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On a Class of Projectively Flat Metrics with Constant Flag Curvature

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Abstract. In this paper, we find equations that characterize locally projectively flat Finsler metrics in the form $F = (\alpha + \beta)^2/\alpha$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form. Then we completely determine the local structure of those with constant flag curvature.

1 Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metrics (with constant flag curvature) on an open domain in R^n. This is Hilbert's fourth problem in the regular case. For a Finsler metric F on a manifold M, the flag curvature $\mathbf{K} = \mathbf{K}(\Pi, y)$ is a function of a tangent plane $\Pi \subset T_xM$ and a non-zero tangent vector $y \in \Pi$. When $F = \sqrt{g_{ij}(x)y^i y^j}$ is a Riemannian metric, $\mathbf{K} = \mathbf{K}(\Pi)$ is independent of y, and is called the sectional curvature. Thus the flag curvature is an analogue of the sectional curvature in Riemannian geometry. Projectively flat Finsler metrics are of scalar flag curvature (*i.e.*, $\mathbf{K}$ is independent of Π containing y for every non-zero tangent vector y), but the flag curvature is not necessarily constant, in contrast to the Riemannian case.

The main purpose of this paper is to study and characterize certain projective flat Finsler metrics (with constant flag curvature).

On every strongly convex domain $\mathcal{U}$ in R^n, Hilbert constructed a complete reversible projectively flat metric $H = H(x, y)$ with negative constant flag curvature $\mathbf{K} = -1$. Then Funk constructed a positively projectively flat metric $\Theta = \Theta(x, y)$ with $\mathbf{K} = -1/4$ on $\mathcal{U}$ such that its symmetrization is just the Hilbert metric, $H(x, y) = \frac{1}{2}(\Theta(x, y) + \Theta(x, -y))$. When $\mathcal{U} = \mathrm{B}^n$ is the unit ball in R^n, the Funk metric is given by

$$\Theta = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2},$$

where $y \in T_x\mathrm{B}^n \approx \mathrm{R}^n$. Here $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product. The Funk metric Θ on B^n is a special Randers metric expressed in the form

$$\Theta = \bar{\alpha} + \bar{\beta}, \tag{1}$$

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where

$$\bar{\alpha} = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \quad \bar{\beta} = \frac{\langle x, y\rangle}{1-|x|^2}.$$

Later, L. Berwald [1] constructed a projectively flat metric with zero flag curvature on the unit ball B^n, given by

$$B = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}},$$

where $y \in T_x\mathrm{B}^n \equiv \mathrm{R}^n$. Berwald's metric can be expressed in the form

$$B = \frac{(\lambda\bar{\alpha} + \lambda\bar{\beta})^2}{\lambda\bar{\alpha}}, \tag{2}$$

where $\lambda = 1/(1-|x|^2)$. Berwald's metric B has been generalized by the first author [13] to an arbitrary convex domain $\mathcal{U} \subset \mathrm{R}^n$ using the Funk metric Θ on $\mathcal{U}$. For example, $\tilde{B} := \Theta\{1 + \Theta_{y^m}x^m\}$ is projectively flat with $\mathbf{K} = 0$.

We can extend the Finsler metrics in (1) or (2) in another way, keeping their expression forms. In [12], the first author showed that a Randers metric on a manifold is locally projectively flat with constant flag curvature if and only if it is locally Minkowskian or up to a scaling and reversing, it is locally isometric to $\Theta_a = \bar{\alpha} + \bar{\beta}_a$, where $\bar{\alpha}$ is defined above and $\bar{\beta}_a$ is given by

$$\bar{\beta}_a := \frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle},$$

where $a \in \mathrm{R}^n$ is a constant vector with $|a| < 1$. The metric Θ_a is projectively flat with $\mathbf{K} = -1/4$.

In [11], we constructed the following metric F_a on $\mathrm{B}^n \subset \mathrm{R}^n$ for any constant vector $a \in \mathrm{R}^n$ with $|a| < 1$:

$$F_a := \frac{(\lambda_a\bar{\alpha} + \lambda_a\bar{\beta}_a)^2}{\lambda_a\bar{\alpha}}, \tag{3}$$

where

$$\lambda_a := \frac{(1+\langle a, x\rangle)^2}{1-|x|^2}.$$

We have proved that the metric F_a in (3) is projectively flat with $\mathbf{K} = 0$. See [11] for a detailed proof. When $a = 0$, the metric in (3) is reduced to (2).

Recently, R. Bryant [2–4] studied and characterized locally projectively flat Finsler metrics with constant flag curvature $\mathbf{K} = 1$. It is clear that Bryant's metrics cannot be expressed in terms of a Riemannian metric and a 1-form as Randers metrics and Berwald's metrics. See [13] for other examples.

The above discussion leads us to study the following function F on the tangent bundle TM of a manifold M,

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \tag{4}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. It is known that F is a Finsler metric if and only if $b(x) := \|\beta_x\|_\alpha < 1$ at any point $x \in M$. A natural question arises: is there any other projectively flat metric in the form (4) with constant flag curvature?

