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AN ANALOGY BETWEEN PRODUCTS OF TWO CONJUGACY CLASSES AND PRODUCTS OF TWO IRREDUCIBLE CHARACTERS IN FINITE GROUPS

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Introduction

It is well-known that the number of irreducible characters of a finite group G is equal to the number of conjugate classes of G. The purpose of this article is to give some analogous properties between these basic concepts.

We present the following theorems:

Theorem A. *If C and D are non-trivial conjugacy classes of a finite group G such that either $CD=mC+nD$ or $CD=mC^{-1}+nD$, where m,n are non-negative integers, then G is not a non-abelian simple group.*

Theorem B. *If χ and ψ are non-trivial irreducible characters of a finite group G such that either $\chi\psi=m\chi+n\psi$ or $\chi\psi=m\bar{\chi}+n\psi$, where m,n are non-negative integers, then G is not a non-abelian simple group.*

Analogous theorems between products of conjugacy classes and products of characters were studied in [**1**–**4**]. For example a finite group G is isomorphic to J_1 (the first Janko group) iff $C^2=G$ for every non-trivial conjugacy class C of the finite group G [**1**]. The analogous theorem is that a finite group G is isomorphic to J_1 iff $\mathrm{Irr}(\chi^2)=\mathrm{Irr}(G)$ (i.e., all the irreducible characters of G are constituents of χ^2) for every non-trivial irreducible character χ of the finite group G [**4**].

The proofs of Theorems A and B are elementary; chapters 1–4 of [**5**] suffice for background.

Our notation is standard and is taken mainly from Isaacs [**5**].

Proofs of theorems

Let $\mathbb{N}$ be the set of all positive integers and set $\mathbb{N}^*$ to be $\mathbb{N}\cup\{0\}$.

Let $\mathrm{Irr}(G)=\{\chi_1,\dots,\chi_k\}$ be the set of all irreducible characters of a finite group G. It is well known that χ is a character if and only if $0\neq\chi=\sum_{i=1}^{k} n_i\chi_i$, where n_i are elements of

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$\mathbb{N}^*$ for $1 \leqq i \leqq k$. If $\chi = \sum_{i=1}^{k} n_i \chi_i$ is a character then those χ_i with $n_i > 0$ are called the irreducible constituents of χ.

Let χ and ψ be characters of G. The fact that $\chi + \psi$ is a character is a triviality. We may define a new class function $\chi\psi$ on G by setting $(\chi\psi)(g) = \chi(g)\psi(g)$. It is true but somewhat less trivial that $\chi\psi$ is a character.

It is well known (see Theorem 4.1 of [5]) that if the $\mathbb{C}[G]$-modules V and W afford characters χ and ψ, respectively, then the tensor product $V \otimes W$ affords the character $\chi\psi$ and is independent of the choice of bases of V and W.

Thus as a consequence of Theorem B we have that for irreducible $\mathbb{C}[G]$-modules V and W of a finite non-abelian simple group

$$V \otimes W \ncong \underbrace{V \oplus \cdots \oplus V}_{m \text{ times}} \oplus \underbrace{W \oplus \cdots \oplus W}_{n \text{ times}}, \quad \text{where} \quad m, n \in \mathbb{N}^*.$$

We can state Theorem B as follows:

Theorem 1. *Let G be a finite non-abelian simple group, $\{n_1, n_2\} \subset \mathbb{N}$ and $\{\psi_1, \psi_2\} \subseteqq \mathrm{Irr}(G) - 1_G$. Then:*

(a) $\psi_1 \psi_2 \neq n_1 \psi_1$ *and* $\psi_1 \psi_2 \neq n_1 \bar{\psi}_1$,

(b) $\psi_1 \psi_2 \neq n_1 \psi_1 + n_2 \psi_2$,

(c) $\psi_1 \psi_2 \neq n_1 \bar{\psi}_1 + n_2 \psi_2$.

Proof of (a). By the First Orthogonality Relation there exists $g \in G - \{1\}$ such that $\psi_1(g) \neq 0$. The simplicity of G implies that $Z(\psi_2) = 1$. Therefore $|\psi_1(g)||\psi_2(g)| \neq |\psi_1(g)||\psi_2(1)|$ and the inequalities of (a) hold.

Proof of (b). Let G be a counterexample with $\psi_1 \psi_2 = n_1 \psi_1 + n_2 \psi_2$ then clearly $\psi_1 \neq \psi_2 \neq \bar{\psi}_1$. We will show:

b(i). $\psi_1 \bar{\psi}_2 = n_1 \psi_1 + n_2 \bar{\psi}_2$.

Proof. Since $n_1 = [\psi_1 \psi_2, \psi_1] = [\psi_1, \psi_1 \bar{\psi}_2]$ and $n_2 = [\psi_1 \psi_2, \psi_2] = [\psi_1 \bar{\psi}_2, \bar{\psi}_2]$ then: $\psi_1 \bar{\psi}_2 = n_1 \psi_1 + n_2 \bar{\psi}_2 + \alpha$, with $[\alpha, \psi_i] = 0$ for $i \in \{1, 2\}$. Since $n_1 \psi_1(1) + n_2 \bar{\psi}_2(1) = n_1 \psi_1(1) + n_2 \psi_2(1) = \psi_1(1) \psi_2(1) = \psi_1(1) \bar{\psi}_2(1) = n_1 \psi_1(1) + n_2 \bar{\psi}_2(1) + \alpha(1)$ then $\alpha(1) = 0$. So $\alpha = 0$.

b(ii). $\psi_i = \bar{\psi}_i$ *for* $i \in \{1, 2\}$.

Proof. By b(i) we get that:

$$\bar{\psi}_2(\psi_1 \psi_2) = \bar{\psi}_2(n_1 \psi_1 + n_2 \psi_2) = n_1(n_1 \psi_1 + n_2 \bar{\psi}_2) + n_2 \psi_2 \bar{\psi}_2$$

and

$$(\bar{\psi}_2 \psi_1) \psi_2 = \psi_2(n_1 \psi_1 + n_2 \bar{\psi}_2) = n_1(n_1 \psi_1 + n_2 \psi_2) + n_2 \psi_2 \bar{\psi}_2.$$

Since $\bar{\psi}_2(\psi_1 \psi_2) = (\bar{\psi}_2 \psi_1) \psi_2$ then $\psi_2 = \bar{\psi}_2$ and similarly $\psi_1 = \bar{\psi}_1$.

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b(iii). $\psi_1^2 = 1_G + n_1\psi_2 + s_1\psi_1 + \alpha_1$, $\psi_2^2 = 1_G + n_2\psi_1 + s_2\psi_2 + \alpha_2$, *where* $s_i \in \mathbb{N}^*$, $\alpha_i \neq 0$ *and* $(\alpha_i, \chi) = 0$ *for* $\chi \in \{1_G, \psi_1, \psi_2\}$, $i \in \{1, 2\}$.

