BI-CONNEXIVE LOGIC, BILATERALISM, AND NEGATION INCONSISTENCY

HEINRICH WANSING
SATORU NIKI
and
SERGEY DROBYSHEVICH
Department of Philosophy I, Ruhr University Bochum, Germany

Abstract. In this paper we study logical bilateralism understood as a theory of two primitive derivability relations, namely provability and refutability, in a language devoid of a primitive strong negation and without a *falsum* constant, $\bot$, and a *verum* constant, $\top$. There is thus no negation that toggles between provability and refutability, and there are no primitive constants that are used to define an "implies falsity" negation and a "co-implies truth" co-negation. This reduction of expressive power notwithstanding, there remains some interaction between provability and refutability due to the presence of (i) a conditional and the refutability condition of conditionals and (ii) a co-implication and the provability condition of co-implications. Moreover, assuming a hyperconnexive understanding of refuting conditionals and a dual understanding of proving co-implications, neither non-trivial negation inconsistency nor hyperconnexivity is lost for unary negation connectives definable by means of certain surrogates of *falsum* and *verum*. Whilst a critical attitude towards $\bot$ and $\top$ can be justified by problematic aspects of the Brouwer-Heyting-Kolmogorov interpretation of the logical operations for these constants, the aim to reduce the availability of a toggling negation and observations on undefinability may also give further reasons to abandon $\bot$ and $\top$.

§1. Introduction. In this paper we will introduce a bilateral connexive logic **B2C** (for "*basic* **2C**"), which can be construed as a bilateral fragment (e.g. [13]) of the logic **2C** introduced in [59]. First, we discuss some related systems and topics.

Logical bilateralism is usually associated with proof-theoretic semantics [21, 52], although the two aspects that are involved can also be seen to have model-theoretic counterparts. In the literature on logical bilateralism, the term 'bilateralism' is used with different meanings: see [5, 12, 13, 19, 20, 26, 46–50, 60] and references therein, a comparative presentation can be found in [64]. The common core of the notion can be described by saying that according to bilateralism, meaning—in particular the meaning of the logical operations—has two dimensions that are of equal importance, namely the dimensions of proof, or verification, or support of truth on the one hand and

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refutation (dual proof), or falsification, or support of falsity on the other. Oftentimes the two dimensions are presented in a way that reflects a 'meaning-as-use' perspective on semantics by focusing on the speech acts of assertion and denial and the distinction between warranted assertability and warranted deniability.[1]

In unilateral proof-theoretic semantics, where just verification, proof, or warranted assertability is considered, the logic that is distinguished as sound is intuitionistic logic, and often attention is restricted to propositional intuitionistic logic, **Int**. It has been maintained[2] that bilateralism enables a proof-theoretic account of the meaning of the logical operations of classical logic [46, 49], however, for the purpose of bilateralism as developed in [49], "co-ordination principles" have to be postulated that specify a suitable interaction between warranted assertability and warranted deniability [50]. Without these additional assumptions, and with the classical understanding of the falsification of conditionals, one naturally arrives (see [19]) at Nelson's paraconsistent four-valued constructive logic [1, 25, 33, 36, 44], often referred to as **N4**. Moreover, while intuitionistic logic is sometimes identified with constructive logic, it has been criticized from a bilateralist perspective for not being sufficiently constructive, and this concern also led Edgar López-Escobar [28] to **N4**.

Unlike **Int**, **N4** is a paraconsistent logic, meaning it allows for non-trivial negation inconsistent theories. Axiomatically, **N4** is obtained from **Int** (formulated in the language $\{\wedge, \vee, \rightarrow, \top, \bot\}$) by removing the *verum* constant $\top$ and the *falsum* constant $\bot$ and adding a primitive negation connective, $\sim$. This connective is often called *strong negation* in the literature; in this paper we will call this and similar connectives *toggling negations*, owing to the fact that they internalize toggling between proofs and refutations. The refutation conditions of compound formulas are then axiomatically expressed by the following equivalences (where $A \leftrightarrow B$ abbreviates the conjunction $(A \rightarrow B) \wedge (B \rightarrow A)$):

$$\begin{array}{ll} {\sim}{\sim}A \leftrightarrow A & {\sim}(A \rightarrow B) \leftrightarrow (A \wedge {\sim}B) \\ {\sim}(A \wedge B) \leftrightarrow ({\sim}A \vee {\sim}B) & {\sim}(A \vee B) \leftrightarrow ({\sim}A \wedge {\sim}B). \end{array}$$

In [57], a new connexive logic, **C**, was introduced as a modification of **N4**. Following ideas by Storrs McCall [30], we say that a system is *connexive (with respect to* $\sim$*)* if the following principles are among its theorems:

AT $\sim(A \rightarrow \sim A)$ (Aristotle's thesis),
AT′ $\sim(\sim A \rightarrow A)$ (Aristotle's thesis′),
BT $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$ (Boethius' thesis),
BT′ $(A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B)$ (Boethius' thesis′),

and it satisfies the condition of non-symmetry of implication, saying that the schema $(A \rightarrow B) \rightarrow (B \rightarrow A)$ is not a theorem.

The logics **N4** and **C** differ from each other with respect to the meaning of negated implications. Axiomatically, the difference is captured by replacing the principle $\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$ with $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$. The latter axiom expresses a

1 Logical bilateralism can be generalized to logical multilateralism by taking into consideration n dimensions ($2 \leq n, n \in \mathbb{N}$) of meaning, see [64]; in this case bilateralism could mean treating not just verification and falsification on a par in the constitution of linguistic meaning, but any two from among the n dimensions.

2 and denied as well, see [22].

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connexive (even "hyperconnexive"[3]) reading of $\to$. One important difference between **N4** and **C** is that **C** is not only paraconsistent but even negation inconsistent in the sense that there is a formula A such that both A and $\sim A$ are theorems of **C**.

The logic **2C** [59] is obtained by adding a *co-implication* connective, $\prec$, that is in a certain sense dual to $\to$, to **C**. In this way the co-implication of **2C** is a connexive version of the co-implication of **2Int** [58]. **2Int**, in turn, was introduced as an alternative to bi-intuitionistic logic (or Heyting-Brouwer logic; see [14, 24, 45] and references therein) **BiInt**. The main idea behind **2Int** was to use a bilateralist perspective to give natural deduction rules for the language of bi-intuitionistic logic, i.e., $\{\wedge, \vee, \to, \prec, \top, \bot\}$. In both **2Int** and **2C** co-implication internalizes the preservation of refutability (support of falsity) in much the same way as implication internalizes the preservation of provability (support of truth). On the other hand, in the logics **N4** and **C** such an inferential relationship can be expressed by negating the premises and the conclusion. In systems with implication, co-implication and truth and falsity constants (such as **BiInt**, **2Int** and **2C**) two negation connectives can be defined naturally: $(A \to \bot)$ ("A implies falsity") and $(\top \prec A)$ ("A co-implies truth"). Obviously, the meanings of these negations in **2Int** and **BiInt** are different.[4]

In the terminology of [41, p. 2 et seq.], the logical bilateralism to be found in **2Int**, **N4**, **C** and **2C** can be seen as supplementing positive reasoning with *negative reasoning*, where negative reasoning is characterized as

> that which involves an inference to the falsity of a proposition, or in modal, epistemic and deontic contexts an inference to impossibility, disbelief, prohibition, etc. (Notice that this may, but need not, be tantamount to reasoning to the negation of the proposition, belief, etc.) Clearly, such reasoning is 'mixed' in the sense that it may be based on inference from the truth of some proposition as well as inference from the falsity of some proposition. We can then distinguish a special case of negative reasoning, which we might call *inverse inference*, as that involving only inference from falsity to falsity.

One may wonder to which extent in logical bilateralism an interaction between positive and negative reasoning depends or ought to depend on the presence of a negation connective. This is one of the main topics of the present investigation.

Since the logic **C** (like **N4**) does without $\bot$ and $\top$, the intuitionistic negation as "implies $\bot$" and the co-negation as "co-implies $\top$" cannot be defined, and instead there is the primitive negation that *toggles* back and forth between verification (support of truth) and falsification (support of falsity). The language of **2C** was introduced including the two constants $\top$ and $\bot$ in order to compare **2C** with **2Int**. The idea now is to drop the toggling negation, $\top$, and $\bot$, and this move will bring us to the logic **B2C**. More concretely, a problem with **2Int** can be seen in that the defined negation $A \to \bot$ of a formula A already is non-constructive and therefore problematic from the point of view of the Brouwer-Heyting-Kolmogorov (BHK) interpretation of **Int**, and that the co-negation of **2Int** is plagued by an analogous problem once the co-conditional

[3] I.e., the converses of **BT** and **BT′** are valid as well. For an overview of the terminology for various connexive principles, see [61, 65].

[4] There is also a system called "da Costa Logic" which contains the co-negation of **BiInt** as primitive but lacks co-implication [43].

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Table 1. *Logics and languages*

Language	Nelsonian conditionals	connexive conditionals
$\{\wedge, \vee, \rightarrow, \sim\}$	**N4**	**C**
$\{\wedge, \vee, \rightarrow, \prec, \top, \bot\}$	**2Int**	
$\{\wedge, \vee, \rightarrow \prec, \top, \bot, \sim\}$		**2C**
$\{\wedge, \vee, \rightarrow, \prec\}$		**B2C**

is dealt with in a BHK-style interpretation for **2Int**. Moreover, if these concerns about *constructiveness* are taken seriously, it may be seen as problematic that the refutability condition of the conditional and the provability condition of the co-implication are not connexive. Constructivity and connexivity are in principle properties that are independent of each other; in [63], however, it has been suggested that constructivists should be (hyper)connexivists.[5]

We summarize key logics and their language in Table 1. When it comes to provability, the logics listed there share a common conjunction/disjunction fragment, $\{\wedge, \vee\}$, namely the distributive lattice logic from [16, Definition 2.1] that characterizes the so-called 'additive' or 'extensional' conjunction and disjunction connectives of intuitionistic logic by a number of postulates and rules:

- $A \vdash A$
- $A \vdash B, B \vdash C / A \vdash C$
- $A \wedge B \vdash A, A \wedge B \vdash B$
- $A \vdash B, A \vdash C \;/\; A \vdash B \wedge C$
- $A \vdash C, B \vdash C \;/\; A \vee B \vdash C$
- $A \vdash A \vee B, B \vdash A \vee B$
- $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$.

The designation of the conditionals of **N4** and **2Int** as Nelsonian stems from their semantic characterization and is mostly used to contrast them with the connexive conditionals of **C**, **2C** and **B2C**.

In this paper we want to explore the expressive capabilities of the language of **B2C**. There are two main themes we want to convey: first, that the language is quite weak, as most of the results of the paper are undefinability results; second, that it is still strong enough to result in a meaningful system. In particular, we will show that **B2C** still can be construed as a connexive (and even "bi-connexive"), negation inconsistent and non-trivial logic—all that despite lacking either a primitive negation, or constants that are usual employed in defining a negation.

Let us outline the plan of the paper. §2 is dedicated to motivating our choice of the language. There we will first discuss the choice of using connexive implication and co-implication connectives and then motivate the lack of truth and falsity constants and of

[5] Relatedly, it is observed in [34] that the expansion $\mathbf{C}^{\bot}$ of **C** by falsum (introduced in [17]) satisfies a constructive property that fails in the corresponding expansion $\mathbf{N4}^{\bot}$ of **N4**. This can be seen to suggests that $\mathbf{C}^{\bot}$ answers some constructive criticisms against **Int** better than $\mathbf{N4}^{\bot}$.

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the toggling negation. In §3 we formally introduce the system **B2C** via a bilateral natural deduction calculus and obtain some initial results regarding it. More specifically, we will discuss its status as a non-trivial negation inconsistent system, show its embedding into the positive fragment of intuitionistic logic and provide it with semantics. The proof of the corresponding soundness and completeness result concludes the section. In §4 we obtain the first non-definability results, showing, in particular, that neither $\top$ and $\bot$, nor the toggling negation $\sim$ are definable in **B2C**. We then show how these connectives could be recovered both proof- and model-theoretically. §5 deals with variants of constants that are available to us proof-theoretically given the bilateralist approach. This part also allows us to comment on some technical details of fashioning natural deduction rules in a bilateral context. Finally, in §6 we investigate various ways of toggling between proofs and refutations that are (for the most part) not available to us, despite clearly non-trivial interaction between them.

§2. On the language of the system. In this section we discuss choices made with regards to the language of the system **B2C**. We start by motivating the presence of connexive implication and its corresponding co-implication and then comment on the absences of both truth and falsity constants and of the toggling negation.

2.1. What speaks for connexive implication and co-implication? Why should one prefer the (hyper)connexive reading of negated implications and co-implications, that is expressed in **2C** by

$$\sim(A \to B) \to (A \to \sim B), \quad (A \to \sim B) \to \sim(A \to B),$$

being provable and

$$\sim(\sim B \prec A) \prec (B \prec A) \quad \sim(B \prec A) \prec (\sim B \prec A),$$

being refutable, over the Nelsonian (and classical) understanding of negated conditionals, that can be seen in **N4** by

$$\sim(A \to B) \to (A \wedge \sim B), \quad (A \wedge \sim B) \to \sim(A \to B),$$

being provable?

This question asks for a motivation of (hyper)connexive logics. A detailed motivation goes beyond the scope of the present paper. One reason, see [59], for considering *connexive* implication, $\to$, and *connexive* co-implication, $\prec$, instead of assuming the familiar understanding of negated implications in Nelson's and other logics, is that one obtains a neat encoding of derivations in the language with $\to$ and $\prec$ by typed λ-terms built up from atomic terms of two sorts, one for proofs and one for refutations (dual proofs), using only

(i) functional application and functional abstraction, and
(ii) certain sort/type-shift operations that turn an encoding of a dual proof of a formula A ($\sim A$) into an encoding of a proof of $\sim A$ (A) and that turn an encoding of a proof of a formula A ($\sim A$) into an encoding of a dual proof of $\sim A$ (A).[6]

[6] In [56] an encoding of derivations in Nelsons's constructive logics **N3** and **N4** was obtained by giving up the unique types of terms. The use of terms of two sorts avoids this: every term is uniquely typed.

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A more recent motivation for first-order connexive logic, in particular for a certain extension of **C** to first order, the system **QC**, is presented in [63]. There the axiom $\sim(A \to B) \leftrightarrow (A \to \sim B)$ is justified by its use in proving the "drinker truism" and the "dual drinker truism" (cf. the "drinker paradox" in classical logic [54]):

$$\sim\exists x(P(x) \to \sim\exists y P(y)), \qquad (\mathbb{DT})$$

("It is false that there is someone such that if she drinks, then it is false that someone drinks.")

$$\sim\exists x(\forall y P(y) \to \sim P(x)). \qquad (\mathbb{DDT})$$

("It is false that there is someone such that if everybody drinks, it is false that she drinks.")