In this paper, we shall first prove the following.

Theorem 1.1 *Let $F = (\alpha + \beta)^2/\alpha$ be a Finsler metric on a manifold M. Then F is projectively flat if and only if*

(i) $b_{i|j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}$,

(ii) *the spray coefficients G^i_α of α are in the form: $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,*

where $b := \|\beta_x\|_\alpha$, $b_{i|j}$ denote the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\theta = a_i(x)y^i$ is a 1-form on M.

In [11], we have already noticed that if α and β satisfy conditions (i) and (ii), then $F = (\alpha + \beta)^2/\alpha$ is locally projectively flat. Theorem 1.1 asserts that the converse is true, too. Theorem 1.1 is a special case of Theorem 3.1 below. There are many non-trivial Finsler metrics satisfying conditions (i) and (ii) of Theorem 1.1. See [11].

By Theorem 1.1, we can completely determine the local structure of a projectively flat Finsler metric F in the form (4) which is of constant flag curvature.

Theorem 1.2 *Let $F = (\alpha + \beta)^2/\alpha$ be a Finsler metric on a manifold M. Then F is locally projectively flat with constant flag curvature if and only if one of the following conditions holds.*

(i) *α is flat and β is parallel with respect to α. In this case, F is locally Minkowskian.*

(ii) *Up to a scaling on x and a scaling on F, F is locally isometric to F_a in* (3).

In either case (i) *or* (ii)*, the flag curvature of F must be zero,* $\mathbf{K} = 0$.

Below is an outline of the proof of Theorem 1.2. By imposing the curvature condition that the flag curvature be constant, we first show that the flag curvature must be zero, $\mathbf{K} = 0$. If $\tau = 0$, then F is locally Minkowskian. In the case when $\tau \neq 0$, we show that

$$d\tau + 2\tau^2\beta = 0, \quad \theta_{x^k}y^k - \theta^2 = 3\tau^2(\alpha^2 - 2\beta^2). \tag{5}$$

Then we show that $\tau\beta$ is closed. Thus there is a local scalar function $\rho = \rho(x)$ such that $\tau\beta = \frac{1}{2}d\rho$ and $\tau = ce^{-\rho}$ for some constant c. Immediately, we can see that $\bar{\alpha} := e^{-\rho}\alpha$ is projectively flat, hence $\bar{\alpha}$ is of constant curvature $\bar{\mathbf{K}} = \mu$ by the Beltrami theorem. The constant μ must be nonpositive. By choosing the projective form of $\bar{\alpha}$, we can solve (5) for ρ. Then we determine α and β. The detailed argument is given in the proof of Theorem 5.1 below.

2 (α, β)-Metrics

The Finsler metric in (4) is a special (α, β)-metric. By definition, an (α, β)-metric is expressed in the form, $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. Then $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_o.$$

It is known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_o$ for any $x \in M$ [7]. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, given by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^k}\}, \quad G^i_\alpha = \frac{a^{il}}{4}\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l}\},$$

where $(g^{ij}) := (\frac{1}{2}[F^2]_{y^i y^j})$ and $(a^{ij}) := (a_{ij})^{-1}$. We have the following.

Lemma 2.1 *The geodesic coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 + r_{00}\}\Big\{b^i - s\frac{y^i}{\alpha}\Big\}, \tag{6}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad J := \frac{\phi'(\phi - s\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$
$$H := \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

and $s := \beta/\alpha$ and $b := \|\beta_x\|_\alpha$.

The formula (6) is given in [7, 14]. A different version of (6) is given in [9, 10].

It is well known that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{7}$$

This is due to G. Hamel [8]. Using (7), we prove the following.

Lemma 2.2 *An (α, β)-metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$, is projectively flat on an open subset $\mathcal{U} \subset \mathrm{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0. \tag{8}$$

Proof Applying (7) to the (α, β)-metric $F = \alpha\phi(s)$ we obtain

$$\begin{aligned}[\alpha_{x^k y^l} y^k - \alpha_{x^l}]\phi + \alpha\phi'[s_{x^k y^l} y^k - s_{x^l}] \\ + \phi'\big[(\alpha_{x^k} y^k)s_{y^l} + (s_{x^k} y^k)\alpha_{y^l}\big] + \alpha\phi''(s_{x^k} y^k)s_{y^l} = 0.\end{aligned} \tag{9}$$