Proof. By b(ii) $1 = [\psi_i, \psi_i] = [\psi_i^2, 1_G]$ for $i \in \{1, 2\}$ and $n_1 = [\psi_1\psi_2, \psi_1] = [\psi_1^2, \psi_2]$ then: $\psi_1^2 = 1_G + n_1\psi_2 + s_1\psi_1 + \alpha_1$. Since $\psi_1(1) > 1$ then by Burnside's theorem ([**5**] (3.15)) there exists $g \in G$ such that $\psi_1(g) = 0$. Hence $0 = \psi_1(g)\psi_2(g) = n_1\psi_1(g) + n_2\psi_2(g) = n_2\psi_2(g)$. So $\psi_2(g) = 0$. It follows that $0 = \psi_1(g)^2 = 1_G + n_1\psi_2(g) + s_1\psi_1(g) + \alpha_1(g) = 1 + \alpha_1(g)$ and then $\alpha_1 \neq 0$. Similarly $\psi_2^2 = 1_G + n_2\psi_1 + s_2\psi_2 + \alpha_2$ with $\alpha_2 \neq 0$.

b(iv). (1) $n_1\alpha_1 = n_1\alpha_2 + \alpha_1\psi_2 - [\alpha_1, \alpha_2]\psi_2$ *and*

(2) $n_2\alpha_2 = n_2\alpha_1 + \alpha_2\psi_1 - [\alpha_1, \alpha_2]\psi_1$.

Proof. By b(iii) we get that:

$$\psi_1^2\psi_2 = (1_G + n_1\psi_2 + s_1\psi_1 + \alpha_1)\psi_2 = \psi_2 + n_1(1_G + n_2\psi_1 + s_2\psi_2 + \alpha_2) + s_1(n_1\psi_1 + n_2\psi_2) + \alpha_1\psi_2$$

$$= n_1 1_G + (n_1n_2 + s_1n_1)\psi_1 + (1 + n_1s_2 + s_1n_2)\psi_2 + n_1\alpha_2 + \alpha_1\psi_2.$$

$$\psi_1(\psi_1\psi_2) = \psi_1(n_1\psi_1 + n_2\psi_2) = n_1(1_G + n_1\psi_2 + s_1\psi_1 + \alpha_1) + n_2(n_1\psi_1 + n_2\psi_2)$$

$$= n_1 1_G + (n_1s_1 + n_1n_2)\psi_1 + (n_1^2 + n_2^2)\psi_2 + n_1\alpha_1.$$

Since $\psi_1^2\psi_2 = \psi_1(\psi_1\psi_2)$ then: $n_1\alpha_1 = n_1\alpha_2 + \alpha_1\psi_2 - [\alpha_1\psi_2, \psi_2]\psi_2$. Since $[\alpha_1\psi_2, \psi_2] = [\alpha_1, \psi_2^2] = [\alpha_1, \alpha_2]$ then (1) holds and similarly also (2).

b(v). *Final contradiction.*

Proof. Let us multiply b(iv)(1) by n_2 and b(iv)(2) by n_1. By adding these equations we get that

$$n_1n_2(\alpha_1 + \alpha_2) = n_1n_2(\alpha_1 + \alpha_2) + n_2\alpha_1\psi_2 + n_1\alpha_2\psi_1 - [\alpha_1, \alpha_2](n_1\psi_1 + n_2\psi_2).$$

Hence $n_1\alpha_2\psi_1 + n_2\alpha_1\psi_2 = [\alpha_1, \alpha_2](n_1\psi_1 + n_2\psi_2)$. It follows that $[\alpha_2\psi_1, \beta] = 0$ for every $\beta \in \mathrm{Irr}(G) - \{\psi_1, \psi_2\}$. Moreover, since $[\alpha_2\psi_1, \psi_2] = [\alpha_2, \psi_1\psi_2] = [\alpha_2, n_1\psi_1 + n_2\psi_2] = 0$ then $\alpha_2\psi_1 = l\psi_1$ with $l \in \mathbb{N}$. Let χ be an irreducible constituent of α_2 then $\chi\psi_1 = k\psi_1$ with $k \in \mathbb{N}$ which contradicts (a).

Proof of (c). Let G be a counterexample with $\psi_1\psi_2 = n_1\bar{\psi}_1 + n_2\psi_2$. By (b) $\psi_1 \neq \bar{\psi}_1$. We consider the following two cases:

c(1) $\psi_2 = \bar{\psi}_2$ and

c(2) $\psi_2 \neq \bar{\psi}_2$.

Case c(1)(i). $\bar{\psi}_1^2 + n_2\psi_1 = \psi_1^2 + n_2\bar{\psi}_1$.

Proof. Since $\bar\psi_1\psi_2=n_1\psi_1+n_2\psi_2$ then

$$\bar\psi_1(\psi_1\psi_2)=\bar\psi_1(n_1\bar\psi_1+n_2\psi_2)=n_1\bar\psi_1\bar\psi_1+n_2(n_1\psi_1+n_2\psi_2)$$

and

$$(\bar\psi_1\psi_2)\psi_1=\psi_1(n_1\psi_1+n_2\psi_2)=n_1\psi_1^2+n_2(n_1\bar\psi_1+n_2\psi_2)$$

then $\bar\psi_1^2+n_2\psi_1=\psi_1^2+n_2\bar\psi_1$.

Case c(1)(ii). $\psi_1^2=n_1\psi_2+n_2\psi_1+\alpha_1$, $\psi_2^2=1_G+n_2(\psi_1+\bar\psi_1)+\alpha_2$ *with* $\alpha_i=\bar\alpha_i$, $[\alpha_i,\chi]=0$ *for* $\chi\in\{1_G,\psi_1,\bar\psi_1,\psi_2\}$, $i\in\{1,2\}$, *and* $n_1\psi_1\bar\psi_1=n_1\psi_2^2+\alpha_1\psi_2$.

Proof. Since $n_1=[\psi_1\psi_2,\bar\psi_1]=[\psi_1^2,\psi_2]$ then $\psi_1^2=n_1\psi_2+l_1\psi_1+l_2\bar\psi_1+\alpha_1$ with $\{l_1,l_2\}\subset\mathbb{N}^*$ and $[\alpha_1,\beta]=0$ for $\beta\in\{1_G,\psi_1,\bar\psi_1,\psi_2\}$. By c(1)(i) $n_1\psi_2+(n_2+l_2)\psi_1+l_1\bar\psi_1+\bar\alpha_1=\bar\psi_1^2+n_2\psi_1=\psi_1^2+n_2\bar\psi_1=n_1\psi_2+l_1\psi_1+(n_2+l_2)\bar\psi_1+\alpha_1$. So $\alpha_1=\bar\alpha_1$ and $l_1=l_2+n_2$.

Now

$$\psi_1^2\psi_2=\psi_2[n_1\psi_2+(l_2+n_2)\psi_1+l_2\bar\psi_1+\alpha_1]$$

$$=n_1\psi_2^2+(l_2+n_2)(n_1\bar\psi_1+n_2\psi_2)+l_2(n_1\psi_1+n_2\psi_2)+\alpha_1\psi_2$$

and

$$\psi_1(\psi_1\psi_2)=\psi_1(n_1\bar\psi_1+n_2\psi_2)=n_1\psi_1\bar\psi_1+n_2(n_1\bar\psi_1+n_2\psi_2)$$

so

$$n_1\psi_1\bar\psi_1=n_1\psi_2^2+l_2n_1(\psi_1+\bar\psi_1)+2l_2n_2\psi_2+\alpha_1\psi_2.$$

Since $[\psi_1\bar\psi_1,\psi_2]=[\psi_1\psi_2,\psi_1]=0$ then $n_2l_2=0$. So $l_2=0$. Moreover $[\psi_1\bar\psi_1,\psi_2]=0$ implies that $[\psi_2^2,\psi_2]=0=[\alpha_1\psi_2,\psi_2]$. Since $n_2=[\psi_1\psi_2,\psi_2]=[\psi_2^2,\psi_1]$ then $\psi_2^2=1_G+n_2(\psi_1+\bar\psi_1)+\alpha_2$.