The two formulas $\mathbb{DT}$ and $\mathbb{DDT}$ are valid in **QC** and have simple derivations in the sequent calculus **G3C** for **QC**. The provability of $\mathbb{DT}$ and $\mathbb{DDT}$ in the Hilbert-style proof system **HQC** for **QC** is guaranteed by the axiom $\sim(A \to B) \leftrightarrow (A \to \sim B)$. With it, $\mathbb{DT}$ is provably interderivable in **HQC** with $\forall x(P(x) \to \exists y P(y))$, and $\mathbb{DDT}$ is provably interderivable in **HQC** with $\forall x(\forall y P(y) \to P(x))$.

The above reading of $\mathbb{DT}$ and $\mathbb{DDT}$ is rather non-idiomatic. Usually, the negated existential $\sim\exists x P(x)$ is read as "No one drinks". Under that reading, it seems that $\mathbb{DT}$ indeed ought to be valid: No one is such that if she drinks, no one drinks.

Another recent motivation for quantified (hyper)connexive logic is given in [38] and [66].

2.2. On dispensing with $\bot$ and $\top$. Irrespective of whether $A \to \bot$ in **Int** is accepted as a negation connective, the presence of $\bot$ leads to a problem with the BHK interpretation of the intuitionistic connectives. Jean-Yves Girard [23] presents the BHK interpretation of **Int** as follows (notation and presentation adjusted):

1. for atomic sentences, we assume that we know intrinsically what a proof is (for example, pencil and paper calculation serves as a proof of "$27 \times 37 = 999$");
2. (a construction) c is a proof of $A \wedge B$ iff $c = \langle d, e \rangle$, where d is a proof of A and e is a proof of B;
3. c is a proof of $A \vee B$ iff $c = \langle i, d \rangle$, where $i = 0$ and d is a proof of A, or $i = 1$ and d is a proof of B;
4. c is a proof of $A \to B$ iff c is a function, which maps any proof of A to a proof of B;
5. no construction is a proof of $\bot$.

In general, the negation $\sim$ is treated as $A \to \bot$. The BHK-interpretation has been criticized for this treatment of negation by Ingebrigt Johansson, Edgar López-Escobar, and others. López-Escobar [28, p. 362 f.] remarks that

> if one accepts that there is no construction that proves an absurdity (as do most people) then a salient property of the construction π that proves "not-A" is that when π is applied to a particular non-existent construction (namely a proof of A) it yields another non-existent construction!

By the BHK-clause for negation, a proof of the intuitionistically valid $\sim(A \wedge \sim A)$ is a function f, which maps each proof π of $A \wedge \sim A$ to a nonexistent proof $f(\pi)$ of $\bot$.

Since $(A \wedge \sim A)$ itself has no proof, any function whatsoever proves $\sim(A \wedge \sim A)$, which seems fairly non-constructive. A similar criticism can be found in the work of George Griss. As Thomas Ferguson [18, p. 3] explains (notation adjusted), for Griss

> [c]onstructions serving to witness a conditional act as transformations whose application to constructions of an antecedent yield constructions of the consequent. In this context, the executability of a construction is interpreted as the possibility of successful acts of transformation. In principle the act of applying a function can only be considered successful in case there exists some operand to which the function is applied. Consequently, Griss' reading requires that the possibility of a construction of A serves as a precondition of the possibility of constructions of $A \rightarrow B$.

López-Escobar [28] suggested to supplement the BHK-interpretation of positive intuitionistic logic with the primitive notion of *refutation* to give a constructively acceptable interpretation for negation. As a result, and upon disregarding $\perp$, one obtains a semantics that is sound for **N4**. López-Escobar gives the following refutation interpretation of the intuitionistic connectives $\wedge$, $\vee$, $\rightarrow$, and the toggling negation $\sim$ (notation and presentation adjusted):

(i) c is a refutation of $A \wedge B$ iff $c = \langle i, d \rangle$, where $i = 0$ and d is a refutation of A or $i = 1$ and d is a refutation of B;
(ii) c is a refutation of $A \vee B$ iff $c = \langle d, e \rangle$ and d is a refutation of A and e is a refutation of B;
(iii) c is a refutation of $A \rightarrow B$ iff $c = \langle d, e \rangle$, d is a proof of A and e is a refutation of B;
(iv) c is a refutation of $\sim A$ iff c is a proof of A.

Definition 2.1. *The* López-Escobar interpretation *of the connectives* $\wedge$, $\vee$, $\rightarrow$, *and* $\sim$ *is obtained from the BHK-interpretation by replacing the clause* 5 *for negation by the above clauses* i) – iv) *and adding the clauses*

(v) *for atomic sentences, we assume that we know intrinsically what a refutation is*;
(vi) c *is a proof of* $\sim A$ *iff* c *is a refutation of* A.

The connexive López-Escobar interpretation [63] *is obtained from the López-Escobar interpretation by replacing clause iii) by the following*

iii') c *is a refutation of* $A \rightarrow B$ *iff* c *is a function, which maps any proof of* A *to a refutation of* B.

Definition 2.2. *The* bi-connexive López-Escobar interpretation *is obtained from the connexive López-Escobar interpretation by adding the following two clauses*:

(vii) c *is a refutation of* $B \prec A$ *iff* c *is a function, which maps any refutation of* A *to a refutation of* B.
(viii) c *is a proof of* $B \prec A$ *iff* c *is a function, which maps any refutation of* A *to a proof of* B.

In bilateral proof systems with such a pair of derivability relations for **N4**, its connexive variant **C**, and the bi-connexive logic **2C**, a direct interaction between the two relations comes with the primitive toggling negation connective, $\sim$. The relationship

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is so tight that in the BHK-style interpretations of **N4**, **C**, and **2C**, proofs of A are seen as identical with refutations of $\sim A$ and refutations of A are identified with proofs of $\sim A$.

Importantly, after discarding $\top$ and $\bot$ we still have access to some connectives similar to two definable negations of **2Int** and **2C**. We can define surrogates of *verum* and *falsum* constants as $\top\!\!\top := p \to p$ and $\bot\!\!\bot := p \prec p$ and define the new *negation* $\neg A := A \to \bot\!\!\bot$ and *co-negation* $- A := \top\!\!\top \prec A$. These two connectives will play an important role in the paper.

2.3. On dispensing with the toggling negation. If assertion and denial are both primitive speech acts that are on a par, then asserting the negation of A can be achieved by denying A, and denying the negation of A can be brought about by asserting A. Similarly, if proofs and refutations are two primitive kinds of derivations that are on a par, a proof of the toggling negation of A amounts to a refutation of A, and a refutation of the toggling negation of A amounts to a proof of A, cf. the discussion in [62]. Yet in the bilateral sequent calculi for **N4** and its modifications **C** and **2C**, there is already some interaction between proofs and refutation even without the toggling negation. Is there, then, indeed *a need* for the toggling negation?

There are various reasons for dispensing with the toggling negation, but most importantly for us is that we take proofs and refutations to be primitive notions and the toggling negation to be just a device for internalizing refutations within proofs and the other way around. Some authors, instead, take incompatibility to be the prime notion behind the concept of negation, see [9–11, 42]. Among these [42] is notable in that it is nevertheless quite close to our version of bilateralism and [9] in that it holds negation to be a modal operator: a view that seems to be incompatible with bilateralism. Let us discuss more specific reasons for abandoning the toggling negation.

In [13] Drobyshevich shows that given a relatively well-behaved semantics, the toggling negation can be introduced conservatively to a bilateral system but argues that this does not constitute a limitation for the bilateral approach. As a demonstration, in the same paper a few examples of non-trivial bilateral systems without the toggling negation are given—both in the form of sequent calculi and via semantics—including one for the bilateral negation-free fragment of **C** (confusingly, under the name **P2C** as opposed to, say, **PC**). Sara Ayhan [6, sec. 3.2] takes an even stronger stance, saying that it is "preferable not to have [the toggling] negation as a primitive connective in the language since, in a way, it stands against the bilateralist idea that refutation (or denial, or rejection, etc.) is a concept prior to negation," and "incorporating [the toggling] negation would mean to have a primitive connective that is basically expressing exactly what is expressed by our derivability relations." She also points to the fact that in Nelson's constructive logics, the toggling negation is non-congruential, namely, that it prevents provable equivalence from being a congruence relation for which a replacement theorem holds.

The toggling negation in **C** and **2C** is also non-congruential because, as in **N4**, for distinct propositional variables p and q, $p \to p$ and $q \to q$ are provably equivalent, whereas $\sim(p \to p)$ and $\sim(q \to q)$ are not. However, one could argue that, exactly from a bilateralist point of view, it is to be expected that provable equivalence *fails to be* and that provable strong equivalence *is* a congruence relation. Interreplaceability of A and B in all linguistic contexts requires that A and B are provably strongly equivalent,

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meaning that A and B as well as $\sim A$ and $\sim B$ are provably equivalent, i.e., that A and B are mutually provable *and* refutable.

Another issue of debate is the nestability of negation versus the non-nestability of speech acts and derivability relations.[7] We remind the reader that two negations $\cdot \to \bot$ and $\top \prec \cdot$ are definable in **2Int**. Ayhan [6, footnote 17] remarks that because of this, "an objection coming from a 'Frege-Geach-point' angle [...], that we need a negation in our language to express it in subclauses of sentences (where an interpretation as refutation would not suffice), does not seem to be a concern" for **2Int**. This gives rise to the question whether it nevertheless might be desirable to embed the *toggling negation* instead of negations $\cdot \to \bot$ and $\top \prec \cdot$, and whether it is justified to refer to $\neg A$ defined as $A \to \perp\!\!\!\perp$ and $-A$ defined as $\top\!\!\!\top \prec A$ as 'negations' of A. There is no general agreement about the properties that a one-place connective should have in order to be justifiably called a *negation*. In [29] some minimal conditions for a unary connective $*$ to qualify as a negation are listed, the chief among them being that (with respect to provability) A cannot be derived from $*A$ and $*B$ cannot be derived from B for some A and B. In our bilateral setting with both proofs and refutations, we may require that either the above condition is satisfied or that with respect to refutability A cannot be derived from $*A$ and $*B$ cannot be derived from B for some A and B. According to this, both $A \to \perp\!\!\!\perp$ and $\top\!\!\!\top \prec A$ can be considered as negations (we make these conditions more formal in Fact 3.5).

Irrespective of such a discussion, we believe that it is interesting to explore what effects dispensing with the toggling negation has for **B2C**. What are the results of letting the interaction between proofs and refutations emerge from the meaning of the binary connectives $\to$ and $\prec$ only? That is, we assume the relations of provability and refutability, internalize them by $\to$ and $\prec$, moreover do without $\top$ and $\bot$, and do not stipulate a direct back and forth between proofs and refutations enabled by the toggling negation. This approach of "better fewer, but better" allows us, at least, to take a fresh look back at the very connectives we abandoned. As we shall see later in §5 and 6, such a perspective can then be utilized in making an informed choice about restoring some of these connectives in the language, if one so desires. Note that the language of **B2C** is in some sense minimal if one i) considers the presence of conjunction, disjunction and implication as essential and ii) takes the desiderata of having proofs and refutations on par with each other very strongly. Observe that having a connective which internalizes the preservation of provability but no connective that internalizes the preservation of refutability can well be construed as a strong preference to provability over refutability. Thus having both implication and co-implication keeps things in perfect balance. Still, it is worth pointing out that as most of the results in our paper are non-definability result, they trivially hold for the co-implication-free fragment of **B2C**.

§3. System B2C: its proof theory and semantics. In this section we will introduce **B2C** starting with a natural deduction system. As is clear from the preceding discussion, the language $\mathcal{L}$ of **B2C** is defined by the following grammar:

$$A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \prec A).$$

[7] There *is* a literature on sequent calculi with embeddable sequent arrows. For an early reference see [27] and [7].

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We denote the denumerable set of propositional variables with Prop and the set of all formulas of $\mathcal{L}$ as Form $\mathcal{L}$ (similar notation will be adopted to other languages introduced later in the paper).

To remind the reader, throughout the paper we will use the following definable connectives:

$$\top\!\!\top := p \rightarrow p; \qquad \bot\!\!\bot := p \prec p; \qquad \neg A := A \rightarrow \bot\!\!\bot; \qquad -A := \top\!\!\top \prec A.$$

We will call $\neg$ *negation*, $-$ *co-negation* and $\sim$ (whenever it is present) the *toggling negation*.

After introducing the natural deduction system we will show how it can be embedded into positive intuitionistic logic, provide its semantics and establish the corresponding completeness result.

***3.1. The natural deduction proof system* NB2C.** The natural deduction proof system **NB2C** for **B2C** is obtained as a simplification and modification of the proof system **N2C** for **2C** from [59]. Derivations in **NB2C** combine *proofs* and *dual proofs* (corresponding to refutations), which are differentiated by whether a single line (for proofs) or a double line (for dual proofs) is drawn above a formula. Moreover, a proof may contain dual proofs as subderivations, and a dual proof may contain proofs as subderivations.

First it will be convenient to introduce some notions. By a *set-pair* we will call a pair of sets of formulas denoted as $(\Delta;\Gamma)$. For two set-pairs $(\Delta;\Gamma)$ and $(\Delta';\Gamma')$ we will write $(\Delta;\Gamma) \leq (\Delta';\Gamma')$ if $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$. In writing set-pairs we will sometimes replace the union sign with a comma and drop curly brackets over finite sets so that, for instance, $(\Delta, A_1, \dots, A_n; \Gamma, \Gamma')$ denotes $(\Delta \cup \{A_1, \dots, A_n\}; \Gamma \cup \Gamma')$. We say that a set-pair is *finite* if both sets in it are finite.

The conclusions of proof and dual proofs depend on finite set-pairs $(\Delta;\Gamma)$ of premises: a set Δ of *assumptions* that are taken to be provable, and a set Γ of *counterassumptions* that are taken to be refutable. Single square brackets [ ] are used to indicate assumptions that may be discharged, and double-square brackets $[\![\ ]\!]$ are used to indicate counterassumptions that may be discharged. We usually write $[A]$ instead of $[\overline{A}]$ and $[\![A]\!]$ instead of $[\![\overline{\overline{A}}]\!]$. Note that we allow for the empty discharge of assumptions and counterassumptions. Finally, in the statement of derivation rules dotted horizontal lines are sometimes used: these stand uniformly for either provability or dual provability in each instance of the rule.

DEFINITION 3.1. *We consider* $\overline{A}$ *as a proof of* A *from* $(\{A\};\varnothing)$ *and* $\overline{\overline{A}}$ *as a dual proof of* A *from* $(\varnothing;\{A\})$. *In addition to these stipulations, the system* **NB2C** *comprises the introduction and elimination rules from Tables 2 and 3. In the names of the rules E stands for "elimination from", I for "introduction into", p for "proofs", and dp for "dual proofs". We write* $(\Delta;\Gamma) \vdash A$ *if there is a proof of* A *from* $(\Delta;\Gamma)$; *and we write* $(\Delta;\Gamma) \vdash^d A$ *if there is a dual proof of* A *from* $(\Delta;\Gamma)$. *We say that a formula* A *is* provable (*in* **NB2C**) *if* $(\varnothing;\varnothing) \vdash A$ *and* refutable (*in* **NB2C**) *if* $(\varnothing;\varnothing) \vdash^d A$.