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We have

$$\alpha_{x^l} = \frac{1}{\alpha}\frac{\partial G^m_\alpha}{\partial y^l} y_m, \quad \alpha_{x^k} y^k = \frac{2}{\alpha} G^m_\alpha y_m, \quad \alpha_{x^k y^l} y^k - \alpha_{x^l} = \frac{2}{\alpha^3}\{a_{ml}\alpha^2 - y_m y_l\} G^m_\alpha,$$

$$s_{x^l} = \frac{1}{\alpha} b_{ml} y^m + \frac{1}{\alpha^2}\{b_m\alpha - s y_m\}\frac{\partial G^m_\alpha}{\partial y^l}, \quad s_{x^k} y^k = \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2}\{b_m\alpha - s y_m\} G^m_\alpha,$$

$$\begin{aligned} s_{x^k y^l} y^k - s_{x^l} &= -\frac{r_{00}}{\alpha^3} y_l + \frac{2}{\alpha} s_{l0} - \frac{4 y_l}{\alpha^4}\{b_m\alpha - s y_m\} G^m_\alpha \\ &\quad + \frac{2}{\alpha^2}\left\{\frac{y_l}{\alpha} b_m - \frac{b_l\alpha - s y_l}{\alpha^2} y_m - s\delta_{ml}\right\} G^m_\alpha, \\ s_{y^l} &= \frac{b_l\alpha - s y_l}{\alpha^2}, \end{aligned}$$

where $y_m := a_{im} y^i$. Plugging these into (9) yields

$$\begin{aligned} (10)\quad 2(\phi - s\phi')\{a_{ml}\alpha^2 - y_m y_l\} G^m_\alpha + 2\phi'\alpha^3 s_{l0} \\ + \phi''\{r_{00}\alpha + 2(b_m\alpha - s y_m) G^m_\alpha\}(b_l\alpha - s y_l) = 0. \end{aligned}$$

Contracting (10) with b^l yields

$$2(b_m\alpha - s y_m) G^m_\alpha = -\frac{2\phi'\alpha^2 s_0 + (b^2 - s^2)\phi''\alpha r_{00}}{\phi - s\phi' + (b^2 - s^2)\phi''}.$$

Substituting it back into (10), we get (8). ∎

3 $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$

In this section, we consider an (α, β)-metric in the following form:

$$F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha},$$

where ε, k are constants with $k \neq 0$, $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form on M. Let $b_o = b_o(\varepsilon, k) > 0$ be the largest number such that

$$(11)\qquad 1 + \varepsilon s + k s^2 > 0, \quad 1 + 2kb^2 - 3ks^2 > 0, \quad |s| \le b < b_o,$$

so that F is a Finsler metric if and only if β satisfies that $b := \|\beta_x\|_\alpha < b_o$ for any $x \in M$.

From now on, we always assume that ε and $k \neq 0$ satisfy (11). By Lemma 2.1, the spray coefficients G^i of F are given by (6) with

$$Q = \frac{\varepsilon + 2ks}{1 - ks^2}, \quad J = \frac{(\varepsilon + 2ks)(1 - ks^2)}{2(1 + \varepsilon s + ks^2)(1 + 2kb^2 - 3ks^2)}, \quad H = \frac{k}{1 + 2kb^2 - 3ks^2}.$$

Equation (8) is reduced to

$$(12)\quad (a_{mi}\alpha^2 - y_m y_i)G^m_\alpha + \frac{\varepsilon + 2ks}{1 - ks^2}\alpha^3 s_{l0} + \frac{k}{1 + 2kb^2 - 3ks^2}\alpha\Big\{-\frac{2(\varepsilon + 2ks)}{1 - ks^2}\alpha s_0 + r_{00}\Big\}(b_i\alpha - sy_i) = 0.$$

By the above identity, we can prove the following.

Theorem 3.1 *Let $k \neq 0$. Then $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ is projectively flat if and only if*
(i) $b_{i|j} = \tau((k^{-1} + 2b^2)a_{ij} - 3b_i b_j)$,
(ii) $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$,
where $\tau = \tau(x)$ and $\theta = a_i(x)y^i$. In this case,

$$(13)\qquad G^i = \{\theta + \tau\chi\alpha\}y^i,$$

where

$$(14)\qquad \chi := \frac{(\varepsilon + 2ks)(1 - ks^2)}{2k(1 + \varepsilon s + ks^2)} - s, \quad s = \frac{\beta}{\alpha}.$$

Proof First, we rewrite (12) as a polynomial in y^i and α that is linear in α. This gives

$$(15)\ [(1 + 2kb^2)\alpha^2 - 3k\beta^2][\alpha^2 - k\beta^2](a_{mi}\alpha^2 - y_m y_i)G^m_\alpha + \alpha^4[(1 + 2kb^2)\alpha^2 - 3k\beta^2][\varepsilon\alpha + 2k\beta]s_{i0} + k\alpha^2[-2\alpha^2(\varepsilon\alpha + 2k\beta)s_0 + (\alpha^2 - k\beta^2)r_{00}](b_i\alpha^2 - \beta y_i) = 0.$$

The coefficients of α must be zero (note: α^{even} is a polynomial in y^i). We obtain

$$\varepsilon[(1 + 2kb^2)\alpha^2 - 3k\beta^2]s_{i0} = \varepsilon[2ks_0(b_i\alpha^2 - \beta y_i)].$$

Suppose that $\varepsilon \neq 0$. Then

$$(16)\qquad [(1 + 2kb^2)\alpha^2 - 3k\beta^2]s_{i0} = 2ks_0(b_i\alpha^2 - \beta y_i).$$

Contracting (16) with b^i yields $(\alpha^2 - k\beta^2)s_0 = 0$. By assumption, for any $y \neq 0$, $\alpha^2 - k\beta^2 \neq 0$. Thus $s_0 = 0$. Then it follows from (16) that

$$(17)\qquad s_{i0} = 0.$$

Thus β is closed.