Case c(1)(iii). $\alpha_1=0$.

Proof. By c(1)(ii)

$$(n_1\bar\psi_1\psi_1)\psi_2=\psi_2(n_1\psi_2^2+\alpha_1\psi_2)=n_1\psi_2[1_G+n_2(\psi_1+\bar\psi_1)+\alpha_2]+\alpha_1\psi_2^2$$

$$=n_1\psi_2+n_1n_2(n_1\psi_1+n_1\bar\psi_1+2n_2\psi_2)+n_1\alpha_2\psi_2+\alpha_1\psi_2^2$$

and

$$n_1\bar\psi_1(\psi_1\psi_2)=n_1\bar\psi_1(n_1\bar\psi_1+n_2\psi_2)=n_1^2(n_1\psi_2+n_2\bar\psi_1+\alpha_1)+n_1n_2(n_1\psi_1+n_2\psi_2).$$

We get that $n_1^2n_2\psi_1=n_1^2n_2\psi_1+[n_1\alpha_2\psi_2+\alpha_1\psi_2^2,\psi_1]\psi_1$ and then $[\alpha_1\psi_2^2,\psi_1]=0$.

But $\alpha_1\psi_2^2=\alpha_1(1_G+n_2\psi_1+n_2\bar\psi_1+\alpha_2)$; so in particular $0=[\alpha_1\bar\psi_1,\psi_1]=[\alpha_1,\psi_1^2]=[\alpha_1,\alpha_1]$. Thus $\alpha_1=0$.

Case c(1)(iv). $\psi_2 \neq \bar{\psi}_2$.

Proof. By c(1)(ii) and c(1)(iii) $\psi_1\bar{\psi}_1 = \psi_2^2$. So

$$\psi_2^2\psi_1 = (1_G + n_2\psi_1 + n_2\bar{\psi}_1 + \alpha_2)\psi_1 = \psi_1 + n_2\psi_1^2 + n_2\psi_2^2 + \alpha_2\psi_1$$

and

$$\psi_2(\psi_2\psi_1) = \psi_2(n_1\bar{\psi}_1 + n_2\psi_2) = n_1(n_1\psi_1 + n_2\psi_2) + n_2\psi_2^2.$$

Hence

$$\psi_1 + n_2(n_1\psi_2 + n_2\psi_1) + \alpha_2\psi_1 = n_1^2\psi_1 + n_1n_2\psi_2.$$

Then $n_1^2\psi_1 = (1+n_2^2)\psi_1 + \alpha_2\psi_1$ implies that $\alpha_2\psi_1 = l\psi_1$ which contradicts (a). Then $\alpha_2 = 0$ and $n_1^2 = 1 + n_2^2$. So ψ_2 is not a real character.

Case c(2). In this case $0 = [\psi_1\psi_2, \psi_1] = [\psi_1\bar{\psi}_1, \bar{\psi}_2] = [\psi_1, \psi_1\bar{\psi}_2]$, $0 = [\psi_1\psi_2, \bar{\psi}_2] = [\psi_2^2, \bar{\psi}_1]$, $n_1 = [\psi_1\psi_2, \bar{\psi}_1] = [\psi_1^2, \bar{\psi}_2]$ and $n_2 = [\psi_1\psi_2, \psi_2] = [\psi_1, \psi_2\bar{\psi}_2] = [\psi_1\bar{\psi}_2, \bar{\psi}_2]$.

Denote by $l_1 = [\psi_1\bar{\psi}_2, \bar{\psi}_1] = [\psi_1^2, \psi_2]$, by $l_2 = [\psi_1\bar{\psi}_2, \psi_2] = [\psi_1, \psi_2^2]$, by $j_1 = [\psi_1^2, \psi_1] = [\psi_1\bar{\psi}_1, \psi_1]$, by $j_2 = [\psi_2^2, \psi_2] = [\psi_2\bar{\psi}_2, \psi_2]$, by $d_1 = [\psi_1^2, \bar{\psi}_1]$ and by $d_2 = [\psi_2^2, \bar{\psi}_2]$. So we have the following table:

	1_G	ψ_1	$\bar{\psi}_1$	ψ_2	$\bar{\psi}_2$	
$\psi_1\psi_2$	0	0	n_1	n_2	0	
$\psi_1\bar{\psi}_2$	0	0	l_1	l_2	n_2	α_{12}
ψ_1^2	0	j_1	d_1	l_1	n_1	α_{11}
ψ_2^2	0	l_2	0	j_2	d_2	α_{22}
$\psi_1\bar{\psi}_1$	1	j_1	j_1	0	0	β_1
$\psi_2\bar{\psi}_2$	1	n_2	n_2	j_2	j_2	β_2

where β_i is real and $[\gamma, \delta] = 0$ for $\gamma \in \{\alpha_{ij}, \beta_i / 1 \leqq i \leqq 2, 1 \leqq j \leqq 2\}$ and $\delta \in \{1_G, \psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2\}$.

Case c(2)(i). $n_1\bar{\psi}_1\bar{\psi}_2 = l_2\psi_2^2 + \alpha_{12}\psi_2$, $l_1 = 0$ *and* $[\alpha_{12}, \beta_2] = 0$.

Proof. Since

$$(\psi_1\psi_2)\bar{\psi}_2 = \bar{\psi}_2(n_1\bar{\psi}_1 + n_2\psi_2) = n_1\bar{\psi}_1\bar{\psi}_2 + n_2\psi_2\bar{\psi}_2 \qquad (*)$$

and

$$(\psi_1\bar{\psi}_2)\psi_2 = \psi_2(l_1\bar{\psi}_1 + l_2\psi_2 + n_2\bar{\psi}_2 + \alpha_{12}) = l_1\bar{\psi}_1\psi_2 + l_2\psi_2^2 + n_2\bar{\psi}_2\psi_2 + \alpha_{12}\psi_2$$

then $n_1\bar{\psi}_1\bar{\psi}_2 = l_1\bar{\psi}_1\psi_2 + l_2\psi_2^2 + \alpha_{12}\psi_2$. Since $[\bar{\psi}_1\bar{\psi}_2, \psi_2] = 0$ then $0 = l_1[\bar{\psi}_1\psi_2, \psi_2] = l_1n_2$, so $l_1 = 0$ and $0 = [\alpha_{12}\psi_2, \psi_2] = [\alpha_{12}, \beta_2]$.

Case c(2)(ii). $\alpha_{12}\psi_2 = [\alpha_{12}, \alpha_{12}]\psi_1 + [\alpha_{12}, \bar{\alpha}_{22}]\bar{\psi}_2$.