Note that the rules $(\vee Ep)$ and $(\wedge Edp)$ generalize those from [59]. Such generalized rules can also be found in the term-annotated natural deduction proof system for **2Int** in [6]. In the latter system the generalized rules are derivable by means of the negation and co-negation operations of **2Int**.

We make some observations regarding **NB2C**.

Table 2. *Introduction and elimination rules of* **NB2C** *with respect to proofs*

$$\begin{array}{c} \begin{array}{cc} (\Delta;\Gamma) & (\Delta';\Gamma') \\ \vdots & \vdots \\ \hline A & B \end{array} \\ \hline A \land B \end{array}\ (\land Ip) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline A \land B \\ \hline A \end{array}\ (\land Ep) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline A \land B \\ \hline B \end{array}\ (\land Ep)$$

$$\begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline A \\ \hline A \lor B \end{array}\ (\lor Ip) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline B \\ \hline A \lor B \end{array}\ (\lor Ip) \qquad \begin{array}{c} \begin{array}{ccc} (\Delta;\Gamma) & ([A],\Delta';\Gamma') & ([B],\Delta'';\Gamma'') \\ \vdots & \vdots & \vdots \\ \hline A \lor B & C & C \end{array} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C \end{array}\ (\lor Ep)$$

$$\begin{array}{c} ([A],\Delta;\Gamma) \\ \vdots \\ \hline B \\ \hline A \to B \end{array}\ (\to Ip) \qquad \begin{array}{c} \begin{array}{cc} (\Delta;\Gamma) & (\Delta';\Gamma') \\ \vdots & \vdots \\ \hline A & A \to B \end{array} \\ \hline B \end{array}\ (\to Ep)$$

$$\begin{array}{c} (\Delta;\Gamma,[\![A]\!]) \\ \vdots \\ \hline B \\ \hline B \prec A \end{array}\ (\prec Ip) \qquad \begin{array}{c} \begin{array}{cc} (\Delta';\Gamma') & (\Delta;\Gamma) \\ \vdots & \vdots \\ \hline B \prec A & \hline\hline A \end{array} \\ \hline B \end{array}\ (\prec Ep)$$

Table 3. *Introduction and elimination rules of* **NB2C** *with respect to dual proofs*

$$\begin{array}{c} \begin{array}{cc} (\Delta;\Gamma) & (\Delta';\Gamma') \\ \vdots & \vdots \\ \hline\hline A & B \end{array} \\ \hline\hline A \lor B \end{array}\ (\lor Idp) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline\hline A \lor B \\ \hline\hline A \end{array}\ (\lor Edp) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline\hline A \lor B \\ \hline\hline B \end{array}\ (\lor Edp)$$

$$\begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline\hline A \\ \hline\hline A \land B \end{array}\ (\land Idp) \qquad \begin{array}{c} (\Delta;\Gamma) \\ \vdots \\ \hline\hline B \\ \hline\hline A \land B \end{array}\ (\land Idp) \qquad \begin{array}{c} \begin{array}{ccc} (\Delta;\Gamma) & (\Delta';\Gamma',[\![A]\!]) & (\Delta'';\Gamma'',[\![B]\!]) \\ \vdots & \vdots & \vdots \\ \hline\hline A \land B & C & C \end{array} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C \end{array}\ (\land Edp)$$

$$\begin{array}{c} (\Delta;\Gamma,[\![A]\!]) \\ \vdots \\ \hline\hline B \\ \hline\hline B \prec A \end{array}\ (\prec Idp) \qquad \begin{array}{c} \begin{array}{cc} (\Delta';\Gamma') & (\Delta;\Gamma) \\ \vdots & \vdots \\ \hline\hline B \prec A & A \end{array} \\ \hline\hline B \end{array}\ (\prec Edp)$$

$$\begin{array}{c} ([A],\Delta;\Gamma) \\ \vdots \\ \hline\hline B \\ \hline\hline A \to B \end{array}\ (\to Idp) \qquad \begin{array}{c} \begin{array}{cc} (\Delta;\Gamma) & (\Delta';\Gamma') \\ \vdots & \vdots \\ \hline A & \hline\hline A \to B \end{array} \\ \hline\hline B \end{array}\ (\to Edp)$$

The logics **C** and **2C** are non-trivial negation inconsistent logics, see [35]. For example, $(A\land \sim A) \to\sim(A\land \sim A)$ and $\sim((A\land \sim A) \to\sim(A\land \sim A))$ are provable in both systems. Meanwhile, in **2C** the formulas $\sim(A\lor \sim A)\prec(A\lor \sim A)$ and $\sim(\sim(A\lor \sim A)\prec(A\lor \sim A))$ are both dually provable. Moreover, **2C** has an even

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more unorthodox property: the co-negation $\top\!\prec A$ is provable and the negation $A \to \bot$ is refutable for any formula A. A similar observation can be made regarding **NB2C**; since the empty discharge of assumptions is permitted,[8] we have the following:

FACT 3.1. *$-A = \Pi \prec A$ is provable and $\neg A = A \to \amalg$ is refutable in* **NB2C** *for any formula A.*

FACT 3.2. *In* **NB2C** *for any formula A we have*:

1. *$A, -A$ are both provable iff A is provable*;
2. *$A, \neg A$ are both refutable iff A is refutable.*

Since there are provable formulas in **NB2C** (e.g. Π), the first item of the last fact implies that **NB2C** is negation inconsistent with respect to $-$: there is a formula A such that both A and $-A$ are provable. Moreover, it even describes how all provable contradictions of this kind in **NB2C** look like; see [35] for a similar inquiry. Similarly, the second item implies a kind of incompleteness statement with respect to $\neg$: there is a formula A (e.g. $\amalg$) such that both A and $\neg A$ are refutable. From the perspective of classical logic this would also constitute a kind of contradiction albeit of a different kind. Note that these two kinds of contradictions are equivalent if the negation in question is the toggling negation. There is another sense in which **B2C** is inconsistent, which is unique to the bilateral presentation of the system; namely

FACT 3.3. *There is a formula A such that A is both provable and refutable in* **NB2C**.

Proof. One can take $A = -\Pi$ as demonstrated by the following derivations:

$$\frac{\dfrac{[p]}{p \to p}}{(p \to p)\prec(p \to p)} \qquad \frac{[\![p \to p]\!]}{\overline{(p \to p)\prec(p \to p)}}\ .$$

Similarly, one can take $A = \neg\amalg$. □

In **N2C** the following dual versions of Aristotle's and Boethius' theses are *refutable*:

dAT $\sim(\sim A \prec A)$,
dAT′ $\sim(A \prec \sim A)$,
dBT $\sim(\sim B \prec A)\prec(B \prec A)$,
dBT′ $\sim(B \prec A)\prec(\sim B \prec A)$.

Moreover, $(A \prec B)\prec(B \prec A)$ is not refutable in **N2C**. We can then say that a logic is *dually connexive (w.r.t.* $\sim$) if its co-implication connective satisfies this non-symmetry of co-implication and if it allows one to derive dAT–dBT′ for the derivability relation internalized by co-implication.

FACT 3.4. *The system* **NB2C** *is bi-connexive. It is connexive w.r.t. co-negation and dually connexive w.r.t. negation.*

8 Whereas the empty discharge of assumptions is rejected by proponents of relevance logic, for constructivists it is unproblematic.

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3.2. Faithful embedding of* B2C *into positive intuitionistic logic. The connexive logic **C** is faithfully embeddable into positive intuitionistic logic, $\mathbf{Int}^{+}$, under a single translation function, τ. This is possible because the toggling negation toggles between provability and refutability, so that, in particular, a refutation of a formula A *is* a proof of the toggling negation $\sim A$ of A. In the absence of the toggling negation, the separate treatment of proofs and refutations requires the use of two translation functions, τ^{+} and τ^{-}.

DEFINITION 3.2. *Let* $\mathsf{Prop}^{-} := \{p^{-} \mid p \in \mathsf{Prop}\}$ *and* $\mathsf{Prop}^{+} := \{p^{+} \mid p \in \mathsf{Prop}\}$. *We define two mappings* τ^{+} *and* τ^{-}, *from* $\mathsf{Form}\,\mathcal{L}$ *based on* Prop *to the set of all formulas of the language of* $\mathbf{Int}^{+}$ *defined over* $\mathsf{Prop}^{+} \cup \mathsf{Prop}^{-}$ *as follows*:

- $\tau^{+}(p) = p^{+}$, $\tau^{-}(p) = p^{-}$ *for* $p \in \mathsf{Prop}$;
- $\tau^{+}(A \circ B) := \tau^{+}(A) \circ \tau^{+}(B)$ *for* $\circ \in \{\rightarrow, \wedge, \vee\}$;
- $\tau^{+}(B \prec A) := \tau^{-}(A) \rightarrow \tau^{+}(B)$;
- $\tau^{-}(A \wedge B) := \tau^{-}(A) \vee \tau^{-}(B)$;
- $\tau^{-}(A \vee B) := \tau^{-}(A) \wedge \tau^{-}(B)$;
- $\tau^{-}(A \rightarrow B) := \tau^{+}(A) \rightarrow \tau^{-}(B)$;
- $\tau^{-}(B \prec A) := \tau^{-}(A) \rightarrow \tau^{-}(B)$.

If $\Delta \subseteq \mathsf{Form}\,\mathcal{L}$ is finite, then $\tau^{*}(\Delta) = \{A^{*} \mid A \in \Delta\}$, for $* \in \{+, -\}$. Taking rules $(\circ Ip)$ and $(\circ Ep)$ for $\circ \in \{\wedge, \vee, \rightarrow\}$ from Table 2 and restricting them to empty sets of counterassumptions and, moreover, restricting $(\vee Ep)$ so that only a single line is permissible above C, results in a notational variant of a natural deduction proof system $\mathbf{NInt}^{+}$, for $\mathbf{Int}^{+}$. We write $\Delta \vdash A$ if there is a proof of A from Δ in $\mathbf{NInt}^{+}$.

THEOREM 3.1. *For any finite* $\{A\} \cup \Delta \cup \Gamma \subseteq \mathsf{Form}\,\mathcal{L}$:

1. $(\Delta; \Gamma) \vdash A$ *in* **NB2C** *iff* $\tau^{+}(\Delta) \cup \tau^{-}(\Gamma) \vdash \tau^{+}(A)$ *in* $\mathbf{NInt}^{+}$;
2. $(\Delta; \Gamma) \vdash^{d} A$ *in* **NB2C** *iff* $\tau^{+}(\Delta) \cup \tau^{-}(\Gamma) \vdash \tau^{-}(A)$ *in* $\mathbf{NInt}^{+}$.

Proof. The directions from right to left are obvious because the rules of $\mathbf{NInt}^{+}$ are restricted versions of the proof rules of **NB2C**. For the directions from left to right, the proof is by simultaneous induction on derivations in **NB2C**. As to the induction bases, i.e., the cases $(\varnothing; \{A\}) \vdash A$ and $(\{A\}; \varnothing) \vdash^{d} A$, we have $\tau^{*}(A) \vdash \tau^{*}(A)$. Simultaneous induction is needed for the cases of rules that make use of both proofs and refutations. We present two cases for claim (1) and derivations ending in a proof.

$(\prec Ep)$. Suppose

$$
\begin{array}{ccc}
(\Delta'; \Gamma') & & (\Delta; \Gamma) \\
\vdots & & \vdots \\
\hline
B \prec A & & \overline{\overline{A}}
\end{array}
\;.
$$

By the two induction hypotheses, we have $\tau^{+}(\Delta) \cup \tau^{-}(\Gamma) \vdash \tau^{-}(A)$ and $\tau^{+}(\Delta') \cup \tau^{-}(\Gamma') \vdash \tau^{+}(B \prec A)$ in $\mathbf{NInt}^{+}$. Since $\tau^{+}(B \prec A) = \tau^{-}(A) \rightarrow \tau^{+}(B)$, we obtain in $\mathbf{NInt}^{+}$:

$$
\dfrac{\begin{array}{c}\tau^{+}(\Delta) \cup \tau^{-}(\Gamma) \\ \vdots \\ \tau^{-}(A)\end{array} \qquad \begin{array}{c}\tau^{+}(\Delta') \cup \tau^{-}(\Gamma') \\ \vdots \\ \tau^{-}(A) \rightarrow \tau^{+}(B)\end{array}}{\tau^{+}(B)}\,(\rightarrow Ep)\;.
$$

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$(\wedge Edp)$. Suppose

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\end{matrix}}{\overline{A\wedge B}} \qquad \dfrac{\begin{matrix}(\Delta';\Gamma',[\![A]\!])\\ \vdots\end{matrix}}{C} \qquad \dfrac{\begin{matrix}(\Delta'';\Gamma'',[\![B]\!])\\ \vdots\end{matrix}}{C} \quad .$$

By the two induction hypotheses, we have $\tau^+(\Delta)\cup\tau^-(\Gamma)\vdash\tau^-(A\wedge B)$, $\tau^+(\Delta')\cup\tau^-(\Gamma')\cup\{\tau^-(A)\}\vdash\tau^+(C)$, and $\tau^+(\Delta'')\cup\tau^-(\Gamma'')\cup\{\tau^-(B)\}\vdash\tau^+(C)$ in **NInt**$^+$. Since $\tau^-(A\wedge B)=\tau^-(A)\vee\tau^-(B)$, we obtain in **NInt**$^+$:

$$\dfrac{\begin{matrix}\tau^+(\Delta)\cup\tau^-(\Gamma)\\ \vdots\\ \tau^-(A)\vee\tau^-(B)\end{matrix} \qquad \begin{matrix}\tau^+(\Delta')\cup\tau^-(\Gamma')\cup\{\tau^-(A)\}\\ \vdots\\ \tau^+(C)\end{matrix} \qquad \begin{matrix}\tau^+(\Delta'')\cup\tau^-(\Gamma'')\cup\{\tau^-(B)\}\\ \vdots\\ \tau^+(C)\end{matrix}}{\tau^+(C)}\,(\vee Ep) \quad .$$

The reasoning for claim (2) is similarly straightforward. □

The functions τ^+ and τ^- could be used to complement the syntactical embedding from Theorem 3.1, by a semantical embedding along the lines of the semantical embedding of first-order **C** into first-order **Int**$^+$ in [57], in order to obtain an embedding-based completeness proof for **NB2C**. The embedding of **B2C** into **Int**$^+$ clarifies the non-trivial negation inconsistency of **NB2C**. Although for every provable formula A, $-A$ is provable and for every refutable formula A, $\neg A$ is refutable, neither is every formula provable in **NB2C** nor is every formula refutable in **NB2C**. Moreover, under both translations the formula $(p\to p)\prec(p\to p)$, for example, that is both provable and refutable in **NB2C**, is mapped to a theorem of **Int**$^+$:

$$\tau^+((p\to p)\prec(p\to p))=(p^+\to p^-)\to(p^+\to p^+),$$
$$\tau^-((p\to p)\prec(p\to p))=(p^+\to p^-)\to(p^+\to p^-).$$

In the next section we shall present a direct completeness proof for **NB2C**. This will demonstrate that our bilateral framework is robust enough to obtain the usual completeness proof via prime theories even in the absence of the toggling negation.