Now equation (15) is reduced to

$$(18)\qquad [(1 + 2kb^2)\alpha^2 - 3k\beta^2](a_{mi}\alpha^2 - y_m y_i)G^m_\alpha + k\alpha^2 r_{00}(b_i\alpha^2 - \beta y_i) = 0.$$

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Contracting (18) with b^i, we get

$$[(1+2kb^2)\alpha^2 - 3k\beta^2](b_m\alpha^2 - y_m\beta)G^m_\alpha = -k\alpha^2(b^2\alpha^2 - \beta^2)r_{00}.$$

Note that the polynomial $(1+2kb^2)\alpha^2 - 3k\beta^2$ is not divisible by α^2 and $b^2\alpha^2 - \beta^2$. Thus $(b_m\alpha^2 - y_m\beta)G^m_\alpha$ is divisible by $\alpha^2(b^2\alpha^2 - \beta^2)$. Therefore, there is a scalar function $\tau = \tau(x)$ such that

$$r_{00} = \frac{\tau}{k}\left[(1+2kb^2)\alpha^2 - 3k\beta^2\right]. \tag{19}$$

By (17) and (19), the formula (6) for G^i can be simplified to

$$G^i = G^i_\alpha + \tau\chi\alpha y^i + \tau\alpha^2 b^i, \tag{20}$$

where χ is given in (14). We know that F is projectively flat if and only if $G^i = Py^i$. By (20), this is equivalent to $G^i_\alpha = \theta y^i - \tau\alpha^2 b^i$, where $\theta = a_i y^i$ is a 1-form. In this case, G^i are given by (13). This proves Theorem 3.1 in the case when $\varepsilon \neq 0$.

Now let us study the case when $\varepsilon = 0$. In this case,

$$F = \alpha + \frac{\beta^2}{\alpha}. \tag{21}$$

First it is easy to verify that under the conditions (i) and (ii) in Theorem 3.1, F in (21) is projectively flat. Conversely, assume that F is locally projectively flat. Then it must be a Douglas metric. By Matsumoto's result on Douglas metrics [10], one can see that α and β must satisfy the condition (i). Since F is locally projectively flat, by a simple argument as above, one can see that the condition (ii) is satisfied. ■

We should point out that the Riemannian metric α in Theorem 3.1 is not locally projectively flat in general.

4 Flag Curvature

In this section, we shall study the following metric with constant flag curvature $\mathbf{K} = \lambda$,

$$F = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha},$$

where ε and k are constants with $k \neq 0$. We assume that F is locally projectively flat, so that in a local coordinate system the spray coefficients of F are in the form (13). It is known that if the spray coefficients of F are in the form $G^i = Py^i$, then F is of scalar curvature with flag curvature

$$\mathbf{K} = \frac{P^2 - P_{x^k}y^k}{F^2}.$$

Then

$$\begin{aligned}\mathbf{K} &= \frac{[\theta + \tau\chi\alpha]^2 - [\theta + \tau\chi\alpha]_{x^k} y^k}{F^2} \\ &= \frac{(\theta + \tau\chi\alpha)^2 - \theta_{x^k} y^k - \tau_{x^k} y^k \chi\alpha - \tau\chi'(s) s_{x^k} y^k \alpha - \tau\chi\alpha_{x^k} y^k}{F^2}.\end{aligned}$$

Observe that

$$\begin{aligned} s_{x^k} y^k &= \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2}\{b_m\alpha - s y_m\} G^m_\alpha \\ &= \frac{\tau}{k}\{(1 + 2kb^2) - 3ks^2\}\alpha + \frac{2}{\alpha^2}\{b_m\alpha - s y_m\}\{\theta y^m - \tau\alpha^2 b^m\} \\ &= \tau\{(1/k + 2b^2) - 3s^2\}\alpha - 2\tau(b^2 - s^2)\alpha \\ &= \tau(1/k - s^2)\alpha. \\ \alpha_{x^k} y^k &= \frac{2}{\alpha} G^m_\alpha y_m = \frac{2}{\alpha}\{\theta y^m - \tau\alpha^2 b^m\} y_m = 2(\theta - \tau\beta)\alpha. \end{aligned}$$

We obtain

$$\mathbf{K} = \frac{\theta^2 - \theta_{x^k} y^k + \tau^2\chi^2\alpha^2 - \chi\tau_0\alpha - \tau^2(1/k - s^2)\chi'\alpha^2 + 2s\tau^2\chi\alpha^2}{F^2}. \tag{22}$$

Lemma 4.1 *Suppose that $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ with $k \neq 0$ is projectively flat with constant flag curvature $\mathbf{K} = \lambda =$ constant; then $\lambda = 0$.*