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Proof. Since $[\alpha_{12}\psi_2, \psi_1] = [\alpha_{12}, \alpha_{12}]$, $[\alpha_{12}\psi_2, \bar{\psi}_1] = [\alpha_{12}, \bar{\psi}_1\bar{\psi}_2] = 0$, $[\alpha_{12}\psi_2, \bar{\psi}_2] = [\alpha_{12}, \bar{\alpha}_{22}]$ and $0 = [\bar{\psi}_1\bar{\psi}_2, \chi]$ for $\chi \in \mathrm{Irr}(G) - \{\psi_1, \bar{\psi}_2\}$ then by c(2)(i) $0 = l_2\alpha_{22} + \alpha_{12}\psi_2 - [\alpha_{12}, \alpha_{12}]\psi_1 - [\alpha_{12}, \bar{\alpha}_{22}]\bar{\psi}_2$.

Case c(2)(iii). $[\alpha_{12}, \alpha_{11}] = 0$.

Proof. By c(2)(ii) $0 = [\alpha_{12}\psi_2, \psi_1\psi_2] = [\alpha_{12}\bar{\psi}_1, \psi_2\bar{\psi}_2]$. Hence, in particular $0 = [\alpha_{12}\bar{\psi}_1, \psi_1] = [\alpha_{12}, \alpha_{11}]$.

Case c(2)(iv). $j_2 = 0$, $[\alpha_{11}, \bar{\alpha}_{22}] = 0$ *and* $\psi_2(j_1\psi_1 + d_1\bar{\psi}_1 + n_1\bar{\psi}_2 + \alpha_{11}) = \psi_1(n_1\bar{\psi}_1 + n_2\psi_2)$.

Proof. By c(2)(i) $\psi_1^2\psi_2 = \psi_2(j_1\psi_1 + d_1\bar{\psi}_1 + n_1\bar{\psi}_2 + \alpha_{11})$ and $(\psi_1\psi_2)\psi_1 = \psi_1(n_1\bar{\psi}_1 + n_2\psi_2)$. Since $[\psi_1\bar{\psi}_1, \bar{\psi}_2] = 0 = [\psi_1\psi_2, \bar{\psi}_2]$ then we obtain that $0 = [\psi_2\bar{\psi}_2, \psi_2] = j_2$ and $0 = [\alpha_{11}\psi_2, \bar{\psi}_2] = [\alpha_{11}, \bar{\alpha}_{22}]$.

Case c(2)(v). $j_1 = n_2$.

Proof. Since $[\psi_1\psi_2, \psi_1] = 0 = [\psi_1\bar{\psi}_2, \psi_1]$, $[\bar{\psi}_1\psi_2, \psi_1] = l_1 = 0$ then c(2)(iv) yields that $n_1 j_1 = n_1[\bar{\psi}_1\psi_1, \psi_1] = n_1[\bar{\psi}_2\psi_2, \psi_1] + [\alpha_{11}\psi_2, \psi_1] = n_1 n_2 + [\alpha_{11}, \alpha_{12}] = n_1 n_2$ by c(2)(iii). So $j_1 = n_2$.

Case c(2)(vi). $d_1 = 0$ *and* $[\alpha_{11}, \beta_2] = 0$.

Proof. Since $[\bar{\psi}_1\psi_1, \psi_2] = 0$, $j_1 = n_2$ then c(2)(iv) yields that $n_2[\psi_1\psi_2, \psi_2] = n_2[\psi_1\psi_2, \psi_2] + d_1[\bar{\psi}_1\psi_2, \psi_2] + [\alpha_{11}\psi_2, \psi_2]$. Hence $0 = d_1[\bar{\psi}_1\psi_2, \psi_2] = d_1 n_2$ so $d_1 = 0$ and $0 = [\alpha_{11}\psi_2, \psi_2] = [\alpha_{11}, \beta_2]$.

Case c(2)(vii). $n_1\beta_1 = n_1\beta_2 + \alpha_{11}\psi_2$.

Proof. By c(2)(iii), c(2)(iv) and c(2)(vi) $[\alpha_{11}\psi_2, \psi_1 + \bar{\psi}_1 + \psi_2 + \bar{\psi}_2] = 0$. It follows from c(2)(iv) that $n_1\beta_1 = n_1\beta_2 + \alpha_{11}\psi_2$.

Case c(2)(viii). $\alpha_{11} = \beta_2 = 0$.

Proof. By c(2)(iv)

$$(\psi_2\bar{\psi}_2)\psi_1 = (1_G + n_2\psi_1 + n_2\bar{\psi}_1 + \beta_2)\psi_1. \qquad (**)$$

Since $[\beta_2\psi_1, \bar{\psi}_2] = [\beta_2, \psi_1\psi_2] = 0$, $[\beta_2\psi_1, \bar{\psi}_1] = [\beta_2, \alpha_{11}] = 0$ by c(2)(vi), $[\beta_2\psi_1, \psi_2] = [\beta_2, \alpha_{12}] = 0$ by c(2)(i) then (*) and (**) yield that $n_2\beta_2 = n_2\alpha_{11} + n_2\beta_1 + \beta_2\psi_1 - [\beta_2\psi_1, \psi_1]\psi_1$. Multiplying this equation by n_1 and using c(2)(vii) we get:

$$n_1 n_2\beta_2 = n_1 n_2\alpha_{11} + n_2(n_1\beta_2 + \alpha_{11}\psi_2) + n_1\beta_2\psi_1 - n_1[\beta_2\psi_1, \psi_1]\psi_1.$$

So $\alpha_{11} = 0$ and $\beta_2\psi_1 = [\beta_2\psi_1, \psi_1]\psi_1$ so by (a) (as in b(v)) $\beta_2 = 0$.

Case c(2)(ix). *Final contradiction.*

Proof. By c(2)(vii) and c(2)(viii) $\beta_1=0$. It follows by c(2)(iv) and c(2)(v) that $\psi_1\bar{\psi}_1=\psi_2\bar{\psi}_2=1_G+n_2(\psi_1+\bar{\psi}_1)$. So

$$(\psi_1\bar{\psi}_1)(\psi_2\bar{\psi}_2)=(1_G+n_2\psi_1+n_2\bar{\psi}_1)^2=1_G+2n_2(\psi_1+\bar{\psi}_1)+n_2^2(\psi_1^2+\bar{\psi}_1^2+2\psi_1\bar{\psi}_1)$$

and

$$(\psi_1\psi_2)(\bar{\psi}_1\bar{\psi}_2)=(n_1\bar{\psi}_1+n_2\psi_2)(n_1\psi_1+n_2\bar{\psi}_2)=n_1^2\psi_1\bar{\psi}_1+n_1n_2(\psi_1\psi_2+\bar{\psi}_1\bar{\psi}_2)+n_2^2\psi_2\bar{\psi}_2.$$

So

$$1+2n_2^2=[\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2,1_G]=n_1^2+n_2^2$$

and then $1+n_2^2=n_1^2$, a contradiction.

Proof of Theorem A. The proof of Theorem A is similar to the proof of Theorem B with a few changes.