3.3. Semantics. The following semantics for **B2C** is obtained by simply dropping the missing connectives from the semantics for **2C** [59].

DEFINITION 3.3. *A* model $\mathcal{M}$ *is a structure* $\langle I,\leq,v^+,v^-\rangle$ *where* (i) I *is a non-empty set of states,* (ii) $\leq$ *is a pre-order on* I *and* (iii) *for* $*\in\{+,-\}$, $v^*:\mathsf{Prop}\to 2^I$ *is such that* $v^*(p)$ *is upward closed for any* $p\in\mathsf{Prop}$ *(i.e.,* $x\in v^*(p)$ *and* $y\geq x$ *implies* $y\in v^*(p)$*). Relations* $\models^+$ (support of truth), $\models^-$ (support of falsity) *between states and formulas are defined by*:

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$$
\begin{array}{lll}
\mathcal{M}, x \models^{+} p & \text{iff} & x \in v^{+}(p); \\
\mathcal{M}, x \models^{-} p & \text{iff} & x \in v^{-}(p); \\
\mathcal{M}, x \models^{+} A \wedge B & \text{iff} & \mathcal{M}, x \models^{+} A \text{ and } \mathcal{M}, x \models^{+} B; \\
\mathcal{M}, x \models^{-} A \wedge B & \text{iff} & \mathcal{M}, x \models^{-} A \text{ or } \mathcal{M}, x \models^{-} B; \\
\mathcal{M}, x \models^{+} A \vee B & \text{iff} & \mathcal{M}, x \models^{+} A \text{ or } \mathcal{M}, x \models^{+} B; \\
\mathcal{M}, x \models^{-} A \vee B & \text{iff} & \mathcal{M}, x \models^{-} A \text{ and } \mathcal{M}, x \models^{-} B; \\
\mathcal{M}, x \models^{+} A \rightarrow B & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^{+} A \text{ implies } \mathcal{M}, y \models^{+} B); \\
\mathcal{M}, x \models^{-} A \rightarrow B & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^{+} A \text{ implies } \mathcal{M}, y \models^{-} B); \\
\mathcal{M}, x \models^{+} B \prec A & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^{-} A \text{ implies } \mathcal{M}, y \models^{+} B); \\
\mathcal{M}, x \models^{-} B \prec A & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^{-} A \text{ implies } \mathcal{M}, y \models^{-} B).
\end{array}
$$

Occasionally, we will use $v^*(A)$ as a shorthand for $\{x : \mathcal{M}, x \models^* A\}$ $(* \in \{+,-\})$.

DEFINITION 3.4. *A model* $\langle I, \leq, v^+, v^- \rangle$ *will be called* classical, *if* $y \geq x$ *implies* $x = y$ *for all* $x, y \in I$.

As usual, we have the following

LEMMA 3.1 (monotonicity). *For any model* $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$, *formula* $A \in \mathsf{Form}\,\mathcal{L}$ *and* $* \in \{+,-\}$:

$$\forall x, y \in I \; (x \leq y \textit{ and } \mathcal{M}, x \vDash^* A \textit{ implies } \mathcal{M}, y \vDash^* A).$$

Proof. By a simple induction on the complexity of A. □

Next, we establish soundness and completeness of **B2C** with respect to this semantics. To do so, let us first define the corresponding consequence relations.

DEFINITION 3.5. *Fix a set-pair* $(\Gamma; \Delta)$ *and a formula* A. *Then put*

- $(\Gamma; \Delta) \vdash^{+}_{\mathbf{B2C}} A$ *if* $(\Gamma_0; \Delta_0) \vdash A$ *in* **NB2C** *for some finite* $(\Gamma_0; \Delta_0) \leq (\Gamma; \Delta)$;
- $(\Gamma; \Delta) \vdash^{-}_{\mathbf{B2C}} A$ *if* $(\Gamma_0; \Delta_0) \vdash^{d} A$ *in* **NB2C** *for some finite* $(\Gamma_0; \Delta_0) \leq (\Gamma; \Delta)$;

Similarly, where $* \in \{+,-\}$ *put* $(\Gamma; \Delta) \vDash^{*}_{\mathbf{B2C}} A$ *if for any model* $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *and any state* $x \in I$, $\mathcal{M}, x \vDash^{+} B$ *for all* $B \in \Gamma$ *and* $\mathcal{M}, x \vDash^{-} C$ *for all* $C \in \Delta$ *implies* $\mathcal{M}, x \vDash^{*} A$.

We list some simple properties of the two syntactic consequence relations:

PROPOSITION 3.1. *Let* $* \in \{+,-\}$.

1. *Reflexivity*: *if* $A \in \Gamma^*$, *then* $(\Gamma^+; \Gamma^-) \vdash^{*}_{\mathbf{B2C}} A$.
2. *Monotonicity*: *if* $(\Gamma^+; \Gamma^-) \vdash^{*}_{\mathbf{B2C}} A$ *and* $(\Gamma^+; \Gamma^-) \leq (\Delta^+; \Delta^-)$ *then* $(\Delta^+; \Delta^-) \vdash^{*}_{\mathbf{B2C}} A$.
3. *Cut*: *if* $(\Gamma^+; \Gamma^-) \vdash^{+}_{\mathbf{B2C}} A$ *and* $(\Gamma^+, A; \Gamma^-) \vdash^{*}_{\mathbf{B2C}} B$, *then* $(\Gamma^+; \Gamma^-) \vdash^{*}_{\mathbf{B2C}} B$.
4. *Dual cut*: *if* $(\Gamma^+; \Gamma^-) \vdash^{-}_{\mathbf{B2C}} A$ *and* $(\Gamma^+; \Gamma^-, A) \vdash^{*}_{\mathbf{B2C}} B$, *then* $(\Gamma^+; \Gamma^-) \vdash^{*}_{\mathbf{B2C}} B$.
5. *Compactness*: *if* $(\Gamma^+; \Gamma^-) \vdash^{-}_{\mathbf{B2C}} A$ *then* $(\Delta^+; \Delta^-) \vdash^{-}_{\mathbf{B2C}} A$ *for some finite* $(\Delta^+; \Delta^-) \leq (\Gamma^+; \Gamma^-)$.

A *bi-theory (in **B2C**)* is a set-pair $(\Gamma^+; \Gamma^-)$ such that for any formula A and $* \in \{+,-\}$ we have

$$(\Gamma^+, \Gamma^-) \vdash^{*}_{\mathbf{B2C}} A \implies A \in \Gamma^*.$$

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A bi-theory $(\Gamma^+;\Gamma^-)$ is *prime* if Γ^+,Γ^- are both non-empty and the following constructive disjunction and conjunction properties are satisfied:

$$A \vee B \in \Gamma^+ \implies A \in \Gamma^+ \text{ or } B \in \Gamma^+; \qquad A \wedge B \in \Gamma^- \implies A \in \Gamma^- \text{ or } B \in \Gamma^-.$$

LEMMA 3.2 (extension). *Suppose $(\Gamma^+;\Gamma^-)$ is a set-pair, A is a formula and $* \in \{+,-\}$. If $(\Gamma^+;\Gamma^-) \nvdash^*_{\mathbf{B2C}} A$ then there is a prime bi-theory (Δ^+,Δ^-) such that $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$ and $(\Delta^+,\Delta^-) \nvdash^*_{\mathbf{B2C}} A$.*

Proof. We consider the case of $* = +$, the other case is similar.

Take X to be the set of all expressions of the form A^+ and A^-, where A is a formula and fix some enumeration $\{x_i \mid i \in \mathbb{N}\}$ of elements of X. Put $\Delta^+_0 := \Gamma^+$, $\Delta^-_0 := \Gamma^-$ and for any $i \in \mathbb{N}$:

$$(\Delta^+_{i+1};\Delta^-_{i+1}) := \begin{cases} (\Delta^+_i, B;\Delta^-_i), & \text{if } x_i = B^+ \text{ and } (\Delta^+_i, B;\Delta^-_i) \nvdash^+_{\mathbf{B2C}} A; \\ (\Delta^+_i;\Delta^-_i, B), & \text{if } x_i = B^- \text{ and } (\Delta^+_i;\Delta^-_i, B) \nvdash^+_{\mathbf{B2C}} A; \\ (\Delta^+_i;\Delta^-_i), & \text{otherwise.} \end{cases}$$

Then put $\Delta^* = \bigcup_{i\in\mathbb{N}} \Delta^*_i$ for $* \in \{+,-\}$. We show that (Δ^+,Δ^-) is the required bi-theory. That $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$ is trivial by the construction.

A simple induction shows that $(\Delta^+_i;\Delta^-_i) \nvdash^+_{\mathbf{B2C}} A$ for any $i \in \mathbb{N}$ and hence that $(\Delta^+;\Delta^-) \nvdash^+_{\mathbf{B2C}} A$ due to compactness.

Suppose $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}} B$ and $B \notin \Delta^+$. Then there is $i \in \mathbb{N}$ such that $B^+ = x_i$ and $(\Delta^+_i, B;\Delta^-_i) \vdash^+_{\mathbf{B2C}} A$. By monotonicity we have $(\Delta^+, B;\Delta^-) \vdash^+_{\mathbf{B2C}} A$ hence by the cut we have $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}} A$, which contradicts the assumption. So $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}} B$ implies $B \in \Delta^+$. Similarly, $(\Delta^+;\Delta^-) \vdash^-_{\mathbf{B2C}} B$ implies $B \in \Delta^-$.

That Δ^+ (Δ^-) is non-empty then follows from the fact that $\top\!\!\top$ $(\perp\!\!\perp)$ is provable (refutable) in **NB2C** using (dual) cut.

Finally, suppose $B \vee C \in \Delta^+$, $B \notin \Delta^+$ and $C \notin \Delta^+$. Again, there are $i, j \in \mathbb{N}$ such that $B^+ = x_i$, $C^+ = x_j$,

$$(\Delta^+_i, B;\Delta^-_i) \vdash^+_{\mathbf{B2C}} A; \qquad (\Delta^+_j, C;\Delta^-_j) \vdash^+_{\mathbf{B2C}} A.$$

Then the following derivation

$$\dfrac{\dfrac{\begin{matrix}(B \vee C;\varnothing)\\ \vdots\end{matrix}}{B \vee C} \qquad \dfrac{\begin{matrix}(\Delta^+_i,[B];\Delta^-_i)\\ \vdots\end{matrix}}{A} \qquad \dfrac{\begin{matrix}(\Delta^+_i,[C];\Delta^-_i)\\ \vdots\end{matrix}}{A}}{A}\,(\vee Ep),$$

shows that $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}} A$. Similarly if $B \wedge C \in \Delta^-$, $B \notin \Delta^-$ and $C \notin \Delta^-$, then there are $i, j \in \mathbb{N}$ such that $B^- = x_i$, $C^- = x_j$,

$$(\Delta^+_i;\Delta^-_i, B) \vdash^+_{\mathbf{B2C}} A; \qquad (\Delta^+_j;\Delta^-_j, C) \vdash^+_{\mathbf{B2C}} A.$$

And hence the following derivation shows that $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}} A$:

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$$\dfrac{\dfrac{\begin{matrix}(\varnothing; B\wedge C)\\ \vdots\end{matrix}}{B\wedge C} \qquad \dfrac{\begin{matrix}(\Delta_i^+;\Delta_i^-,[\![B]\!])\\ \vdots\end{matrix}}{A} \qquad \dfrac{\begin{matrix}(\Delta_i^+;\Delta_i^-,[\![C]\!])\\ \vdots\end{matrix}}{A}}{A}\;(\wedge Edp) .$$

□

The *canonical model (for **B2C**)* is $\mathcal{M}_{\mathbf{B2C}} = \langle I_{\mathbf{B2C}}, \leq, v^+_{\mathbf{B2C}}, v^-_{\mathbf{B2C}}\rangle$, where

- $I_{\mathbf{B2C}}$ is the set of all prime bi-theories for **B2C**;
- $\leq$ is the order relation on set-pairs defined above;
- $(\Gamma^+;\Gamma^-) \in v^+_{\mathbf{B2C}}(p)$ iff $p \in \Gamma^+$ for $p \in \mathsf{Prop}$;
- $(\Gamma^+;\Gamma^-) \in v^-_{\mathbf{B2C}}(p)$ iff $p \in \Gamma^-$ for $p \in \mathsf{Prop}$.

LEMMA 3.3 (truth). *For any* $(\Gamma^+;\Gamma^-) \in I_{\mathbf{B2C}}$ *and any formula* A *we have*:

$$\mathcal{M}_{\mathbf{B2C}}, (\Gamma^+;\Gamma^-) \models^+ A \textit{ iff } A \in \Gamma^+; \qquad \mathcal{M}_{\mathbf{B2C}}, (\Gamma^+;\Gamma^-) \models^- A \textit{ iff } A \in \Gamma^-.$$

Proof. By induction on the complexity of A. The basis of induction follows from the definitions. Let us consider the cases of conjunction and co-implication, the other two cases are dual.

$(\wedge, +)$. By the induction hypothesis it is enough to show that

$$A \wedge B \in \Gamma^+ \text{ iff } A \in \Gamma^+ \text{ and } B \in \Gamma^+.$$

To show that, it is enough to use $(\wedge Ip)$ and $(\wedge Ep)$.

$(\wedge, -)$. It is enough to show that

$$A \wedge B \in \Gamma^- \text{ iff } A \in \Gamma^- \text{ or } B \in \Gamma^-.$$

For the right-to-left direction use $(\wedge Idp)$, the left-to-right direction follows from the definition of a prime bi-theory.

$(\prec, +)$. It is enough to show that

$$B \prec A \in \Gamma^+ \text{ iff } \forall(\Delta^+;\Delta^-) \in I_{\mathbf{B2C}}\,((\Gamma^+,\Gamma^-) \leq (\Delta^+,\Delta^-) \text{ and } A \in \Delta^- \text{ implies } B \in \Delta^+).$$

Suppose $B \prec A \in \Gamma^+$, $(\Gamma^+,\Gamma^-) \leq (\Delta^+,\Delta^-)$ and $A \in \Delta^-$. Then $B \prec A \in \Delta^+$, $A \in \Delta^-$ and one application of $(\prec Ep)$ shows that $B \in \Delta^+$.

For the other direction assume that $B \prec A \notin \Gamma^+$. If, additionally, $(\Gamma^+;\Gamma^-, A) \vdash^+_{\mathbf{B2C}} B$, then by $(\prec Ip)$ we have $(\Gamma^+;\Gamma^-) \vdash^+_{\mathbf{B2C}} B \prec A$, hence $B \prec A \in \Gamma^+$ by the definition of a prime bi-theory. This contradicts the assumption. Hence $(\Gamma^+;\Gamma^-, A) \nvdash^+_{\mathbf{B2C}} B$ and by the extension lemma there is $(\Delta^+;\Delta^-) \in I_{\mathbf{B2C}}$ such that $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$, $A \in \Delta^-$ and $(\Delta^+;\Delta^-) \nvdash^+_{\mathbf{B2C}} B$. The latter implies that $B \notin \Delta^+$ and concludes the proof.