Proof First by (22), the equation $\mathbf{K} = \lambda$ multiplied by $k^2\alpha^4F^4$ yields:

$$A\alpha^3 + B\alpha^2 - 4\varepsilon\lambda k^5\beta^7\alpha - \lambda k^6\beta^8 = 0,$$

where A and B are homogeneous polynomials in y of degrees 5 and 6, respectively. Rewriting the above equation as $\{A\alpha^2 - 4\varepsilon\lambda k^5\beta^7\}\alpha + \{B\alpha^2 - \lambda k^6\beta^8\} = 0$. We must have

$$A\alpha^2 - 4\varepsilon\lambda k^5\beta^7 = 0, \quad B\alpha^2 - \lambda k^6\beta^8 = 0. \tag{23}$$

Since β^2 is not divisible by α, we conclude from the second identity in (23) that $\lambda = 0$. ■

Now we consider the trivial case when $\tau = 0$. In this case, $b_{i|j} = 0$, and $G^i = G^i_\alpha = \theta y^i$. By Lemma 4.1, F has zero flag curvature, thus α has zero sectional curvature. Thus α is locally isometric to the Euclidean metric. We have proved the following.

Proposition 4.2 *Let $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$, where $k \neq 0$. Suppose that F is a locally projectively flat metric with zero flag curvature. If $\tau = 0$, then α is flat metric and β is parallel. In this case, F is locally Minkowskian.*

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The case when $\tau \neq 0$ is more complicated. First we have the following.

Proposition 4.3 *Suppose that $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ with $k \neq 0$ is projectively flat with zero flag curvature and $\tau \neq 0$. Then*
(i) $\varepsilon^2 = 4k$,
(ii) $\tau_{x^i} + 2\tau^2 b_i = 0$,
(iii) $\theta_{x^i} y^i - \theta^2 = 3\tau^2(k^{-1}\alpha^2 - 2\beta^2)$.

Proof Under the assumption that $\mathbf{K} = 0$, we obtain

$$\Phi\alpha + \Psi = 0, \tag{24}$$

where

$$\Phi := -\{2k\varepsilon P\alpha^2 - 4k^2\varepsilon\beta^2 P + 8k^2\varepsilon\beta Q\}\alpha^2 + 14k^3\varepsilon\beta^4 P - 8k^3\varepsilon\beta^3 Q,$$
$$\begin{aligned}\Psi := {} & 3\tau^2(\varepsilon^2 - 4k)\alpha^6 - \{4Qk^2 + 2k\varepsilon^2\beta P + 6k\tau^2(\varepsilon^2 - 4k)\beta^2\}\alpha^4 \\ & + \{3\tau^2 k^2(\varepsilon^2 - 4k)\beta^4 + 2k^2(3\varepsilon^2 + 4k)P\beta^3 - 4k^2(\varepsilon^2 + 2k)Q\beta^2\}\alpha^2 \\ & + 8k^4\beta^5 P - 4k^4\beta^4 Q,\end{aligned}$$

where $P := \tau_{x^k} y^k + 2\tau^2\beta$ and $Q = \theta_{x^k} y^k - \theta^2 - 3\tau^2(k^{-1}\alpha^2 - 2\beta^2)$. Note that Φ and Ψ are homogeneous polynomials in y and $\alpha = \sqrt{a_{ij}y^i y^j}$ is in a radical form. Equation (24) implies that $\Phi = 0$, and $\Psi = 0$. First we consider the equation $\Phi = 0$. It can be written as

$$\{2k\varepsilon P\alpha^2 - 4k^2\varepsilon\beta^2 P + 8k^2\varepsilon\beta Q\}\alpha^2 = \{14k^3\varepsilon\beta P - 8k^3\varepsilon Q\}\beta^3. \tag{25}$$

Since α^2 does not contain the factor β, there is a scalar function $c_1 = c_1(x)$ such that

$$14\varepsilon\beta P - 8\varepsilon Q = c_1\alpha^2. \tag{26}$$

Then (25) becomes

$$2\varepsilon P\alpha^2 = k\{4\varepsilon\beta P - 8\varepsilon Q + c_1 k\beta^2\}\beta. \tag{27}$$

Since α^2 is not divisible by β, there is a scalar function $c_2 = c_2(x)$ such that

$$4\varepsilon\beta P - 8\varepsilon Q + c_1 k\beta^2 = c_2\alpha^2. \tag{28}$$

Then (27) is reduced to

$$2\varepsilon P = c_2 k\beta. \tag{29}$$

It follows from (26) and (28) that

$$10\varepsilon\beta P - c_1 k\beta^2 = (c_1 - c_2)\alpha^2. \tag{30}$$

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Plugging (29) into (30) yields $k(5c_2 - c_1)\beta^2 = (c_1 - c_2)\alpha^2$. Thus $c_1 = c_2 = 0$, and

$$\varepsilon P = 0, \quad \varepsilon Q = 0. \tag{31}$$

First we assume that $\varepsilon \neq 0$. Then (31) implies that $P = Q = 0$. The formula for Ψ is reduced to $\Psi = 3(\varepsilon^2 - 4k)\tau^2\alpha^2(\alpha^2 - k\beta^2)^2$. Under the assumption that $\tau \neq 0$, the equation $\Psi = 0$ implies that $\varepsilon^2 = 4k$.

