(S_1)$ precedes $m_2(S_2)$ in the convex order.

In this paper we first propose an alternative sufficient condition of crossing type for $m_1(S_1)$ and $m_2(S_2)$ to be ordered in the convex sense. As suggested in Mizuno (2006, p. 1185), the assumptions of his Proposition 1 are strong and can be relaxed when the analysis is restricted to identically distributed signals. In fact, we show that if the bivariate (2,1)-increasing convex order introduced in Denuit *et al.* (1999) holds between (X, S_1) and (X, S_2), then $m_1(S_1)$ and $m_2(S_2)$ are ordered in the convex sense. This result turns out to be useful for the applications as most bivariate models can be ordered in the bivariate (2,1)-increasing convex order (which is weaker than the concordance order, or (1,1)-increasing convex order). Finally, we examine the closeness (in the L^2 distance) of X to its conditional expectation given signal when the strength of dependence is increased (in the sense of the (2,1)-increasing convex order).

2. Crossing-type condition for the conditional expectations

Let us now propose an alternative to the sufficient condition in Proposition 1 of Mizuno (2006) in the special case of identically distributed signals S_1 and S_2. Instead of considering the function $x \mapsto G_1(s_1 \mid x) - G_2(s_2 \mid x)$ for arbitrary s_1 and s_2, we use here the difference $s \mapsto m_2(s) - m_1(s)$ of the conditional expectations of X given signals. In the next result, we show that it suffices that the difference $m_1 - m_2$ exhibits a single sign change for the conditional expectations $m_1(S_1)$ and $m_2(S_2)$ to be ordered in the convex sense.

Proposition 2.1. *Assume that m_1 and m_2 are nondecreasing. If $s \mapsto m_2(s) - m_1(s)$ changes sign at most once from negative to positive as s increases, that is,*

$$m_2(s) \geq m_1(s) \quad \Longrightarrow \quad m_2(s') \geq m_1(s') \quad \text{for all } s \leq s', \tag{2.1}$$

then $m_1(S_1) \preceq_{\mathrm{cx}} m_2(S_2)$.

Proof. Note that $\mathrm{E}[m_1(S_1)] = \mathrm{E}[m_2(S_2)] = \mathrm{E}[X]$. Condition (2.1) ensures that the functions m_1 and m_2 cross at most once, and that m_2 dominates m_1 for sufficiently large arguments. Hence, the distribution functions of $m_1(S_1)$ and $m_2(S_2)$ cross exactly once (because of equal expectations), the distribution function of $m_1(S_1)$ dominating for sufficiently large arguments. By Theorem 3.A.44 of Shaked and Shanthikumar (2007), this ensures that $m_1(S_1) \preceq_{\mathrm{cx}} m_2(S_2)$, as announced.

The single crossing condition (2.1) is not on the conditional distributions, as in Mizuno (2006), but involves the conditional expectations. It is useful in some applications, for instance in auction theory. A related result has been obtained in Ganuza and Penalva (2010, Lemma 1) where the dispersive order between $m_1(S_1)$ and $m_2(S_2)$ is obtained under a stronger condition than (2.1).

3. Bivariate (2,1)-increasing convex order

For some positive integer s, let $\mathcal{U}_{s-\mathrm{icx}}$ be the class of measurable functions with nonnegative derivatives of degrees 1 to s. Recall from Denuit *et al.* (1998) that given two random variables Y and Z, Y is said to be smaller than Z in the s-increasing convex sense, denoted by $Y \preceq_s Z$, when $\mathrm{E}[g(Y)] \le \mathrm{E}[g(Z)]$ for all the functions $g \in \mathcal{U}_{s-\mathrm{icx}}$ such that the expectations exist. For $s = 1$, '$\preceq_1$' coincides with the usual stochastic dominance. For $s = 2$, '$\preceq_2$' is the increasing convex order. Furthermore, $Y \preceq_2 Z$ and $\mathrm{E}[Y] = \mathrm{E}[Z]$ if and only if $Y \preceq_{\mathrm{cx}} Z$.