$(\prec, -)$. Again, it is enough to show that

$$B \prec A \in \Gamma^- \text{ iff } \forall(\Delta^+;\Delta^-) \in I_{\mathbf{B2C}}\,((\Gamma^+,\Gamma^-) \leq (\Delta^+,\Delta^-) \text{ and } A \in \Delta^- \text{ implies } B \in \Delta^-).$$

The left-to-right direction is obtained easily using $(\prec Edp)$. For the other direction we assume $B \prec A \notin \Gamma^+$. As before, if $(\Gamma^+;\Gamma^-, A) \vdash^-_{\mathbf{B2C}} B$ then an application of $(\prec Idp)$ gives us a contradiction with $B \prec A \notin \Gamma^+$. Then $(\Gamma^+;\Gamma^-, A) \nvdash^-_{\mathbf{B2C}} B$ and applying the extension lemma we obtain $(\Delta^+;\Delta^-) \in I_{\mathbf{B2C}}$ such that $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$, $A \in \Delta^-$ and $B \notin \Delta^-$. □

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THEOREM 3.2 (soundness and completeness). *Where* $(\Gamma;\Delta)$ *is a set-pair,* $A \in \mathsf{Form}\,\mathcal{L}$ *and* $* \in \{+,-\}$:

$$(\Gamma^+;\Gamma^-) \vdash^*_{\mathbf{B2C}} A \text{ iff } (\Gamma^+;\Gamma^-) \models^*_{\mathbf{B2C}} A.$$

Proof. The soundness part can be established directly by an induction on the height of the corresponding derivation. We skip this part of the proof as it follows from the results in [59].

For the completeness part, assume that $(\Gamma^+;\Gamma^-) \nvdash^*_{\mathbf{B2C}} A$. Then by the extension lemma there is a prime bi-theory $(\Delta^+;\Delta^-)$ such that $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$ and $A \notin \Delta^*$. According to the truth lemma we have $\mathcal{M}_{\mathbf{B2C}}, (\Delta^+;\Delta^-) \models^{*'} B$ for every $B \in \Delta^{*'}$ and $*' \in \{+,-\}$, where $\mathcal{M}_{\mathbf{B2C}} = \langle I_{\mathbf{B2C}}, \leq, v^+_{\mathbf{B2C}}, v^-_{\mathbf{B2C}}\rangle$ is the canonical model for **B2C**. By the definition of semantic consequence relations this demonstrates that $(\Gamma^+;\Gamma^-) \nvDash^*_{\mathbf{B2C}} A$. This concludes the proof. □

COROLLARY 3.1. *A formula A is provable (refutable) in* **NB2C** *iff for any model* $\mathcal{M} = \langle I, \leq, v^+, v^-\rangle$ *and any* $x \in I$ *we have* $\mathcal{M}, x \models^+ A$ $(\mathcal{M}, x \models^- A)$.

We are now ready to clarify the status of $\neg$ and $-$ in **B2C** as negations. First, for a unary connective θ put $\theta^{(0)}A := A$ and $\theta^{(n+1)}A := \theta\theta^{(n)}A$ where $n \geq 1$. The following is established by a straightforward application of the completeness result above.

FACT 3.5. *For every* $n \in \mathbb{N}$ *the following conditions hold (where* $p \neq q$*):*

(i)	$(\neg^{(n+1)}p;\varnothing) \nvdash^+_{\mathbf{B2C}} q;$	(ii)	$(\neg^{(n+1)}p;\varnothing) \nvdash^+_{\mathbf{B2C}} \neg^{(n)}p;$
(iii)	$(\varnothing;\varnothing) \nvdash^+_{\mathbf{B2C}} \neg^{(n+1)}p;$	(iv)	$(\neg^{(n)}p;\varnothing) \nvdash^+_{\mathbf{B2C}} \neg^{(n+1)}p;$
(v)	$(-^{(n+1)}p;\varnothing) \nvdash^-_{\mathbf{B2C}} q;$	(vi)	$(-^{(n+1)}p;\varnothing) \nvdash^-_{\mathbf{B2C}} -^{(n)}p;$
(vii)	$(\varnothing;\varnothing) \nvdash^-_{\mathbf{B2C}} -^{(n+1)}p;$	$(viii)$	$(-^{(n)}p;\varnothing) \nvdash^-_{\mathbf{B2C}} -^{(n+1)}p.$

Conditions (i)–(iv) are natural variations of requirements for a minimal negation put forward in [29] (under the name *mid-negation*, where mid stands for "minimally decent"), while conditions (v)–$(viii)$ are their natural duals.

§4. Expansions.

We will now consider expansions of the language $\mathcal{L}$ of **B2C** by constants $\top$, $\bot$, **n**, the unary connective $\sim$, as well as the binary connectives $\Rightarrow$ and $\Rightarrow\!\!\!\!\!-$. The toggling negation $\sim$ as well as the constants $\top$ and $\bot$ we have already discussed at length. Constant **n** (for *neither*) corresponds to a statement for which neither support of truth nor support of falsity are available. Along with its dual constant **b** (for *both*; corresponds to a statement for which both support of truth and support of falsity are available) it is a very natural addition given the informational interpretation of our semantics [8]. Our interest in them here stems from the fact that the addition of both constants seems to often lead to a stark increase in expressive power (e.g. [2, 3, 37]). The reason we do not consider an expansion of **B2C** with **b** is that this constant turns out to be definable in the logic, as we will see later. Likewise, *strong implication* is an important connective for logics with toggling negation as it is involved in defining a congruence relation on the set of formulas and, as a consequence, in obtaining algebraizability results (see, e.g. [17] for the case of **C**). In the presence of the toggling negation, strong implication can be defined as $A \Rightarrow B := (A \to B) \wedge (\sim B \to \sim A)$. Since we are working in a language without strong implication it makes sense to consider this connective

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separately, as a primitive one. Meanwhile, strong co-implication $\Lleftarrow$ can be construed as its natural dual by having the definition $A \Lleftarrow B := (A \prec B) \vee ({\sim}B \prec {\sim}A)$ in mind.

Then denote by $\mathcal{L}_{all}$ the language

$$A ::= p \mid \top \mid \bot \mid \mathbf{n} \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A) \mid (A \Rightarrow A) \mid (A \Lleftarrow A).$$

We will sometimes use the notation $\mathcal{L}_X^Y$, where $X, Y \subseteq \mathcal{L}_{all}$, $X \cap Y = \varnothing$, to denote the result of adding connectives in X to and simultaneously removing connectives in Y from $\mathcal{L}$. Thus, for instance, $\mathcal{L}_{\{\top,\bot\}}^{\rightarrow}$ represents the language with connectives from $\{\wedge, \vee, \prec, \top, \bot\}$.

4.1. Non-definability. Before we introduce these new connectives via natural deduction rules it is imperative to show that they are not already definable in **B2C** via a formula. The argument will be semantical, so by a connective $c(p_1, \dots, p_n)$ here we will understand a pair of support of truth and support of falsity conditions of the form

$$\mathcal{M}, x \vDash^+ c(A_1, \dots, A_n) \text{ iff } S^+(x, A_1, \dots, A_n);$$
$$\mathcal{M}, x \vDash^- c(A_1, \dots, A_n) \text{ iff } S^-(x, A_1, \dots, A_n).$$

Then by $c(p_1, \dots, p_n)$ being *definable* we will mean that there is a formula $B(p_1, \dots, p_n, q_1, \dots, q_m)$ (where $q_1, \dots, q_m$ are parameters) in the given language such that

$$\mathcal{M}, x \vDash^+ B(A_1, \dots, A_n, q_1, \dots, q_m) \text{ iff } S^+(x, A_1, \dots, A_n);$$
$$\mathcal{M}, x \vDash^- B(A_1, \dots, A_n, q_1, \dots, q_m) \text{ iff } S^-(x, A_1, \dots, A_n)$$

holds for any model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$, any $x \in I$ and any formulas $A_1, \dots, A_n$. In particular, we will sometimes talk of definability in classical models, when we restrict our attention to those models.

To proceed we list the support of truth and falsity conditions for these connectives; later in the section we will provide them with natural deduction rules and prove the corresponding completeness results. Then assuming $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ to be a model put

$$\begin{array}{lll}
\mathcal{M}, x \models^+ \top & \text{and } \mathcal{M}, x \not\models^- \top; & \\
\mathcal{M}, x \not\models^+ \bot & \text{and } \mathcal{M}, x \models^- \bot; & \\
\mathcal{M}, x \not\models^+ \mathbf{n} & \text{and } \mathcal{M}, x \not\models^- \mathbf{n}; & \\
\mathcal{M}, x \models^+ \mathbf{b} & \text{and } \mathcal{M}, x \models^- \mathbf{b}; & \\
\mathcal{M}, x \models^+ {\sim}A & \text{iff} & \mathcal{M}, x \models^- A; \\
\mathcal{M}, x \models^- {\sim}A & \text{iff} & \mathcal{M}, x \models^+ A; \\
\mathcal{M}, x \models^+ A \Rightarrow B & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^+ A \text{ implies } \mathcal{M}, y \models^+ B) \text{ and} \\
 & & \forall y \geq x(\mathcal{M}, y \models^- B \text{ implies } \mathcal{M}, y \models^- A); \\
\mathcal{M}, x \models^- A \Rightarrow B & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^+ A \text{ implies } \mathcal{M}, y \models^- B) \text{ or} \\
 & & \forall y \geq x(\mathcal{M}, y \models^- B \text{ implies } \mathcal{M}, y \models^+ A); \\
\mathcal{M}, x \models^+ B \Lleftarrow A & \text{iff} & \forall y \geq x(\mathcal{M}, y \models^- A \text{ implies } \mathcal{M}, y \models^+ B) \text{ or} \\
 & & \forall y \geq x(\mathcal{M}, y \models^+ B \text{ implies } \mathcal{M}, y \models^- A);
\end{array}$$

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$$\mathcal{M}, x \models^- B \Rightarrow\!\!\prec A \text{ iff } \forall y \geq x(\mathcal{M}, y \models^- A \text{ implies } \mathcal{M}, y \models^- B) \text{ and }$$
$$\forall y \geq x(\mathcal{M}, y \models^+ B \text{ implies } \mathcal{M}, y \models^+ A).$$

As we discussed at the beginning of the section, the constant **b** turns out to be definable in **B2C**. In fact, we have already established this in Fact 3.3.

LEMMA 4.1. *The* **b** *constant is definable in* $\mathcal{L}$ *as* $-\Pi$ (*or as* $\neg \amalg$).

Proof. Follows from Fact 3.3 and the completeness result in the previous section. □

We proceed to the non-definability results. The main technical lemma is the following:

LEMMA 4.2. *Suppose* $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *is a model.*

1. *If* $\mathcal{M}, x \vDash^+ p$ *for any* $p \in \mathsf{Prop}$ *and* $x \in I$, *then* $\mathcal{M}, x \vDash^+ A$ *for any formula* A *and* $x \in I$.
2. *If* $\mathcal{M}, x \vDash^- p$ *for any* $p \in \mathsf{Prop}$ *and* $x \in I$, *then* $\mathcal{M}, x \vDash^- A$ *for any formula* A *and* $x \in I$.

Proof. Both items are proved via a simple induction on the complexity of A (simultaneously for all x). Consider the first item and the case of $A = B \prec C$. By the induction hypothesis we have $\mathcal{M}, x \vDash^+ B$ for all $x \in I$. But then trivially $\mathcal{M}, x \vDash^- C$ implies $\mathcal{M}, x \vDash^+ B$ for all $x \in I$, hence $\mathcal{M}, x \vDash^+ B \prec C$ for all $x \in I$. □

Note that the claims of Lemma 4.2 do not hold for **2Int** and **N4**.

THEOREM 4.1. *The connectives* $\Rightarrow$, $\Rightarrow\!\!\prec$, $\bot$, $\top$, **n** *and* $\sim$ *are not definable in* **B2C**.

Proof. $\Rightarrow$. Consider a model $\mathcal{M} = \langle \{x\}, \leq, v^+, v^- \rangle$ such that $x \in v^+(r)$ for all $r \in \mathsf{Prop}$, $x \in v^-(q)$ and $x \notin v^-(p)$. Then we should have $\mathcal{M}, x \nvDash^+ p \Rightarrow q$, yet according to the previous lemma we have $\mathcal{M}, x \vDash^+ A$ for any formula A.

$\Rightarrow\!\!\prec$. Similarly, consider a model $\mathcal{M} = \langle \{x\}, \leq, v^+, v^- \rangle$ such that $x \in v^+(q)$, $x \notin v^+(p)$ and $x \in v^-(r)$ for all $r \in \mathsf{Prop}$. Then $\mathcal{M}, x \nvDash^- q \Rightarrow\!\!\prec p$ according to the support of falsity clause for $\Rightarrow\!\!\prec$ and $\mathcal{M}, x \vDash^- A$ for any formula A according to the previous lemma.

$\sim$. Consider a model $\mathcal{M} = \langle \{x\}, \leq, v^+, v^- \rangle$ such that $x \in v^+(r)$ for all $r \in \mathsf{Prop}$ and $x \notin v^-(p)$. Then we should have $\mathcal{M}, x \nvDash^+ \sim p$, yet $\mathcal{M}, x \vDash^+ A$ for any formula A.

The constants are considered similarly. □

Next, we shall check that the choice of having both implication and co-implication in $\mathcal{L}$ is a reasonable one. This will be established by a McKinsey-style argument [31] showing that implication and co-implication are inter-independent in the absence of the toggling negation, even when $\bot$ and $\top$ are present.

PROPOSITION 4.1. *The following statements hold.*

(i) $\prec$ *is not definable in* $\mathcal{L}^{\prec}_{\{\top,\bot\}}$.
(ii) $\rightarrow$ *is not definable in* $\mathcal{L}^{\rightarrow}_{\{\top,\bot\}}$.

Proof. (i) Let $I := \{x, y\}$ and $\leq := \{\langle x, x\rangle, \langle y, y\rangle\}$ and $\mathcal{M} := \langle I, \leq, v^+, v^- \rangle$ be such that $\langle v^+(p), v^-(p)\rangle \in \{\langle I, I\rangle, \langle I, \{x\}\rangle, \langle I, \emptyset\rangle, \langle \emptyset, I\rangle\}$ for all $p \in \mathsf{Prop}$. Then we have the

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next tables of $\langle v^+(A), v^-(A)\rangle$ for each formula A in $\mathcal{L}^{\prec}_{\{\top,\bot\}}$: note that $\top$ and $\bot$ are assigned the pairs $\langle I, \emptyset\rangle$ and $\langle \emptyset, I\rangle$, respectively.