Now we assume that $\varepsilon = 0$. We are going to show that this is impossible. The formula for Ψ is reduced to

$$\Psi = -4k\{3\tau^2\alpha^4 + k(Q - 6\tau^2\beta^2)\alpha^2 + k^2(3\tau^2\beta^2 - 2P\beta + 2Q)\beta^2\}\alpha^2 \\ + 4k^4(2\beta P - Q)\beta^4.$$

Then $\Psi = 0$ implies that there is a scalar function $\delta_1 = \delta_1(x)$ such that

$$2\beta P - Q = \delta_1\alpha^2, \tag{32}$$

$$\{3\tau^2\alpha^2 + k(Q - 6\tau^2\beta^2)\}\alpha^2 + k^2\{(3\tau^2\beta^2 - 2P\beta + 2Q) - k\delta_1\beta^2\}\beta^2 = 0. \tag{33}$$

It follows from (33) that there is a scalar function $\delta_2 = \delta_2(x)$ such that

$$3\tau^2\beta^2 - 2P\beta + 2Q = k\delta_1\beta^2 - \delta_2\alpha^2, \tag{34}$$

$$3\tau^2\alpha^2 + k\{Q - 6\tau^2\beta^2\} = k^2\delta_2\beta^2. \tag{35}$$

It follows from (32) and (34) that $Q = (\delta_1 - \delta_2)\alpha^2 + (k\delta_1 - 3\tau^2)\beta^2$. Substituting it into (35) yields that $\{3\tau^2 + k(\delta_1 - \delta_2)\}\alpha^2 = k\{9\tau^2 - k(\delta_1 - \delta_2)\}\beta^2$. We conclude that $3\tau^2 + k(\delta_1 - \delta_2) = 0$, $9\tau^2 - k(\delta_1 - \delta_2) = 0$. This is impossible, since $\tau \neq 0$. Therefore, $\varepsilon \neq 0$. ∎

5 Solving the Equations

In this section, we assume that $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ is projectively flat with zero flag curvature $\mathbf{K} = 0$ and $\tau \neq 0$. By Proposition 4.3, $\varepsilon^2 = 4k > 0$. Then

$$F = \frac{(\alpha \pm \sqrt{k}\beta)^2}{\alpha}.$$

We shall prove the following .

Theorem 5.1 *Let $k > 0$. Let $F = (\alpha \pm \sqrt{k}\beta)^2/\alpha$ be locally projectively flat with $\tau \neq 0$. Suppose that F has constant flag curvature. Then the flag curvature $\mathbf{K} = 0$ and one of the following holds:*

(i) *F is locally Minkowskian.*

(ii) *At every point there is a local coordinate system* (x^i) *in which* α *and* β *are given by*

$$\alpha = \frac{(\delta + \langle a, x\rangle)^2}{1 - c^2|x|^2} \frac{\sqrt{(1 - c^2|x|^2)|y|^2 + c^2\langle x, y\rangle^2)}}{1 - c^2|x|^2},$$

$$\beta = \frac{1}{c\sqrt{k}} \frac{(\delta + \langle a, x\rangle)^2}{1 - c^2|x|^2} \left\{ \frac{\langle a, y\rangle}{\delta + \langle a, x\rangle} + \frac{c^2\langle x, y\rangle}{1 - c^2|x|^2} \right\},$$

where δ *and* c *are non-zero constants and* $a \in \mathrm{R}^n$ *is a constant vector.*

Proof Without loss of generality, we may assume that $k = 1$, thus $\varepsilon = \pm 2$ and

$$F = \frac{(\alpha \pm \beta)^2}{\alpha}.$$

By Theorem 3.1 and Proposition 4.3,

$$b_{i|j} = \tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\}, \tag{36}$$

$$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i, \tag{37}$$

$$\tau_{x^i} + 2\tau^2 b_i = 0, \tag{38}$$

$$\theta_{x^i} y^i - \theta^2 = 3\tau^2(\alpha^2 - 2\beta^2). \tag{39}$$

We are going to solve (36)–(39) for α and β.

It follows from (36) and (38) that

$$(\tau b_i)_{|j} - (\tau b_j)_{|i} = \tau(b_{i|j} - b_{j|i}) + \tau_{x^j} b_i - \tau_{x^i} b_j = 0.$$

Thus $\tau\beta$ is closed. Locally, there is a scalar function $\rho = \rho(x)$ such that

$$\tau b_i = \frac{1}{2}\rho_{x^i}. \tag{40}$$

Substituting it into (38) yields $\tau_{x^i} + \tau\rho_{x^i} = 0$. We obtain

$$\tau = ce^{-\rho}, \tag{41}$$

where $c =$ constant.