Let $g_1,\dots,g_k$ be representatives of the conjugacy classes of a finite group G. Let $K=\sum_{i=1}^k n_iCl(g_i)$ with $n_i\in N^*$ for $1\leqq i\leqq k$. Define: $(K,Cl(g_i))=n_i$. Clearly, $(Cl(g_i),Cl(g_j))=\delta_{ij}$. So $n_i=(K,Cl(g_i))=\sum_{j=1}^k n_j(Cl(g_j),\ Cl(g_i))$. Let $L=\sum_{i=1}^k m_iCl(g_i)$ with $m_i\in\mathbb{N}^*$, we extend the above definition by:

$$(K,L)=\sum_{i=1}^k m_i(K,Cl(g_i))=\sum_{i=1}^k n_im_i=(L,K).$$

Let $j_i\in\{1,\dots,k\}$ for $1\leqq i\leqq s$ and let

$$\prod_{i=j_1}^{j_s} Cl(g_i)=K=\sum_{i=1}^k n_iCl(g_i).$$

Clearly, $n_i\in\mathbb{N}^*$ and it is known that

$$n_l=|G|^{-1}\prod_{i=j_1}^{j_s}|Cl(g_i)|\sum_{\chi\in\mathrm{Irr}(G)}\bar{\chi}(g_l)\chi(1)^{1-s}\prod_{i=j_1}^{j_s}\chi(g_i).$$

In particular, if D_1,D_2,D_3 are conjugacy classes then:

(i) $(D_1D_2,D_3)=(D_1^{-1}D_2^{-1},D_3^{-1})$.

It is easy to compute that:

(ii) $(D_1D_2,D_3)=|D_2||D_3|^{-1}(D_1D_3^{-1},D_2^{-1})$.

For $D_1=D_3$ we get that:

(iii) $(D_1D_2,D_1)=|D_2||D_1|^{-1}(D_1D_1^{-1},D_2^{-1})=(D_2D_1^{-1},D_1^{-1})=(D_2^{-1}D_1,D_1)$.

It is appropriate to introduce here the following:

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Definition. The covering number, $cn(G)$, of a group G is the smallest positive integer n, such that $C^n = G$ for all non-identity conjugacy classes, C, of G. If no such integer exists we say that the covering number is infinite. The notion of a covering number was mentioned in [3] where it is shown:

Lemma. *A finite group has a finite covering number if and only if it is a nonabelian simple group.*

(In [4] we proved the analogous lemma for character covering numbers.)

Now we can state Theorem A as follows:

Theorem 2. *Let D_1, D_2 be conjugacy classes of a finite nonabelian simple group G with $D_1 \neq Cl(1) \neq D_2$ and $\{n_1, n_2\} \subset \mathbb{N}$. Then*

(a) $D_1 D_2 \neq n_1 D_1$ *and* $D_1 D_2 \neq n_1 D_1^{-1}$.

(b) $D_1 D_2 \neq n_1 D_1 + n_2 D_2$.

(c) $D_1 D_2 \neq n_1 D_1^{-1} + n_2 D_2$.

Proof of (a). Let $D_1 D_2 = n_1 D_1$ be a counterexample then $D_1 D_2^2 = n_1 D_1 D_2 = n_1^2 D_1$. By induction $D_1 D_2^s = n_1^s D_1$ for every $s \in \mathbb{N}$ which contradicts the lemma.

Similarly, if $D_1 D_2 = n_1 D_1^{-1}$ then $D_1 D_2 D_2^{-1} = n_1 D_1^{-1} D_2^{-1} = n_1^2 D_1$. By induction $D_1 (D_2^{-1})^s D_2^s = n_1^{2s} D_1$, the same contradiction.

Proof of (b). Let $D_1 D_2 = n_1 D_1 + n_2 D_2$ be a counterexample. We will show:

b(i). $D_1 D_2^{-1} = n_1 D_1 + n_2 D_2^{-1}$.

Proof. By (iii) $n_1 = (D_1 D_2, D_1) = (D_1 D_2^{-1}, D_1)$ and $n_2 = (D_1 D_2, D_2) = (D_1 D_2^{-1}, D_2^{-1})$. So $D_1 D_2^{-1} = n_1 D_1 + n_2 D_2^{-1} + T$ with $(T, L) = 0$ for $L \in \{D_1, D_2^{-1}\}$. Since

$$n_1|D_1| + n_2|D_2^{-1}| = n_1|D_1| + n_2|D_2| = |D_1||D_2| = |D_1||D_2^{-1}| = n_1|D_1| + n_2|D_2^{-1}| + |T|$$

then $T = 0$.

b(ii). $D_i = D_i^{-1}$ *for* $1 \leqq i \leqq 2$.

Proof. By b(i)

$$(n_1 D_1 + n_2 D_2^{-1}) D_2 = (D_1 D_2^{-1}) D_2 = (D_1 D_2) D_2^{-1} = (n_1 D_1 + n_2 D_2) D_2^{-1}.$$

So $D_1 D_2^{-1} = D_1 D_2$ or equivalently $n_1 D_1 + n_2 D_2^{-1} = n_1 D_1 + n_2 D_2$ then $D_2 = D_2^{-1}$ and similarly $D_1 = D_1^{-1}$.

b(iii). $D_1^2 = |D_1| Cl(1) + n_1 |D_1| |D_2|^{-1} D_2 + s_1 D_1 + M_1$

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and

$$D_2^2 = |D_2|Cl(1) + n_2|D_2||D_1|^{-1}D_1 + s_2D_2 + M_2$$

where $s_i \in N^*$, $M_i \neq 0$ *and* $(M_i, C) = 0$ *for* $C \in \{Cl(1), D_j\}$, $i, j \in \{1, 2\}$.

Proof. Since

$$(D_1^2, Cl(1)) = |D_1|(D_1Cl(1), D_1) = |D_1|$$

and by (ii)

$$(D_1^2, D_2) = |D_1||D_2|^{-1}(D_1D_2, D_1) = |D_1||D_2|^{-1}n_1$$

then $D_1^2 = |D_1|Cl(1) + n_1|D_1||D_2|^{-1}D_2 + s_1D_1 + M_1$. If $M_1 = 0$, by the lemma $G = 1 \cup D_1 \cup D_2$ which contradicts the assumption that G is a nonabelian simple group. So $M_1 \neq 0$. Similarly for D_2^2.

b(iv).

$$n_1M_1 = n_1|D_1||D_2|^{-1}M_2 + M_1D_2 - (M_1D_2, D_2)D_2$$

and

$$n_2M_2 = n_2|D_2||D_1|^{-1}M_1 + M_2D_1 - (M_2D_1, D_1)D_1.$$

Proof. By b(iii)

$$D_1(D_1D_2) = D_1(n_1D_1 + n_2D_2) = n_1(|D_1|Cl(1) + n_1|D_1||D_2|^{-1}D_2 + s_1D_1 + M_1)$$

$$+ n_2(n_1D_1 + n_2D_2)$$

and

$$D_1^2D_2 = (|D_1|Cl(1) + n_1|D_1||D_2|^{-1}D_2 + s_1D_1 + M_1)D_2$$

$$= |D_1|D_2 + n_1|D_1||D_2|^{-1}(|D_2|Cl(1) + n_2|D_2||D_1|^{-1}D_1 + s_2D_2 + M_2)$$

$$+ s_1(n_1D_1 + n_2D_2) + M_1D_2.$$

Since $D_1(D_1D_2) = D_1^2D_2$, $0 = (M_1D_2, Cl(1))$ and $(M_1D_2, D_1) = |M_1||D_1|^{-1}(M_1, D_1D_2) = 0$ then $n_1M_1 = n_1|D_1||D_2|^{-1}M_2 + M_1D_2 - (M_1D_2, D_2)D_2$. Similarly $n_2M_2 = n_2|D_2||D_1|^{-1}M_1 + M_2D_1 - (M_2D_1, D_1)D_1$.