$\wedge$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\vee$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,I\rangle$
$\langle I,\{x\}\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$	$\langle \emptyset,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{x\}\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$
$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$

$\rightarrow$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$
$\langle I,\{x\}\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$
$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$

Thus any formula in $\mathcal{L}^{\prec}_{\{\top,\bot\}}$ has one of the four pairs of values. Now suppose that $\langle v^+(p), v^-(p)\rangle = \langle I, \{x\}\rangle$ and $\langle v^+(q), v^-(q)\rangle = \langle \emptyset, I\rangle$. Then $v^+(q \mathrel{-\!\!\prec} p) = \{y\}$ and $v^-(q \mathrel{-\!\!\prec} p) = I$, but $\langle \{y\}, I\rangle$ does not belong to the collection of pairs. Therefore it cannot be defined in $\mathcal{L}^{\prec}_{\{\top,\bot\}}$.

(ii) This time, let v^+, v^- be s.t. $\langle v^+(p), v^-(p)\rangle \in \{\langle I,I\rangle, \langle I,\emptyset\rangle, \langle \{x\}, I\rangle, \langle \emptyset, I\rangle\}$ for all $p \in \mathsf{Prop}$. Then we obtain the following tables for $\langle v^+(A), v^-(A)\rangle$ for each formula A in $\mathcal{L}^{\rightarrow}_{\{\top,\bot\}}$.

$\wedge$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$	$\vee$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$
$\langle \{x\},I\rangle$	$\langle \{x\},I\rangle$	$\langle \{x\},I\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$	$\langle \{x\},I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \{x\},I\rangle$
$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$

$\mathrel{-\!\!\prec}$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle \{x\},I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$
$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \{x\},I\rangle$	$\langle \emptyset,I\rangle$

Thus any formula in $\mathcal{L}^{\rightarrow}_{\{\top,\bot\}}$ has one of the four pairs of values. Now suppose that $\langle v^+(p), v^-(p)\rangle = \langle \{x\}, I\rangle$ and $\langle v^+(q), v^-(q)\rangle = \langle I, \emptyset\rangle$. Then $v^+(p \rightarrow q) = I$ and $v^-(p \rightarrow q) = \{y\}$, but $\langle I, \{y\}\rangle$ does not belong to the collection of pairs. Therefore it cannot be defined in $\mathcal{L}^{\rightarrow}_{\{\top,\bot\}}$. □

4.2. Natural deduction rules and completeness. We are ready to introduce natural deduction rules corresponding to the new connectives introduced in this section. These rules are listed in Tables 4 and 5.

The subsystems/expansions of **B2C** defined with some of these rules will be denoted similarly to their linguistic counterpart: e.g. $\mathbf{B2C}^{\rightarrow}_{\{\top,\bot\}}$ is a system in $\mathcal{L}^{\rightarrow}_{\{\top,\bot\}}$ obtained by removing $\rightarrow$-related rules from **B2C** and adding the $\top$- and $\bot$-related rules from Table 4.

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Table 4. *Natural deduction rules for* $\top$, $\bot$, $\mathbf{n}$ *and* $\sim$

$$\dfrac{(\varnothing;\varnothing)}{\top}\,(\top Ip) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \top\\ \cdot\;\cdot\;\cdot\\ A\end{array}\,(\top Edp) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \bot\\ \cdot\;\cdot\;\cdot\\ A\end{array}\,(\bot Ep) \qquad \begin{array}{c}(\varnothing;\varnothing)\\ \hline\hline \bot\end{array}\,(\bot Idp)$$

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}\\ \cdot\;\cdot\;\cdot\\ A\end{array}\,(\mathbf{n}Ep) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}\\ \cdot\;\cdot\;\cdot\\ A\end{array}\,(\mathbf{n}Edp)$$

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline A\\ \hline {\sim}A\end{array}\,({\sim}\, Ip) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline {\sim}\, A\\ \hline\hline A\end{array}\,({\sim}\, Ep) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline A\\ \hline\hline {\sim}\, A\end{array}\,({\sim}\, Idp) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline {\sim}\, A\\ \hline A\end{array}\,({\sim}\, Edp)$$

REMARK 4.1. *Notice that implication and co-implication appear explicitly in rules* $(\Rightarrow Edp)$ *and* $(\Rightarrow\!\!\!\!-\, Ep)$, *which makes these rules not 'pure'* [15, *p.* 257]. *Although this feature does not cause a problem in our current enquiry, the rules still make it appear that strong implication/co-implication conceptually depend on implication/co-implication, which may not necessarily be the case. One way to rectify this would be to use* higher-order rules [51]. *On the other hand, the use of a more complex framework, if it turns out to be essential, may also provide a reason not to include strong implication/co-implication in one's preferred language.*

We proceed to extend our completeness result to the new connectives. Suppose $\mathcal{L}_i$ is a language such that $\mathcal{L} \subseteq \mathcal{L}_i \subseteq \mathcal{L}_{all}$ and $\mathbf{B2C}_i$ is its corresponding expansion.

We have to modify the notion of a prime bi-theory. Note that for **B2C** itself there is a prime bi-theory $\langle \Gamma^+, \Gamma^- \rangle$ such that both Γ^+ and Γ^- coincide with the set of all formulas. This is consistent with the semantics since we can have support of truth and support of falsity for all formulas in a given state. This is no longer the case once we introduce one of the constants $\top$, $\bot$ or $\mathbf{n}$. On the other hand, the presence of $\Rightarrow$ or $\Rightarrow\!\!\!\!-$ forces us to adopt properties similar to those of the constructive properties for conjunction and disjunction.

Then a *bi-theory (for* $\mathbf{B2C}_i$*)* is a set-pair $(\Gamma^+;\Gamma^-)$ such that for any formula A and $* \in \{+,-\}$ we have

$$(\Gamma^+,\Gamma^-) \vdash^*_{\mathbf{B2C}_i} A \text{ implies } A \in \Gamma^*.$$

A bi-theory $(\Gamma^+;\Gamma^-)$ is *prime (for* $\mathbf{B2C}_i$*)* if Γ^+, Γ^- are both non-empty; Γ^+ satisfies the constructive disjunction property, Γ^- satisfies the constructive conjunction property and, additionally:

1. $\Gamma^+ \neq \mathsf{Form}\,\mathcal{L}_i$ in case $\bot \in \mathcal{L}_i$ or $\mathbf{n} \in \mathcal{L}_i$;
2. $\Gamma^- \neq \mathsf{Form}\,\mathcal{L}_i$ in case $\top \in \mathcal{L}_i$ or $\mathbf{n} \in \mathcal{L}_i$;

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Table 5. *Natural deduction rules for strong implication and strong co-implication*

$$\begin{array}{c}\begin{array}{c}([A],\Gamma;\Delta)\\ \vdots \\ \hline B\end{array}\qquad\begin{array}{c}(\Gamma';\Delta',[\![B]\!])\\ \vdots \\ \hline\hline A\end{array}\\ \hline A\Rightarrow B\end{array}\ (\Rightarrow Ip)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta;\Gamma)\\ \vdots \\ \hline A\Rightarrow B\end{array}\qquad\begin{array}{c}(\Delta';\Gamma')\\ \vdots \\ \hline A\end{array}\\ \hline B\end{array}\ (\Rightarrow Ep)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta;\Gamma)\\ \vdots \\ \hline A\Rightarrow B\end{array}\qquad\begin{array}{c}(\Delta';\Gamma')\\ \vdots \\ \hline\hline B\end{array}\\ \hline\hline A\end{array}\ (\Rightarrow Ep)$$

$$\begin{array}{c}([A],\Delta;\Gamma)\\ \vdots \\ \hline\hline B\\ \hline\hline A\Rightarrow B\end{array}\ (\Rightarrow Idp)
\qquad
\begin{array}{c}(\Delta;\Gamma,[\![B]\!])\\ \vdots \\ \hline A\\ \hline\hline A\Rightarrow B\end{array}\ (\Rightarrow Idp)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta;\Gamma)\\ \vdots \\ \hline\hline A\Rightarrow B\end{array}\qquad\begin{array}{c}(\Delta';\Gamma',[\![A\rightarrow B]\!])\\ \vdots \\ C\end{array}\qquad\begin{array}{c}([A\mathrel{-\!\!\prec} B],\Delta'';\Gamma'')\\ \vdots \\ C\end{array}\\ \hdashline C\end{array}\ (\Rightarrow Edp)$$

$$\begin{array}{c}(\Delta;\Gamma,[\![A]\!])\\ \vdots \\ \hline B\\ \hline B\mathrel{=\!\!\!\prec} A\end{array}\ (\mathrel{=\!\!\!\prec} Ip)
\qquad
\begin{array}{c}([B],\Delta;\Gamma)\\ \vdots \\ \hline\hline A\\ \hline B\mathrel{=\!\!\!\prec} A\end{array}\ (\mathrel{=\!\!\!\prec} Ip)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta,\Gamma)\\ \vdots \\ \hline B\mathrel{=\!\!\!\prec} A\end{array}\qquad\begin{array}{c}([B\mathrel{-\!\!\prec} A],\Delta';\Gamma')\\ \vdots \\ C\end{array}\qquad\begin{array}{c}(\Delta'';\Gamma'',[\![B\rightarrow A]\!])\\ \vdots \\ C\end{array}\\ \hdashline C\end{array}\ (\mathrel{=\!\!\!\prec} Ep)$$

$$\begin{array}{c}\begin{array}{c}(\Delta;\Gamma,[\![A]\!])\\ \vdots \\ \hline\hline B\end{array}\qquad\begin{array}{c}([B],\Delta';\Gamma')\\ \vdots \\ \hline A\end{array}\\ \hline\hline B\mathrel{=\!\!\!\prec} A\end{array}\ (\mathrel{=\!\!\!\prec} Idp)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta;\Gamma)\\ \vdots \\ \hline\hline B\mathrel{=\!\!\!\prec} A\end{array}\qquad\begin{array}{c}(\Delta';\Gamma')\\ \vdots \\ \hline\hline A\end{array}\\ \hline\hline B\end{array}\ (\mathrel{=\!\!\!\prec} Edp)
\qquad
\begin{array}{c}\begin{array}{c}(\Delta;\Gamma)\\ \vdots \\ \hline\hline B\mathrel{=\!\!\!\prec} A\end{array}\qquad\begin{array}{c}(\Delta';\Gamma')\\ \vdots \\ \hline B\end{array}\\ \hline A\end{array}\ (\mathrel{=\!\!\!\prec} Edp)$$

3. $A \Rightarrow B \in \Gamma^-$ implies $A \to B \in \Gamma^-$ or $A \prec B \in \Gamma^+$ in case $\Rightarrow \in \mathcal{L}_i$;
4. $A \Yleft B \in \Gamma^+$ implies $A \prec B \in \Gamma^+$ or $A \to B \in \Gamma^-$ in case $\Yleft\ \in \mathcal{L}_i$.

The *canonical model (for* **B2C**$_i$*)* is defined exactly as for **B2C**.

First, we see how this modification changes the proof of the extension lemma.

Lemma 4.3 (extension; for expansions). *Fix some expansion* **B2C**$_i$ *where* $\mathcal{L} \subseteq \mathcal{L}_i \subseteq \mathcal{L}_{all}$. *Suppose* $(\Gamma^+;\Gamma^-)$ *is a set-pair, A is a formula and* $* \in \{+,-\}$. *If* $(\Gamma^+;\Gamma^-) \nvdash^*_{\mathbf{B2C}_i} A$ *then there is a prime bi-theory* (Δ^+,Δ^-) *such that* $(\Gamma^+;\Gamma^-) \leq (\Delta^+;\Delta^-)$ *and* $(\Delta^+,\Delta^-) \nvdash^*_{\mathbf{B2C}_i} A$.

Proof. We consider the case $* = +$ as an example. So suppose $(\Gamma^+;\Gamma^-) \nvdash^+_{\mathbf{B2C}_i} A$ and from there (Δ^+,Δ^-) is obtained like in the proof of Lemma 3.2. That is $\Delta^* = \bigcup_{i\in\mathbb{N}} \Delta^*_i$ for $* \in \{+,-\}$, where $\Delta^+_0 = \Gamma^+$, $\Delta^-_0 = \Gamma^-$ and for any $i \in \mathbb{N}$:

$$(\Delta^+_{i+1};\Delta^-_{i+1}) := \begin{cases} (\Delta^+_i, B;\Delta^-_i), & \text{if } x_i = B^+ \text{ and } (\Delta^+_i, B;\Delta^-_i) \nvdash^+_{\mathbf{B2C}_i} A;\\ (\Delta^+_i;\Delta^-_i, B), & \text{if } x_i = B^- \text{ and } (\Delta^+_i;\Delta^-_i, B) \nvdash^+_{\mathbf{B2C}_i} A;\\ (\Delta^+_i;\Delta^-_i), & \text{otherwise;} \end{cases}$$

and $\{x_i \mid i \in \mathbb{N}\}$ is some enumeration of the set X of all expressions of the form A^+ and A^- ($A \in \mathsf{Form}\,\mathcal{L}_i$).

All of the properties established in Lemma 3.2 are proved in exactly the same way, so we consider them to be already established. We show the additional properties.

1. Suppose $\mathbf{n} \in \mathcal{L}_i$ and $\Delta^+ = \mathsf{Form}\,\mathcal{L}_i$. Then $\mathbf{n} \in \Delta^+$ and by $(\mathbf{n}Ep)$ we have $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}_i} A$, which contradicts the already established. The case $\bot \in \mathcal{L}_i$ is treated similarly.

2. Suppose $\mathbf{n} \in \mathcal{L}_i$ and $\Delta^- = \mathsf{Form}\,\mathcal{L}_i$. Then $\mathbf{n} \in \Delta^-$ and by $(\mathbf{n}Edp)$ we have $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}_i} A$. The case $\top \in \mathcal{L}_i$ is similar.

3. Suppose $\Rightarrow \in \mathcal{L}_i$. Assume additionally that $B \Rightarrow C \in \Delta^-$ but $B \to C \notin \Delta^-$ and $B \prec C \notin \Delta^+$. Then there are $i, j \in \mathbb{N}$ such that $x_i = B \to C^-$, $x_j = B \prec C^+$ and

$$(\Delta^+_i;\Delta^-_i, B \to C) \vdash^+_{\mathbf{B2C}} A; \qquad (\Delta^+_j, B \prec C;\Delta^-_j) \vdash^+_{\mathbf{B2C}} A.$$

Then the following derivation

$$\dfrac{\dfrac{\dfrac{(\varnothing; B \Rightarrow C)}{\vdots}}{B \Rightarrow C} \qquad \dfrac{\dfrac{(\Delta^+_i;\Delta^-_i, [\![B \to C]\!])}{\vdots}}{A} \qquad \dfrac{\dfrac{(\Delta^+_i, [B \prec C];\Delta^-_i)}{\vdots}}{A}}{A}\,(\Rightarrow Edp),$$

shows that $(\Delta^+;\Delta^-) \vdash^+_{\mathbf{B2C}_i} A$, which gives us a contradiction. The case of $\Yleft\ \in \mathcal{L}_i$ is considered similarly. □

Lemma 4.4 (truth; for expansions). *Fix some expansion* **B2C**$_i$ *where* $\mathcal{L} \subseteq \mathcal{L}_i \subseteq \mathcal{L}_{all}$. *For any* $(\Gamma^+;\Gamma^-) \in I_{\mathbf{B2C}_i}$ *and any* $A \in \mathsf{Form}\,\mathcal{L}_i$ *we have*:

$$\mathcal{M}_{\mathbf{B2C}_i}, (\Gamma^+;\Gamma^-) \vDash^+ A \textit{ iff } A \in \Gamma^+; \qquad \mathcal{M}_{\mathbf{B2C}_i}, (\Gamma^+;\Gamma^-) \vDash^- A \textit{ iff } A \in \Gamma^-.$$

Proof. We consider the case of $\Rightarrow$, the rest is either similar or simple.