Let $\bar{\alpha} = e^{-\rho}\alpha$. Then $G^i_{\bar{\alpha}} = G^i_\alpha - \rho_0 y^i + \frac{1}{2}\rho^i\alpha^2$, where $\rho^i := \rho_{x^i} a^{ik}$.

By (37) and (40), we get

$$G^i_{\bar{\alpha}} = (\theta - \rho_0) y^i. \tag{42}$$

Thus $\bar{\alpha}$ is projectively flat. By the Beltrami theorem, $\bar{\alpha}$ has constant sectional curvature $\mathbf{K}_{\bar{\alpha}} = \mu$. We may assume that

$$\bar{\alpha} = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}.$$

We have $G^i_{\bar\alpha} = -\frac{\mu\langle x,y\rangle}{1+\mu|x|^2}\, y^i$. Substituting it into (42), we obtain

$$\theta = [\rho - \ln\sqrt{1+\mu|x|^2}]_0.$$

Then (39) is reduced to

$$\rho_{x^ix^j} = -\frac{1}{2}\rho_{x^i}\rho_{x^j} - \mu\frac{x^i\rho_{x^j} + x^j\rho_{x^i}}{1+\mu|x|^2} + (3c^2+\mu)\bar a_{ij},$$

where $\bar a_{ij}$ are the coefficients of $\bar\alpha$. Let $\varphi := e^{\rho/2}$. Then

$$\varphi_{x^ix^j} + \mu\frac{x^i\varphi_{x^j} + x^j\varphi_{x^j}}{1+\mu|x|^2} = \frac{3c^2+\mu}{2}\bar a_{ij}\varphi.$$

Let

$$\xi := \sqrt{1+\mu|x|^2}\,\varphi. \tag{43}$$

Then

$$\xi_{x^ix^j} = \frac{3}{2}(c^2+\mu)\sqrt{1+\mu|x|^2}\,\bar a_{ij}\varphi. \tag{44}$$

Let $h := \frac{|x|^2}{\sqrt{1+\mu|x|^2+1}}$. Then

$$\sqrt{1+\mu|x|^2}\,\bar a_{ij} = h_{x^ix^j}. \tag{45}$$

It follows from (44) that $\xi_{x^ix^j} = \frac{3(c^2+\mu)}{2}h_{x^ix^j}\varphi$. Differentiating it yields

$$\xi_{x^ix^jx^k} = \frac{3(c^2+\mu)}{2}\{h_{x^ix^jx^k}\varphi + h_{x^ix^j}\varphi_{x^k}\}.$$

Under our assumption $\tau \neq 0$, we claim that $c^2+\mu = 0$. Suppose that $c^2+\mu \neq 0$. Then by symmetry, we get $h_{x^ix^j}\varphi_{x^k} = h_{x^ix^k}\varphi_{x^j}$. Thus, by (45) $\bar a_{ij}\varphi_{x^k} = \bar a_{ik}\varphi_{x^j}$. Contracting it with $\bar a^{im}$, we get $\delta^m_j\varphi_{x^k} = \delta^m_k\varphi_{x^j}$. This implies that $\varphi_{x^k} = 0$. That is, $\varphi = e^{\rho/2} =$ constant. Then by (41), $\tau = ce^{-\rho} =$ constant and by (40), $\tau b_i = \frac{1}{2}\rho_{x^i} = 0$. Since $\beta \neq 0$, we must have $\tau = 0$ and $c = 0$. This contradicts our assumption.

Now we have that $c^2+\mu = 0$. Then $\mu = -c^2 \leq 0$. If $c = 0$, then by (41), $\tau = 0$. This is a contradiction. Thus $\mu = -c^2 < 0$. In this case, (44) is reduced to $\xi_{x^ix^j} = 0$. We get $\xi = \delta + \langle a, x\rangle$, where δ is a constant and $a \in \mathrm{R}^n$ is a constant vector. Then by (43) and (41),

$$\rho = \ln\varphi^2 = \ln\frac{\xi^2}{1+\mu|x|^2} = \ln\frac{(\delta+\langle a,x\rangle)^2}{1+\mu|x|^2},$$

$$\tau = ce^{-\rho} = c\frac{1+\mu|x|^2}{(\delta+\langle a,x\rangle)^2}.$$

We obtain

$$\alpha = e^{\rho}\bar{\alpha} = \frac{(\delta + \langle a, x\rangle)^2}{1 - c^2|x|^2}\,\frac{\sqrt{(1 - c^2|x|^2)|y|,^2 + c^2\langle x, y\rangle^2)}}{1 - c^2|x|^2},$$

$$\beta = \frac{1}{2\tau}\rho_0 = \frac{1}{c}\,\frac{(\delta + \langle a, x\rangle)^2}{1 - c^2|x|^2}\Big\{\frac{\langle a, y\rangle}{\delta + \langle a, x\rangle} + \frac{c^2\langle x, y\rangle}{1 - c^2|x|^2}\Big\}. \qquad \blacksquare$$