b(v). *Final contradiction.*

Proof. By b(iv)

$$n_1n_2M_2=n_1n_2|D_2||D_1|^{-1}M_1+n_1(M_2D_1-(M_2D_1,D_1)D_1)$$
$$=n_2|D_2||D_1|^{-1}(n_1|D_1||D_2|^{-1}M_2+M_1D_2-(M_1D_2,D_2)D_2)$$
$$+n_1M_2D_1-n_1(M_2D_1,D_1)D_1.$$

It follows that $n_2|D_2||D_1|^{-1}M_1D_2+n_1M_2D_1=l_1D_1+l_2D_2$ for $\{l_1,l_2\}\subset\mathbb{N}^*$. In particular, $M_1D_2=m_1D_1+m_2D_2$ for $\{m_1,m_2\}\subset\mathbb{N}^*$. Since $(M_1D_2,D_1)=0$ the $M_1D_2=m_2D_2$. Let $M_1=\sum_{i=1}^k d_iCl(g_i)$, choose $Cl(g_j)$ for $d_j>0$ then $Cl(g_j)D_2=mD_2$ which contradicts (a).

Proof of (c). Let $D_1D_2=n_1D_1^{-1}+n_2D_2$ be a counterexample. By (b) $D_1\neq D_1^{-1}$. We consider the following two cases:

case c(1) $D_2=D_2^{-1}$ and case c(2) $D_2\neq D_2^{-1}$.

Case c(1)

c(1)(i). $(D_1^{-1})^2+n_2D_1=D_1^2+n_2D_1^{-1}$.

Proof. Since $(D_1D_2,D_i)=(D_1^{-1}D_2,D_i^{-1})$ then

$$D_1^{-1}(D_1D_2)=D_1^{-1}(n_1D_1^{-1}+n_2D_2)=n_1(D_1^{-1})^2+n_2(n_1D_1+n_2D_2)$$

and

$$D_1(D_1^{-1}D_2)=D_1(n_1D_1+n_2D_2)=n_1D_1^2+n_2(n_1D_1^{-1}+n_2D_2).$$

It follows that $(D_1^{-1})^2+n_2D_1=D_1^2+n_2D_1^{-1}$.

c(1)(ii). $D_1^2=n_1|D_1||D_2|^{-1}D_2+n_2D_1+M$ *with* $M=M^{-1}$, $0=(M,C)$ *for* $C\in\{Cl(1),D_1,D_1^{-1},D_2\}$, $n_1D_1D_1^{-1}=n_1|D_1||D_2|^{-1}D_2^2+MD_2$ *and* $(D_2^2,D_2)=0$.

Proof. Since $n_1=(D_1D_2,D_1^{-1})=|D_2||D_1|^{-1}(D_1^2,D_2)$ then $D_1^2=n_1|D_1||D_2|^{-1}D_2+l_1D_1+l_2D_1^{-1}+M$ with $\{l_1,l_2\}\subset\mathbb{N}^*$. By c(1)(i)

$$n_1|D_1||D_2|^{-1}D_2+(l_2+n_2)D_1+l_1D_1^{-1}+M^{-1}=(D_1^{-1})^2+n_2D_1=D_1^2+n_2D_1^{-1}$$
$$=n_1|D_1||D_2|^{-1}D_2+l_1D_1+(l_2+n_2)D_1^{-1}+M.$$

So $M=M^{-1}$ and $l_1=l_2+n_2$.

Now

$$(D_1^2)D_2=(n_1|D_1||D_2|^{-1}D_2+(l_2+n_2)D_1+l_2D_1^{-1}+M)D_2$$
$$=n_1|D_1||D_2|^{-1}D_2^2+(l_2+n_2)(n_1D_1^{-1}+n_2D_2)+l_2(n_1D_1+n_2D_2)+MD_2$$

and

$$D_1(D_1D_2)=D_1(n_1D_1^{-1}+n_2D_2)=n_1D_1D_1^{-1}+n_2(n_1D_1^{-1}+n_2D_2).$$

Since $D_1^2D_2=D_1(D_1D_2)$ then

$$n_1D_1D_1^{-1}=n_1|D_1||D_2|^{-1}D_2^2+l_2n_1(D_1+D_1^{-1})+2l_2n_2D_2+MD_2.$$

Since $0=(D_1D_1^{-1},D_2)$ then $(D_2^2,D_2)=0$ and $l_2=0$ thus $l_1=n_2$.

c(1)(iii). *$D_2^2=|D_2|Cl(1)+n_2|D_2||D_1|^{-1}(D_1+D_1^{-1})+L$ with $L=L^{-1}$ and $0=(L,C)$ for $C\in\{Cl(1),D_1,D_1^{-1},D_2\}$.*

Proof. Since $(D_2^2,D_1^{-1})=|D_2||D_1|^{-1}(D_2D_1,D_2)=n_2|D_2||D_1|^{-1}$ and by c(1)(ii) $0=(D_2^2,D_2)$ then $D_2^2=|D_2|Cl(1)+n_2|D_2||D_1|^{-1}(D_1+D_1^{-1})+L$.

c(1)(iv). *$M=0$, $|D_1|=|D_2|$ and $D_1D_1^{-1}=D_2^2$.*

Proof. By c(1)(ii) and c(1)(iii)

$$\begin{aligned}(n_1D_1D_1^{-1})D_2&=(n_1|D_1||D_2|^{-1}D_2^2+MD_2)D_2\\&=n_1|D_1||D_2|^{-1}D_2(|D_2|Cl(1)+n_2|D_2||D_1|^{-1}(D_1+D_1^{-1})+L)+MD_2^2\\&=n_1|D_1|D_2+n_1n_2(n_1D_1+n_1D_1^{-1}+2n_2D_2)+n_1|D_1||D_2|^{-1}D_2L+MD_2^2\end{aligned}$$

and

$$\begin{aligned}n_1D_1^{-1}(D_1D_2)=n_1D_1^{-1}(n_1D_1^{-1}+n_2D_2)&=n_1^2(n_1|D_1||D_2|^{-1}D_2+n_2D_1^{-1}+M)\\&+n_1n_2(n_1D_1+n_2D_2).\end{aligned}$$

So

$$n_1^2n_2=(n_1D_1^{-1}D_1D_2,D_1)=n_1^2n_2+(n_1|D_1||D_2|^{-1}D_2L+MD_2^2,D_1).$$

Hence $(D_2L,D_1)=0=(MD_2^2,D_1)$.

Since $MD_2^2=M[|D_2|Cl(1)+n_2|D_2||D_1|^{-1}(D_1+D_1^{-1})+L]$ then $(MD_2^2,D_1)=0$ implies, in particular, that $0=(MD_1^{-1},D_1)=|M||D_1|^{-1}(D_1^2,M)=|M||D_1|^{-1}(M,M)$. So $M=0$ and then by c(1)(ii) $D_1D_1^{-1}=|D_1||D_2|^{-1}D_2^2$. Hence $|D_1|^2=|D_1||D_1^{-1}|=|D_1||D_2|^{-1}|D_2|^2=|D_1||D_2|$. Thus $|D_1|=|D_2|$.

c(1)(iv). *$L=0$.*

Proof.

$$D_2^2D_1=(|D_2|Cl(1)+n_2(D_1+D_1^{-1})+L)D_1=|D_2|D_1+n_2D_1^2+n_2D_2^2+LD_1$$

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and

$$D_2(D_1D_2)=D_2(n_1D_1^{-1}+n_2D_2)=n_1(n_1D_1+n_2D_2)+n_2D_2^2.$$

So

$$|D_2|D_1+n_2(n_1D_2+n_2D_1)+LD_1=n_1^2D_1+n_1n_2D_2.$$

Thus $(|D_2|+n_2^2)D_1+LD_1=n_1^2D_1$. Hence $LD_1=kD_1$ and by (a) $L=0$.

c(1)(v). $D_2\neq D_2^{-1}$.