$(\Rightarrow, +)$. It is enough to show that for $(\Gamma^+; \Gamma^-) \in \boldsymbol{I}_{\mathbf{B2C}_i}$

$$A \Rightarrow B \in \Gamma^+ \text{ iff } \forall(\Delta^+; \Delta^-) \geq (\Gamma^+; \Gamma^-) \big((A \in \Delta^+ \text{ implies } B \in \Delta^+) \text{ and } (B \in \Delta^- \text{ implies } A \in \Delta^-)\big).$$

The left-to-right direction follows easily from the definition of $\leq$ using rule $(\Rightarrow Ep)$. For right-to-left, assume that $A \Rightarrow B \notin \Gamma^+$. Suppose also that both of the following conditions hold:

$$(i) \quad (\Gamma^+, A; \Gamma^-) \vdash^+_{\mathbf{B2C}_i} B; \qquad (ii) \quad (\Gamma^+; \Gamma^-, B) \vdash^-_{\mathbf{B2C}_i} A.$$

Then by $(\Rightarrow I)$ we conclude that $(\Gamma^+; \Gamma^-) \vdash^+_{\mathbf{B2C}_i} A \Rightarrow B$, thus $A \Rightarrow B \in \Gamma^+$, which contradicts the assumption. Then either (i) or (ii) does not hold. In the former case by the extension lemma we obtain $(\Delta^+; \Delta^-) \in \boldsymbol{I}_{\mathbf{B2C}_i}$ such that $(\Gamma^+; \Gamma^-) \leq (\Delta^+; \Delta^-)$, $A \in \Delta^+$ and $B \notin \Delta^+$; in the latter case by again the extension lemma we find $(\Delta^+; \Delta^-) \in \boldsymbol{I}_{\mathbf{B2C}_i}$ such that $(\Gamma^+; \Gamma^-) \leq (\Delta^+; \Delta^-)$, $B \in \Delta^-$ and $A \notin \Delta^-$, as required.

$(\Rightarrow, -)$. Similarly, it is enough to show that

$$A \Rightarrow B \in \Gamma^- \text{ iff } \quad \forall(\Delta^+; \Delta^-) \geq (\Gamma^+; \Gamma^-)\ (A \in \Delta^+ \text{ implies } B \in \Delta^-) \text{ or } \forall(\Delta^+; \Delta^-) \geq (\Gamma^+; \Gamma^-)\ (B \in \Delta^- \text{ implies } A \in \Delta^+).$$

If $A \Rightarrow B \in \Gamma^-$, then according to the definition of prime bi-theories (in case $\Rightarrow \in \mathcal{L}_i$) we have $A \rightarrow B \in \Gamma^-$ or $A \prec B \in \Gamma^+$. As already established, the former implies the first disjunct and the latter implies the second disjunct of the condition on the right-hand side.

Suppose $A \Rightarrow B \notin \Gamma^-$. If, additionally, $(\Gamma^+, A; \Gamma^-) \vdash^-_{\mathbf{B2C}_i} B$, then by $(\Rightarrow Idp)$ we conclude that $(\Gamma^+; \Gamma^-) \vdash^-_{\mathbf{B2C}_i} A \Rightarrow B$, which contradicts our assumption. Then applying the extension lemma we obtain $(\Delta^+; \Delta^-) \in \boldsymbol{I}_{\mathbf{B2C}_i}$ such that $(\Gamma^+; \Gamma^-) \leq (\Delta^+; \Delta^-)$, $A \in \Delta^+$ and $B \notin \Delta^-$, as required.

Similarly $(\Gamma^+; \Gamma^-, B) \vdash^+_{\mathbf{B2C}_i} A$ cannot be the case by $(\Rightarrow Idp)$ and the assumption, so one application of the extension lemma gives us $(\Delta^+; \Delta^-) \in \boldsymbol{I}_{\mathbf{B2C}_i}$ such that $(\Gamma^+; \Gamma^-) \leq (\Delta^+; \Delta^-)$, $B \in \Delta^-$ and $A \notin \Delta^+$. This concludes the proof. □

THEOREM 4.2 (soundness and completeness). *Fix some expansion* $\mathbf{B2C}_i$ *where* $\mathcal{L} \subseteq \mathcal{L}_i \subseteq \mathcal{L}_{all}$. *Suppose* $(\Gamma; \Delta)$ *is a set-pair in* $\mathcal{L}_i$, $A \in \mathsf{Form}\,\mathcal{L}_i$ *and* $* \in \{+, -\}$. *Then*

$$(\Gamma^+; \Gamma^-) \vdash^*_{\mathbf{B2C}} A \textit{ iff } (\Gamma^+; \Gamma^-) \vDash^*_{\mathbf{B2C}} A.$$

Proof. The soundness part is established as usual via a routine induction on the height of the corresponding derivation tree. The completeness part is the same as in Theorem 3.2. □

§5. Restricted rules for constants. In the rules for $\top$ and $\bot$ given in §3.1, we made use of dotted lines to permit an inference step with respect to both proofs and dual proofs. This formulation is necessary to make the natural deduction system complete with respect to the semantics in absence of the toggling negation. On the other hand, it is perhaps up to debate whether this formulation is always acceptable. $(\top Ep)$, for instance, allows one to infer from a dual proof of $\top$ *both* the proof and dual proof of any formula. This is unlike $(\top Ip)$ that only concerns proofs. Some may find this discrepancy undesirable. Or from a different perspective, consider a scenario in which

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an intuitionistic logician learns about bilateralist perspectives, and accordingly decides to extend one's logical framework (here taken to be a natural deduction system) so as to incorporate the dimension of refutation. The simplest way to achieve this would be to add rules about dual proof for the intuitionistic connectives without making any changes to the familiar rules about proof. In particular, the proof rules for $\bot$ should remain unchanged, only allowing one to infer arbitrary proofs from a proof for $\bot$. A logician adopting this kind of approach would thus fail to arrive at the rules with dotted lines.[9]

This motivates us to study alternative rules for constants $\top$, $\bot$ and **n**. For $\top$, there can be three forms of $(\top Edp)$:

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \top_1\\ \cdots\cdots\\ A\end{array}\ (\top_1 Edp) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \top_2\\ \hline A\end{array}\ (\top_2 Edp) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \top_3\\ \hline\hline A\end{array}\ (\top_3 Edp),$$

where $(\top_1 Edp)$ is identical to the rule $(\top Edp)$ in Table 4. Notice that they become inter-derivable if the toggling negation is present.

We can also think of three kinds of $(\bot Ep)$:

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \bot_1\\ \cdots\cdots\\ A\end{array}\ (\bot_1 Ep) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \bot_2\\ \hline A\end{array}\ (\bot_2 Ep) \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \bot_3\\ \hline\hline A\end{array}\ (\bot_3 Ep),$$

where $(\bot_1 Ep)$ is identical to $(\bot Ep)$ in Table 4. For **n**, we have three choices for both $(\mathbf{n}Ep)$ and $(\mathbf{n}Edp)$, resulting in nine notions. Not all of them are distinct, however. For example, a pair of rules:

9 It must be admitted that the same reasoning can be used against the dotted lines in $(\vee Ep)$. Although we will not address this question for $(\vee Ep)$ (and $(\wedge Edp)$) in the present paper, it is certainly of interest to consider different forms of disjunction/conjunction rules as well. At the same time, it might be possible to defend the current rules on the ground of an analogy with the fact that e.g. the generalized rule for conjunction (cf. e.g. [32]):

$$\begin{array}{ccc}(\Delta;\Gamma) & & ([A],[B],\Delta';\Gamma')\\ \vdots & & \vdots\\ \cline{1-1} A\wedge B & & C\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ & C & \end{array}$$

is admissible in **NB2C**, independent of how the dual proof rules for conjunction are formulated.

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$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{c}\\ \cdots\\ A\end{array} \qquad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{c}\\ \hline A\end{array},$$

gives the constant in Table 4: a dual proof of the constant enables to prove any formula as well.

$$\begin{array}{c}\hline\hline \mathbf{c}\\ \hline \mathbf{c}\\ \cdots\\ A\end{array}$$

Consequently, we end up instead with six distinct variations of **n** listed below:

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_1\\ \cdots\\ A\end{array}(\mathbf{n}_1Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_1\\ \cdots\\ A\end{array}(\mathbf{n}_1Edp) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_2\\ \cdots\\ A\end{array}(\mathbf{n}_2Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_2\\ \hline\hline A\end{array}(\mathbf{n}_2Edp)$$

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_3\\ \hline A\end{array}(\mathbf{n}_3Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_3\\ \cdots\\ A\end{array}(\mathbf{n}_3Edp) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_4\\ \hline A\end{array}(\mathbf{n}_4Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_4\\ \hline A\end{array}(\mathbf{n}_4Edp)$$

$$\begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_5\\ \hline A\end{array}(\mathbf{n}_5Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_5\\ \hline\hline A\end{array}(\mathbf{n}_5Edp) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline \mathbf{n}_6\\ \hline\hline A\end{array}(\mathbf{n}_6Ep) \quad \begin{array}{c}(\Delta;\Gamma)\\ \vdots\\ \hline\hline \mathbf{n}_6\\ \hline\hline A\end{array}(\mathbf{n}_6Edp).$$

A natural question now is how these constants can be captured semantically. If we use, like before, the absence of the support of truth/falsity at a state as forcing conditions for them, then it does not seem obvious e.g. how a support of truth of $\bot_2$ gives only the supports of truth for other formulas, and not of the falsity (assuming, at least, the classical metalogic).

For this reason, we shall employ a non-standard Kripke semantics introduced by K. Segerberg [53] and W. Veldman [55]. This type of semantics allows a constant, e.g. $\bot$ of intuitionistic logic, to be forced in a state. The set of such worlds can be subjected to additional conditions, resulting in different logics.

Here we shall concentrate on the case for $\mathbf{n}_5$, and show how a system (whose syntactic consequence relation will be denoted by $\vdash_q^*$) with the constant (in $\mathcal{L}_{\mathbf{n}_5}$) corresponds to a non-standard semantics.

Definition 5.1. *Let a* queer model[10] $\mathcal{M}_q$ *be a quadruple* $\langle I, \leq, Q^+, Q^-, v^+, v^-\rangle$ *s.t.* (*i*) $\langle I, \leq, v^+, v^-\rangle$ *is a model,* (*ii*) $Q^+, Q^- \subseteq I$ *are upward closed and* (*iii*) $Q^* \subseteq v^*(p)$ *for any*

[10] Segerberg uses the term *queer* in a more limited sense, to refer to a model s.t. $I = Q$.

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$p \in \mathsf{Prop}$ and $* \in \{+,-\}$. We define the support of truth $\models^+_q$ and support of falsity $\models^-_q$ relations as before, except that:

$$\mathcal{M}_q, x \models^+_q \mathbf{n}_5 \text{ iff } x \in Q^+$$
$$\mathcal{M}_q, x \models^-_q \mathbf{n}_5 \text{ iff } x \in Q^-.$$

The corresponding semantical consequence relations $\vDash^+_q$ an $\vDash^-_q$ are defined as in Definition 3.5.

It is not difficult to recognize from this example how non-standard semantics for other restricted constants can be formulated. When it comes to other types of neither constants, for $\mathbf{n}_2$ ($\mathbf{n}_3$), we can additionally impose $Q^+ \subseteq Q^-$ ($Q^- \subseteq Q^+$). Then for $\mathbf{n}_6$ ($\mathbf{n}_4$), we have to drop the condition about Q^+ from v^+ (Q^- from v^-) in the semantics of $\mathbf{n}_2$ ($\mathbf{n}_3$).

The above clause for $\mathbf{n}_5$ clearly succeeds in restricting the constant to imply only one of support of truth/support of falsity for propositional variables. What becomes crucial then is that this property can be generalized to all formulas in $\mathcal{L}_{\mathbf{n}_5}$.

LEMMA 5.1. *For any formula A, $(\emptyset;\emptyset) \models^+_q \mathbf{n}_5 \to A$ and $(\emptyset;\emptyset) \models^-_q A \prec \mathbf{n}_5$.*

Proof. Let $\mathcal{M}_q$ be a queer model, and $x \in I$. We shall show the statement by induction on A. If $A = \mathbf{n}_5$, then the statements follow immediately.

For a propositional variable p, if $\mathcal{M}_q, y \models^+_q \mathbf{n}_5$ for $y \geq x$ then $y \in Q^+ \subseteq v^+(p)$ by definition of v^+. Thus $\mathcal{M}_q, y \models^+_q p$ and so $\mathcal{M}_q, x \models^+_q \mathbf{n}_5 \to p$. The case for $p \prec \mathbf{n}_5$ is shown analogously.

Cases for other connectives then follow straightforwardly from the I.H.. □

REMARK 5.1. *In the above lemma, it is crucial that the support of falsity conditions are based on those of* **2C**. *If instead those of* **2Int** *were used, then for implication we would have*:

$$\mathcal{M}_q, x \models^-_q A \to B \textit{ iff } \mathcal{M}_q, x \models^+_q A \textit{ and } \mathcal{M}_q, x \models^-_q B.$$

Then a countermodel for $\models^-_q (p \to p) \prec \mathbf{n}_5$ is easily constructed, by means of a state which is in Q^- but not supporting the truth of p. (*Cf. Corollary 5.1 below.*) *For a similar reason, we cannot allow strong implication/co-implication in the current setting. Thus, restricted constants can provide useful information when the acceptability of strong implication/co-implication is not clear.*

The soundness of the system with the non-standard semantics now follows by standard induction.

THEOREM 5.1. *If $(\Gamma^+;\Gamma^-) \vdash^*_q A$ then $(\Gamma^+;\Gamma^-) \models^*_q A$, where $* \in \{+,-\}$.*

COROLLARY 5.1. $(\emptyset;\emptyset) \nvdash^-_q \mathbf{n}_5 \to A$ *and* $(\emptyset;\emptyset) \nvdash^+_q A \prec \mathbf{n}_5$.