Let us now consider the bivariate case and denote by $g^{(i,j)}$ the (i, j)th mixed partial derivative of g with respect to x_1 and x_2, that is, $g^{(i,j)} = (\partial^{i+j}/\partial x_1^i \partial x_2^j)g$. For some positive integers s_1 and s_2, let $\mathcal{U}_{(s_1,s_2)-\mathrm{icx}}$ be the class of measurable functions g such that $g^{(k_1,k_2)} \ge 0$ for all $k_1 = 0, \dots, s_1$ and $k_2 = 0, \dots, s_2$, such that $k_1 + k_2 \ge 1$. Recall from Denuit *et al.* (1999) that (X, S_1) is said to be smaller than (X, S_2) in the bivariate (s_1, s_2)-increasing convex order, denoted by $(X, S_1) \preceq_{(s_1,s_2)} (X, S_2)$, when $\mathrm{E}[g(X, S_1)] \le \mathrm{E}[g(X, S_2)]$ for all $g \in \mathcal{U}_{(s_1,s_2)-\mathrm{icx}}$ such that the expectations exist.

For $s_1 = s_2 = 1$, '$\preceq_{(1,1)}$' coincides with the concordance order. Concordance conveys the idea of clustering of large and small events. Large and small values tend to be more often associated under the distribution that dominates the other one in the concordance order. The '$\preceq_{(1,1)}$' order can be characterized by

$$
\begin{aligned}
&(X, S_1) \preceq_{(1,1)} (X, S_2)\\
&\quad\Longleftrightarrow\quad \Pr[X > t_1,\ S_1 > t_2] \le \Pr[X > t_1,\ S_2 > t_2] \quad \text{for all } t_1, t_2\\
&\quad\Longleftrightarrow\quad \mathrm{E}[h_1(X)h_2(S_1)] \le \mathrm{E}[h_1(X)h_2(S_2)] \quad \text{for all nondecreasing } h_1, h_2 \ge 0\\
&\quad\Longleftrightarrow\quad \mathrm{E}[g(X, S_1)] \le \mathrm{E}[g(X, S_2)] \quad \text{for all nondecreasing } g \text{ such that } g^{(1,1)} \ge 0.
\end{aligned}
$$

Since (X, S_1) and (X, S_2) have identical marginal distributions, we also have

$$
\begin{aligned}
&(X, S_1) \preceq_{(1,1)} (X, S_2)\\
&\quad\Longleftrightarrow\quad \mathrm{cov}[h_1(X), h_2(S_1)] \le \mathrm{cov}[h_1(X), h_2(S_2)] \quad \text{for all nondecreasing } h_1, h_2 \ge 0\\
&\quad\Longleftrightarrow\quad \Pr[X > t_1 \mid S_1 > t_2] \le \Pr[X > t_1 \mid S_2 > t_2]\\
&\qquad\qquad\qquad \text{for all } t_1 \text{ and } t_2 \text{ provided that } \Pr[S_i > t_2] > 0,\ i = 1, 2.
\end{aligned}
$$

The latter inequality intuitively means that the knowledge that S_2 is large (that is, $S_2 > t_2$) increases the probability that X is also large (that is, $X > t_1$) compared to (X, S_1).

Now, for $s_1 = 2$ and $s_2 = 1$, the stochastic order relation '$\preceq_{(2,1)}$' is weaker than the concordance order '$\preceq_{(1,1)}$'. Denoting as $\mathbf{1}(A)$ the indicator function of the event A (equal to 1 if A is realized and to 0 otherwise), and remembering that (X, S_1) and (X, S_2) have identical marginal distributions, it can be characterized by

$$
\begin{aligned}
&(X, S_1) \preceq_{(2,1)} (X, S_2)\\
&\quad\Longleftrightarrow\quad \int_{t_1}^{\infty} \Pr[X > \xi,\ S_1 > t_2]\,\mathrm{d}\xi \le \int_{t_1}^{\infty} \Pr[X > \xi,\ S_2 > t_2]\,\mathrm{d}\xi \quad \text{for all } t_1, t_2\\
&\quad\Longleftrightarrow\quad \mathrm{E}[(X - t_1)_+ \mathbf{1}(S_1 > t_2)] \le \mathrm{E}[(X - t_1)_+ \mathbf{1}(S_2 > t_2)] \quad \text{for all } t_1, t_2\\
&\quad\Longleftrightarrow\quad \mathrm{E}[h_1(X)h_2(S_1)] \le \mathrm{E}[h_1(X)h_2(S_2)]\\
&\qquad\qquad\qquad \text{for all nondecreasing } h_1, h_2 \ge 0 \text{ with } h_1 \text{convex}
\end{aligned}
$$

$$\begin{aligned}
\iff\quad & \mathrm{cov}[h_1(X), h_2(S_1)] \le \mathrm{cov}[h_1(X), h_2(S_2)] \\
& \qquad\qquad \text{for all nondecreasing } h_1, h_2 \ge 0 \text{ with } h_1 \text{ convex} \\
\iff\quad & \mathrm{E}[g(X, S_1)] \le \mathrm{E}[g(X, S_2)] \\
& \qquad \text{for all nondecreasing } g \text{ such that } g^{(2,0)} \ge 0,\ g^{(1,1)} \ge 0,\ \text{and } g^{(2,1)} \ge 0 \\
\iff\quad & \mathrm{E}[(X - t_1)_+ \mid S_1 > t_2] \le \mathrm{E}[(X - t_1)_+ \mid S_2 > t_2] \qquad (3.1) \\
& \qquad \text{for all } t_1 \text{ and } t_2 \text{ provided that } \Pr[S_i > t_2] > 0,\ i = 1, 2.
\end{aligned}$$