Proof. Since $L=0=M$ then $D_2^n=\alpha(n)Cl(1)+\beta(n)D_1+\delta(n)D_1^{-1}+\gamma(n)D_2$ where $\{\alpha(n),\beta(n),\delta(n),\gamma(n)\}\subset\mathbb{N}^*$ for every $n\in\mathbb{N}$. By the lemma we get that $G=1\cup D_1\cup D_1^{-1}\cup D_2$. Since $|D_1^{-1}|=|D_1|=|D_2|$ then $|G|=1+3|D_1|$ which contradicts a consequence of Lagrange's theorem that $|D_i|/|G|$.

Case c(2). In this case we have

$$0=(D_1D_2,D_1)=|D_2||D_1|^{-1}(D_1D_1^{-1},D_2^{-1})=|D_2||D_1|^{-1}(D_1D_1^{-1},D_2)=(D_1D_2^{-1},D_1),$$

$$0=(D_1D_2,D_2^{-1})=|D_1||D_2|^{-1}(D_2^2,D_1^{-1}),$$

$$n_1=(D_1D_2,D_1^{-1})=|D_2||D_1|^{-1}(D_1^2,D_2^{-1}),$$

$$n_2=(D_1D_2,D_2)=(D_1D_2^{-1},D_2^{-1})=|D_1||D_2|^{-1}(D_2D_2^{-1},D_1^{-1}).$$

We denote: $l_1=(D_1D_2^{-1},D_1^{-1})=|D_2||D_1|^{-1}(D_1^2,D_2)$, $l_2=(D_1D_2^{-1},D_2)=|D_1||D_2|^{-1}(D_2^2,D_1)$, $j_1=(D_1^2,D_1)=(D_1D_1^{-1},D_1)$, $j_2=(D_2^2,D_2)=(D_2D_2^{-1},D_2^{-1})$, $d_1=(D_1^2,D_1^{-1})$ and $d_2=(D_2^2,D_2^{-1})$. Therefore we have the following table:

	$Cl(1)$	D_1	D_1^{-1}	D_2	D_2^{-1}	
D_1D_2	0	0	n_1	n_2	0	
$D_1D_2^{-1}$	0	0	l_1	l_2	n_2	M_{12}
D_1^2	0	j_1	d_1	$l_1\lvert D_1\rvert\lvert D_2\rvert^{-1}$	$n_1\lvert D_1\rvert\lvert D_2\rvert^{-1}$	M_{11}
D_2^2	0	$l_2\lvert D_2\rvert\lvert D_1\rvert^{-1}$	0	j_2	d_2	M_{22}
$D_1D_1^{-1}$	$\lvert D_1\rvert$	j_1	j_1	0	0	N_1
$D_2D_2^{-1}$	$\lvert D_2\rvert$	$n_2\lvert D_2\rvert\lvert D_1\rvert^{-1}$	$n_2\lvert D_2\rvert\lvert D_1\rvert^{-1}$	j_2	j_2	N_2

where $N_i=N_i^{-1}$ and $(L,C)=0$ for $C\in\{Cl(1),D_k,D_k^{-1}\}$, $L\in\{M_{ij},N_i\}$ for every $k,i,j\in\{1,2\}$.

We will show:

c(2)(i). $j_2=0$ *and* $(M_{11},M_{22}^{-1})=0$.

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Proof. Since $(D_1D_2, D_1^{-1}D_2^{-1})=0$ then

$$0=(D_1^2,(D_2^{-1})^2)=d_1l_2|D_2||D_1|^{-1}+d_2l_1|D_1||D_2|^{-1}+j_2n_1|D_1||D_2|^{-1}+(M_{11},M_{22}^{-1})$$

then, in particular $j_2=0=(M_{11},M_{22}^{-1})$.

c(2)(ii). $l_1=0$ *and* $(M_{12},N_2)=0$.

Proof. By c(2)(i) $(D_1D_2,D_2^2)=0$ thus $0=(D_1D_2^{-1},D_2D_2^{-1})=l_1n_2|D_2||D_1|^{-1}+(M_{12},N_2)$. It follows that $l_1=0=(M_{12},N_2)$.

c(2)(iii). $l_2M_{22}=0$, $M_{12}D_2=(n_1^2-l_2^2|D_2||D_1|^{-1})D_1+(n_1n_2-l_2d_2)D_2^{-1}$ *and* $0=(M_{12},M_{11})$.

Proof. We compute

$$(D_1D_2)D_2^{-1}=(n_1D_1^{-1}+n_2D_2)D_2^{-1} \tag{*}$$

and $(D_1D_2^{-1})D_2=(l_2D_2+n_2D_2^{-1}+M_{12})D_2$. Hence $n_1D_1^{-1}D_2^{-1}=l_2D_2^2+M_{12}D_2$. This means that

$$n_1(n_1D_1+n_2D_2^{-1})=l_2(l_2|D_2||D_1|^{-1}D_1+d_2D_2^{-1}+M_{22})+M_{12}D_2.$$

In particular $M_{12}D_2=(n_1^2-l_2^2|D_2||D_1|^{-1})D_1+(n_1n_2-l_2d_2)D_2^{-1}$ and $l_2M_{22}=0$. Moreover $(M_{12}D_2,D_1D_2)=0$ so $0=(M_{12}D_1^{-1},D_2D_2^{-1})$. Hence, in particular, $0=(M_{12}D_1^{-1},D_1)$ which implies that $0=(M_{12},D_1^2)=(M_{12},M_{11})$.

c(2)(iv). $j_1=n_2$ *and* $n_1D_1D_1^{-1}=d_1D_1^{-1}D_2+n_1|D_1||D_2|^{-1}D_2D_2^{-1}+M_{11}D_2$.

Proof. Since

$$D_1(n_1D_1^{-1}+n_2D_2)=D_1(D_1D_2)=D_1^2D_2=D_2(j_1D_1+d_1D_1^{-1}+n_1|D_1||D_2|^{-1}D_2^{-1}+M_{11})$$

and $(D_1D_2,D_1)=0=(D_2D_1^{-1},D_1)$ then by c(2)(ii) and c(2)(iii) we get that

$$n_1j_1=n_1(D_1D_1^{-1},D_1)=(D_1^2D_2,D_1)=n_1|D_1||D_2|^{-1}(D_2D_2^{-1},D_1)+(M_{11}D_2,D_1)=n_1n_2.$$

So $j_1=n_2$.

c(2)(v). $d_1=0$ *and* $(M_{11},N_2)=0$.