6 Some Properties of F_a

In this last section, we are going to say a few words about the special metric F_a in (3). F_a is given by $F_a := \frac{(\alpha+\beta)^2}{\alpha}$, where $\alpha := \lambda_a\bar{\alpha}$, $\beta := \lambda_a\bar{\beta}_a$, $\lambda_a := (1+\langle a, x\rangle)^2/(1-|x|^2)$. First, it is easy to get $|\beta\|_\alpha = 1 - \frac{1-|a|^2}{\lambda_a}$. Let $g_{ij} := \frac{1}{2}[F_a^2]_{y^iy^j}$ and $a_{ij} := \frac{1}{2}[\alpha^2]_{y^iy^j}$. We have

$$\det(g_{ij}) = \Big(\frac{\alpha^2 - \beta^2}{\alpha^3}\Big)^n F^{n+1}\,\frac{[(1 + 2\|\beta\|_\alpha^2)\alpha^2 - 3\beta^2]\alpha}{(\alpha^2 - \beta^2)^2}\,\det(a_{ij}).$$

Thus, if $|a| < 1$, F_a is a Finsler metric on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$.

Let $b \in \mathrm{R}^n$ be an arbitrary unit vector, *i.e.*, $|b| = 1$ and $m := \langle a, b\rangle$. We have $|m| \leq |a| < 1$. Let $c(t) := bt$. The F_a-length of $c'(t) = b$ is given by

$$F_a(c(t), c'(t)) = \frac{(1 + m)^2}{(1 - t)^2}.$$

Thus, the F_a-lengths of $C_- : c(t), -1 < t \leq 0$ and $C_+ : c(t), 0 \leq t < 1$ are given by

$$\mathrm{Length}(C_-) = \frac{(1 + m)^2}{2}, \quad \mathrm{Length}(C_+) = +\infty.$$

This shows that F_a is positively complete, but not complete.

The Cartan torsion is unbounded. But the formula for the bound of the Cartan torsion is very complicated.

At the origin $x = 0$ and $x = -a$,

$$F_a(0, y) = \frac{(|y| + \langle a, y\rangle)^2}{|y|}, \quad F_a(-a, y) = \sqrt{(1 - |a|^2)|y|^2 + \langle a, y\rangle^2}.$$

Note that F_a is Euclidean at $x = -a$. When a changes, the "Euclidean center", $-a$, of F_a moves.

We conjecture that F_a is a projective representation of Berwald's metric F_0 at $x = -a$. However, we could not find a diffeomorphism $\varphi_a\colon \mathrm{B}^n \to \mathrm{B}^n$ with the following properties: (i) φ maps lines to lines, (ii) $\varphi(0) = -a$, and (ii) $\varphi^*F_0 = F_a$.

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References

[1] L. Berwald, *Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind.* Math. Z. **30**(1929), no. 1, 449–469.
[2] R. Bryant, *Finsler structures on the 2-sphere satisfying $K = 1$.* In: Finsler Geometry, Contemp. Math. 196, American Mathematical Society, Providence, RI, 1996, pp. 27-42.
[3] ———, *Projectively flat Finsler 2-spheres of constant curvature.* Selecta Math. **3**(1997), no. 2, 161–203.
[4] ———, *Some remarks on Finsler manifolds with constant flag curvature.* Houston J. Math. **28**(2002), no. 2, 221–262.
[5] X. Chen, X. Mo, and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature.* J. London Math. Soc. **68**(2003), no. 3, 762–780.
[6] X. Chen and Z. Shen, *Projectively flat Finsler metrics with almost isotropic S-curvature.* Acta Math. Sci. **26**(2006), 307–313.
[7] S. S. Chern and Z. Shen, *Riemann-Finsler geometry.* Nankai Tracts in Mathematics 6, World Scientific, Hackensack, NJ, 2005.
[8] G. Hamel, *Über die Geometrieen, in denen die Geraden die Kürzesten sind.* Math. Ann. **57**(1903), no. 2, 231–264.
[9] M. Kitayama, M. Azuma, and M. Matsumoto, *On Finsler spaces with (α,β)-metric. Regularity, geodesics and main scalars.* J. Hokkaido Univ. Ed. Sect. II A **46**(1995), no. 1, 1–10.
[10] M. Matsumoto, *Finsler spaces with (α,β)-metric of Douglas type.* Tensor **60**(1998), no. 2, 123–134.
[11] X. Mo, Z. Shen, and C. Yang, *Some constructions of projectively flat Finsler metrics.* Sci. China Ser. A **49**(2006), no. 5, 703–714.
[12] Z. Shen, *Projectively flat Randers metrics of constant curvature.* Math. Ann. **325**(2003), no. 1, 19–30.
[13] ———, *Projectively flat Finsler metrics of constant flag curvature.* Trans. Amer. Math. Soc. **355**(2003), no. 4, 1713–1728 (electronic).
[14] Z. Shen, *Landsberg curvature, S-curvature and Riemann curvature.* In: A Sampler of Riemann-Finsler Geometry, Math. Sci. Res. Inst. Publ. 50, Cambridge University Press, Cambridge, 2004.

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