Inequality (3.1) shows that $(X, S_1) \preceq_{(2,1)} (X, S_2)$ means that the knowledge that S_2 is large (that is, $S_2 > t_2$) increases the average part of X above any threshold t_1 compared to (X, S_1). This also means that the conditional distribution of X given $S_2 > t_2$ dominates the conditional distribution of X given $S_1 > t_2$ in the '$\preceq_2$' order.

We are now in a position to establish the following result.

Proposition 3.1. *If* $(X, S_1) \preceq_{(2,1)} (X, S_2)$ *then* $m_1(S_1) \preceq_{\mathrm{cx}} m_2(S_2)$.

Proof. From Shaked and Shanthikumar (2007, Equation (3.A.41)) we know that $Y \preceq_{\mathrm{cx}} Z$ if and only if $\mathrm{E}[Y] = \mathrm{E}[Z]$ and

$$\mathrm{E}[Y \mid Y \ge F_Y^{-1}(p)] \le \mathrm{E}[Z \mid Z \ge F_Z^{-1}(p)] \quad \text{for all } p \in [0, 1),$$

where F_Y^{-1} and F_Z^{-1} are the quantile functions associated with the distribution functions F_Y and F_Z, respectively. Denoting by $G_{m_1(S_1)}$ the distribution function of $m_1(S_1)$, we have

$$\begin{aligned}
\mathrm{E}[m_1(S_1) \mid m_1(S_1) \ge G^{-1}_{m_1(S_1)}(p)] &= \mathrm{E}[\mathrm{E}[X \mid S_1] \mid S_1 \ge G^{-1}(p)] \\
&= \mathrm{E}[X \mid S_1 \ge G^{-1}(p)].
\end{aligned}$$

As S_1 and S_2 are identically distributed, we see that $m_1(S_1) \preceq_{\mathrm{cx}} m_2(S_2)$ holds if and only if the inequality

$$\mathrm{E}[X \mid S_1 \ge t] \le \mathrm{E}[X \mid S_2 \ge t] \qquad (3.2)$$

is valid for all t, which is the case if $(X, S_1) \preceq_{(2,1)} (X, S_2)$ by virtue of (3.1).

Note that condition (3.2) also appears in Muliere and Petrone (1992) in their study of dependence orderings based on generalized Lorenz curves.

4. The L^2 distance between the signal and conditional expectation

The stochastic inequality $(X, S_1) \preceq_{(2,1)} (X, S_2)$ means that S_2 is a better, or more informative, signal for X than S_1. The next result formalizes this intuitive idea by showing that $m_2(S_2)$ is closer to X than $m_1(S_1)$ in the L^2 distance.

Proposition 4.1. *If* $s \mapsto m_i(s)$ *is nondecreasing for* $i = 1, 2$ *then*

$$(X, S_1) \preceq_{(2,1)} (X, S_2) \Rightarrow \mathrm{E}[(X - m_2(S_2))^2] \le \mathrm{E}[(X - m_1(S_1))^2],$$

that is, X *is closer to* $m_2(S_2)$ *in the* L^2*-norm.*

Proof. The result is a direct consequence of Proposition 3.1 since

$$\mathrm{var}[X] = \mathrm{E}[\mathrm{var}[X \mid S_i]] + \mathrm{var}[m_i(S_i)] \quad \text{and} \quad \mathrm{E}[\mathrm{var}[X \mid S_i]] = \mathrm{E}[(X - m_i(S_i))^2]$$

hold for $i = 1, 2$, and $m_1(S_1) \preceq_{\mathrm{cx}} m_2(S_2)$ implies that $\mathrm{var}[m_1(S_1)] \le \mathrm{var}[m_2(S_2)]$.

Note that the '$\preceq_{(2,1)}$'-ranking needed in this result is rather weak. Most parametric families of bivariate distributions are '$\preceq_{(1,1)}$'-monotonic in their parameters so that the result of Proposition 3.1 generally applies.

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