Proof. Since $(D_1^{-1}D_1,D_2)=0$ then by c(2)(iv)

$$0=d_1(D_1^{-1}D_2,D_2)+n_1|D_1||D_2|^{-1}(D_2^{-1}D_2,D_2)+(M_{11}D_2,D_2)=d_1n_2+(M_{11}D_2,D_2).$$

We conclude that $d_1=0=(M_{11},N_2)$.

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c(2)(vi). $n_1N_1=n_1|D_1||D_2|^{-1}N_2+M_{11}D_2$.

Proof. By c(2)(i), c(2)(iii) and c(2)(v), $(M_{11}D_2, D_1+D_1^{-1}+D_2+D_2^{-1})=0$. It follows from c(2)(iv) that $n_1N_1=n_1|D_1||D_2|^{-1}N_2+M_{11}D_2$.

c(2)(vii). $M_{11}=N_2=N_1=0$.

Proof. By the above

$$(D_2D_2^{-1})D_1=[|D_2|Cl(1)+n_2|D_2||D_1|^{-1}(D_1+D_1^{-1})+N_2]D_1$$

$$=|D_2|D_1+n_2|D_2||D_1|^{-1}(D_1^2+D_1^{-1}D_1)+N_2D_1 \qquad (**)$$

Since $(N_2, D_1D_2)=0$, $(N_2, M_{12})=0$ by c(2)(ii), and $(N_2, M_{11})=0$ by c(2)(v) then $(N_2D_1, D_2^{-1}+D_2+D_1^{-1})=0$. So (*) and (**) yield that

$$n_2N_2=n_2|D_2||D_1|^{-1}(M_{11}+N_1)+N_2D_1-(N_2D_1, D_1)D_1.$$

By c(2)(vi)

$$n_1n_2N_2=n_1n_2|D_2||D_1|^{-1}M_{11}+n_2|D_2||D_1|^{-1}(n_1|D_1||D_2|^{-1}N_2$$

$$+M_{11}D_2)+n_1N_2D_1-n_1(N_2D_1, D_1)D_1.$$

Therefore $M_{11}=0$ and $N_2D_1=(N_2D_1, D_1)D_1$. By (a) we conclude that $N_2=0$ and then by c(2)(vi) $N_1=0$.

c(2)(viii). $|D_1|=|D_2|$ *and* $D_1D_1^{-1}=D_2D_2^{-1}$.

Proof. By c(2)(vii) $D_2D_2^{-1}=|D_2||D_1|^{-1}D_1D_1^{-1}$. Then

$$|D_2|^2=|D_2||D_2^{-1}|=|D_2||D_1|^{-1}|D_1|^2=|D_1||D_2|.$$

Thus $|D_1|=|D_2|$.

c(2)(ix). $n_1=1+n_2$ *and* $|M_{12}|=(1+n_2-l_2)|D_1|$.

Proof. Since $(n_1+n_2)|D_1|=|D_1||D_2|=|D_1||D_1^{-1}|=|D_1|(1+2n_2)$, therefore $n_1=1+n_2$. Also $(n_1+n_2)|D_1|=|D_1||D_2|=|D_1||D_2^{-1}|=(l_2+n_2)|D_1|+|M_{12}|$. Hence $|M_{12}|=(n_1-l_2)|D_1|$.

c(2)(x). $|D_1|=1+2n_2$.

Proof. We compute

$$(D_1D_2)(D_1^{-1}D_2^{-1})=(n_1D_1^{-1}+n_2D_2)(n_1D_1+n_2D_2^{-1})$$

$$=(n_1^2+n_2^2)D_1D_1^{-1}+n_1n_2(D_1D_2+D_1^{-1}D_2^{-1}).$$

Since

$$(D_1D_2)(D_1^{-1}D_2^{-1})=(D_1D_1^{-1})(D_2D_2^{-1})=[|D_1|Cl(1)+n_2(D_1+D_1^{-1})]^2$$

$$=|D_1|^2Cl(1)+n_2^2(D_1^2+(D_1^{-1})^2+2D_1D_1^{-1})+2n_2|D_1|(D_1+D_1^{-1}) \qquad (***)$$

then $(n_1^2+n_2^2)|D_1|=(D_1D_2D_1^{-1}D_2^{-1},Cl(1))=|D_1|^2+2n_2^2|D_1|$. Thus $|D_1|=1+2n_2$.

c(2)(xi). $l_2+d_2=1+2n_2$ *and* $M_{22}=0$.

Proof. By c(2)(iii) and c(2)(ix), $(1+n_2-l_2)|D_1|^2=|M_{12}||D_2|=(n_1^2-l_2^2+n_1n_2-l_2d_2)|D_1|$. Therefore, by c(2)(ix) and c(2)(x), $(1+n_2-l_2)(1+2n_2)=(1+2n_2+n_2^2-l_2^2+n_2+n_2^2-l_2d_2)$. Then $1+2n_2=l_2+d_2$. Since $n_1+n_2=1+2n_2$ and $(n_1+n_2)|D_1|=|D_1||D_2|=|D_2^2|=(l_2+d_2)|D_2|+|M_{22}|$ then $M_{22}=0$.

c(2)(xii). $l_2=n_2$, $M_{12}=M_{12}^{-1}$, $D_1^2=D_2^2$ *and* $M_{12}D_2=|D_1|D_1$.

Proof. Since

$$(D_1D_2)D_1^{-1}=(n_1D_1^{-1}+n_2D_2)D_1^{-1}=n_1(n_2D_1^{-1}+n_1D_2)+n_2(n_2D_2+l_2D_2^{-1}+M_{12}^{-1})$$

and

$$D_1D_1^{-1}D_2=D_2D_2^{-1}D_2=D_2^2D_2^{-1}=(l_2D_1+d_2D_2^{-1})D_2^{-1}$$

$$=l_2(l_2D_2+n_2D_2^{-1}+M_{12})+d_2(l_2D_1^{-1}+d_2D_2)$$

then

$$(n_1n_2-l_2d_2)D_1^{-1}+(n_1^2+n_2^2-l_2^2-d_2^2)D_2=l_2M_{12}-n_2M_{12}^{-1}.$$

Since $(M_{12},D_i)=0=(M_{12},D_i^{-1})$ for $1\leqq i\leqq 2$ then $l_2=n_2$, $M_{12}=M_{12}^{-1}$ ($M_{12}\neq 0$ by the same arguments as in **c(1)(v)**. Therefore $d_2=n_1$ and $D_1^2=D_2^2$ and by **c(2)(iii)** $M_{12}D_2=(n_1^2-l_2^2)D_1=|D_1|D_1$.

c(2)(xiii). *Final contradiction.*

Proof. Let us compute

$$(D_1D_2^{-1})^2=(n_2D_2+n_2D_2^{-1}+M_{12})^2=n_2^2(D_2^2+(D_2^{-1})^2+2D_2D_2^{-1})$$

$$+2n_2(M_{12}D_2+M_{12}D_2^{-1})+M_{12}^2$$

$$=n_2^2(D_1^2+(D_1^{-1})^2+2D_1D_1^{-1})+2n_2|D_1|(D_1+D_1^{-1})+M_{12}^2.$$

Since $(D_1D_2^{-1})^2=D_1^2(D_2^{-1})^2=D_1^2(D_1^{-1})^2=(D_1D_1^{-1})^2$ then by (***) $M_{12}^2=|D_1|^2Cl(1)$, which contradicts the lemma.

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