Proof. Here we show the first part. Let $\mathcal{M}_q = \langle \{x\}, \{\langle x, x\rangle\}, \{x\}, \emptyset, v^+, v^-\rangle$ be a queer model s.t.: $v^+(q) = \{x\}$ and $v^-(q) = \emptyset$ for any propositional variable q. Note in particular that the conditions for v^+ and v^- are satisfied. Now $\mathcal{M}_q, x \models^+ \mathbf{n}_5$ but $\mathcal{M}_q, x \not\models^- p$; so the statement follows. □

Next, we move on to show the completeness direction.

THEOREM 5.2. $(\Gamma^+;\Gamma^-) \models^*_q A$ *then* $(\Gamma^+;\Gamma^=) \vdash^*_q A$, *where* $* \in \{+,-\}$.

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Proof. The basic outline is the same as the one for **B2C** in §3.3. In particular, the construction of prime bi-theory goes through as in Lemma 3.2, and the notion of the canonical model is almost identical to the one in §3.3 except that the sets Q^+ and Q^- are defined by:

- $(\Gamma^+, \Gamma^-) \in Q^+$ iff $\mathbf{n}_5 \in \Gamma^+$.
- $(\Gamma^+, \Gamma^-) \in Q^-$ iff $\mathbf{n}_5 \in \Gamma^-$.

Then the truth lemma for $\mathbf{n}_5$ follows easily, and otherwise the lemma is shown as in Lemma 3.3. □

Completeness for systems with other restricted constants can similarly be shown: the same construction of their canonical models assures the conditions $Q^- \subseteq Q^+$ (for $\mathbf{n}_3, \mathbf{n}_4$) or $Q^+ \subseteq Q^-$ (for $\mathbf{n}_2, \mathbf{n}_6$).

The difference between the two semantics can shed light on whether an intuitionistic logician should prefer $\bot_1$ or $\bot_2$. If one's intended semantics is that of queer models (for intuitionistic logic, obtained by removing notions related to $\models^-$), then adding the dimension of supports of falsity validates $(\bot_2 Ep)$ but not $(\bot_1 Ep)$; so $\bot_2$ will be the natural choice for such a logician. In contrast, an analogous extension for the standard intuitionistic Kripke semantics does validate $(\bot_1 Ep)$; so if this is the intended intuitionistic semantics, then an intuitionistic logician has no problems in accepting $\bot_1$.

§6. Expansions of B2C and weaker notions of toggling.

We have earlier discussed reasons why the choice of connectives in the language $\mathcal{L}$ may be preferable, in particular on why the constants $\bot$ and $\top$ as well as the toggling negation $\sim$ need not be included. Between these decisions, the second one is perhaps more novel, because $\bot$ and $\top$ are often excluded in constructive logics such as **N4**. Indeed, if we take **N4** as paradigmatic (cf. [36]), then it makes some sense, at least technically, to take the inclusion of the constants as optional, whereas the rejection of the toggling negation is taken as more fundamental. More generally, one may think of adding other connectives as long as it does not reproduce the toggling negation as a definable connective.

One thing to note here is that the last requirement still allows connectives exhibiting a "partial" characteristics of the toggling negation to be in the framework. For instance, recall that the defined negation $\neg A$ and co-negation $-A$ in **2Int** each shows one side of the "toggling" property of the toggling negation: a dual proof of $\neg A$ is inter-derivable with a proof of A, and a proof of $-A$ is inter-derivable with a dual proof of A. Even though these connectives play essential roles for the theoretical purpose of **2Int**, it is at the same time at least imaginable that somebody who abhors the toggling negation finds the two connectives unwelcome as well.

Extrapolating from this example, we shall study the acceptability of expansions of $\mathcal{L}$ relative to the degree of toggling they introduce. With duality in mind, we concentrate on the pairs $\bot$&$\top$, $\Rightarrow$ &$\Leftarrow$ and $\mathbf{n}$.

6.1. Half-toggling connectives.

Semantically, the characteristic property of the toggling negation is that both $x \models^+ {\sim}A$ iff $x \models^- A$ for all A, and $x \models^- {\sim}A$ iff $x \models^+ A$ for all A. One way to liberalize it is then to require only one of the equivalences.

Let us call a unary connective $\sim_*$ *half-toggling* if it shares the condition for the support of truth or the support of falsity with the toggling negation: i.e., it satisfies either $\mathcal{M}, x \models^+ \sim_* A$ iff $\mathcal{M}, x \models^- A$ for all A, or $\mathcal{M}, x \models^- \sim_* A$ iff $\mathcal{M}, x \models^+ A$ for all

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A.[11] When we want to be more specific, the former will be called *positively half-toggling* and the latter will be called *negatively half-toggling*. Proof-theoretically, we may understand these by the admissibility of the next pairs of rules, respectively.

$$\begin{array}{c}\overline{\overline{A}}\\ \hline \sim_* A\end{array}, \quad \begin{array}{c}\overline{\sim_* A}\\ \hline\hline A\end{array} \qquad \text{or} \qquad \begin{array}{c}\overline{A}\\ \hline\hline \sim_* A\end{array}, \quad \begin{array}{c}\overline{\overline{\sim_* A}}\\ \hline A\end{array}.$$

For some immediate examples, the toggling negation is a half-toggling connective, and negation and co-negation in **2Int** are each negatively and positively half-toggling. In addition, the variants of the **FDE**-negation discussed in [40] are all positively half-toggling.[12] In particular, among them is a connective named $\sim_4$, which has the next support of truth/falsity condition.

$$\mathcal{M}, x \models^+ \sim_4 A \text{ iff } \mathcal{M}, x \models^- A.$$
$$\mathcal{M}, x \models^- \sim_4 A.$$

The connective $\sim_4$ in addition satisfies Belnap's criterion [8] that an operator in the bilattice *FOUR* must be monotone with respect to the informational ordering (*Scott's thesis*). Therefore there are good reasons to think that it is a legitimate connective. In terms of natural deduction, $\sim_4$ can be defined by adding to the rules for positively half-toggling connectives the next 0-premise rule.

$$\begin{array}{c}(\emptyset;\emptyset)\\ \hline\hline \sim_4 A\end{array}$$

This consideration informs us about the acceptability of adding strong implication (along with strong co-implication) to **B2C** as well, for it will allow us to define $\sim_4$.

PROPOSITION 6.1. *$\sim_4$ is definable in $\mathcal{L}_{\{\Rightarrow, =\!\!\prec\}}$.*

Proof. The following shows that $A \Rightarrow \mathbf{b}$ serves the purpose. (See Lemma 4.1 for a definition of $\mathbf{b}$.)

$$\begin{array}{c}\overline{\mathbf{b}} \quad \overline{\overline{A}}\\ \hline A \Rightarrow \mathbf{b}\end{array} \qquad \begin{array}{c}\overline{A \Rightarrow \mathbf{b}} \quad \overline{\overline{\mathbf{b}}}\\ \hline\hline A\end{array} \qquad \begin{array}{c}\overline{\overline{\mathbf{b}}}\\ \hline\hline A \Rightarrow \mathbf{b}\end{array}.$$

□

Note that the proof above only uses strong implication. One may also obtain a negatively half-toggling connective by means of strong co-implication alone, using $\mathbf{b} =\!\!\prec A$.

On the other hand, if we stay in $\mathcal{L}$ then we are saved from half-toggling connectives. In fact, the addition of $\mathbf{n}$ to the language does not affect the undefinability of such connectives.

[11] Cf. the notions of *positive/negative definability* in a classical setting [39].

[12] Indeed, the enquiry in [40] is based on the idea of A. Avron [4] that positive half-toggling characterizes the intuitive notion of negation in the four-valued setting.

PROPOSITION 6.2. *A half-toggling connective is not definable in* $\mathcal{L}_{\mathbf{n}}$.

Proof. The argument is like for Proposition 4.1: we offer countermodels for both positively and negatively half-toggling connectives. Let $I := \{x, y\}$ and $\leq := \{\langle x, x\rangle, \langle x, y\rangle, \langle y, y\rangle\}$. Let $\mathcal{M}_1 := \langle I, \leq, v_1^+, v_1^-\rangle$ be s.t.

$$\langle v_1^+(p), v_1^-(p)\rangle \in \{\langle I, I\rangle, \langle I, \{y\}\rangle, \langle I, \emptyset\rangle, \langle \emptyset, I\rangle, \langle \emptyset, \{y\}\rangle, \langle \emptyset, \emptyset\rangle\},$$

for all $p \in$ Prop. Note that $\mathbf{n}$ is assigned the pair $\langle \emptyset, \emptyset\rangle$. Then we obtain the following tables.

$\wedge$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$
$\langle I,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\{y\}\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$
$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\{y\}\rangle$
$\langle \emptyset,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$

$\vee$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$
$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$
$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$
$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,\emptyset\rangle$	$\langle \emptyset,\emptyset\rangle$	$\langle \emptyset,\emptyset\rangle$

$\rightarrow$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle \emptyset,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle \emptyset,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$

$\prec$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle \emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\{y\}\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\emptyset\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,I\rangle$	$\langle \emptyset,\emptyset\rangle$
$\langle \emptyset,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$

So the set $\{\langle I, I\rangle, \langle I, \{y\}\rangle, \langle I, \emptyset\rangle, \langle \emptyset, I\rangle, \langle \emptyset, \{y\}\rangle, \langle \emptyset, \emptyset\rangle\}$ is closed under the operations. Now if a positively half-toggling connective is definable, then $v_1^+(A) = \{y\}$ must be possible; a contradiction. Next, Let $\mathcal{M}_2 := \langle I, \leq, v_2^+, v_2^-\rangle$ be a model such that

$$\langle v_2^+(p), v_2^-(p)\rangle \in \{\langle I, I\rangle, \langle I, \emptyset\rangle, \langle \{y\}, I\rangle, \langle \{y\}, \emptyset\rangle, \langle \emptyset, I\rangle, \langle \emptyset, \emptyset\rangle\},$$

for all $p \in$ Prop. Then we obtain the following tables.

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$\wedge$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$
$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, I\rangle$
$\langle \emptyset, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$

$\vee$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$
$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I. \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$
$\langle \{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$
$\langle \{y\}, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$
$\langle \emptyset, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \emptyset, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, \emptyset\rangle$	$\langle \emptyset, \emptyset\rangle$

$\rightarrow$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \{y\}, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \emptyset, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle \emptyset, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$

$-\!\!\!<$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle \{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \{y\}, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle \emptyset, I\rangle$	$\langle I, I\rangle$	$\langle I, \emptyset\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, \emptyset\rangle$	$\langle \emptyset, I\rangle$	$\langle \emptyset, \emptyset\rangle$
$\langle \emptyset, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$

Hence the set $\{\langle I, I\rangle, \langle I, \emptyset\rangle, \langle \{y\}, I\rangle, \langle \{y\}, \emptyset\rangle, \langle \emptyset, I\rangle, \langle \emptyset, \emptyset\rangle\}$ is closed under the operations. Now if a negatively half-toggling connective is definable, then $v_2^-(A) = \{y\}$ must be possible; a contradiction. □

COROLLARY 6.1. *A half-toggling connective is not definable in* $\mathcal{L}_{\{\top,\bot\}}$.

Proof. This follows immediately by noticing that $\top$ and $\bot$ become definable once we have **b** and **n** available to us, respectively by $\mathbf{b} \wedge \mathbf{n}$ and $\mathbf{b} \vee \mathbf{n}$. □

REMARK 6.1. *If we restrict our attention to classical models, then a different picture emerges. The formula* $(\bot -\!\!\!< A) \rightarrow \bot$ *in* $\mathcal{L}_{\{\top,\bot\}}$ *then has the next forcing condition in a model* $\mathcal{M}$ *where* $I = \{x\}$.

$$
\begin{aligned}
\mathcal{M}, x \models^+ (\bot -\!\!\!< A) \rightarrow \bot \text{ iff } & \mathcal{M}, x \not\models^+ \bot -\!\!\!< A. \\
\text{iff } & \mathcal{M}, x \models^- A \text{ and } \mathcal{M}, x \not\models^+ \bot. \\
\text{iff } & \mathcal{M}, x \models^- A.
\end{aligned}
$$

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Similarly, the formula $\top \mathrel{-\!\!\prec} (A \to \top)$ *gives*:

$$\begin{aligned}\mathcal{M}, x \models^{-} \top \mathrel{-\!\!\prec} (A \to \top) &\text{ iff } \mathcal{M}, x \models^{-} A \to \top.\\ &\text{ iff } \mathcal{M}, x \models^{+} A \text{ and } \mathcal{M}, x \not\models^{-} \top.\\ &\text{ iff } \mathcal{M}, x \models^{+} A.\end{aligned}$$

Now it is also easily checked that $(A \to B) \wedge ({\sim_*} B \to {\sim_*} A)$ *defines strong implication when* ${\sim_*}$ *is positively half-toggling, and* $(B \mathrel{-\!\!\prec} A) \vee ({\sim_*} A \mathrel{-\!\!\prec} {\sim_*} B)$ *defines strong co-implication when* ${\sim_*}$ *is negatively half-toggling. (Note that classical models always support the falsity of* $A \Rightarrow B$ */the truth of* $B \mathrel{=\!\!\prec} A$*.)*

6.2. The toggling connective. Our second concern is the definability of the toggling negation. We shall first observe that the language $\mathcal{L}_{\{\top,\bot,\Rightarrow,\mathrel{=\!\!\prec}\}}$ does not define the toggling connective. We utilize the fact that the languages $\mathcal{L}_{\{\top,\bot\}}$ and $\mathcal{L}_{\{\top,\bot,\Rightarrow,\mathrel{=\!\!\prec}\}}$ have no difference in classical models in their expressiveness.

PROPOSITION 6.3. *The toggling negation is not definable in* $\mathcal{L}_{\{\top,\bot\}}$ *in a classical model.*

Proof. We shall use the same model as for Proposition 4.1, except that for all $p \in$ Prop:

$$\langle v^{+}(p), v^{-}(p)\rangle \in \{\langle I, I\rangle, \langle I, \{x\}\rangle, \langle I, \{y\}\rangle, \langle I, \emptyset\rangle, \langle\{x\}, I\rangle, \langle\{x\}, \{y\}\rangle, \langle\{y\}, I\rangle, \langle\emptyset, I\rangle\}.$$

Then we obtain the following tables for $\langle v^{+}(A), v^{-}(A)\rangle$ for each formula A in $\mathcal{L}_{\{\top,\bot\}}$.

$\wedge$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle I,\{x\}\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle I,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$
$\langle\{x\},\{y\}\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$
$\langle\{y\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset.I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$	$\langle\emptyset,I\rangle$

$\vee$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$
$\langle I,I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle I,I\rangle$	$\langle I,I\rangle$
$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{x\}\rangle$
$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\{y\}\rangle$
$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\emptyset\rangle$
$\langle\{x\},I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle I,I\rangle$	$\langle\{x\},I\rangle$
$\langle\{x\},\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle I,\{y\}\rangle$	$\langle\{x\},\{y\}\rangle$
$\langle\{y\},I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle I,I\rangle$	$\langle I,\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\{y\},I\rangle$
$\langle\emptyset,I\rangle$	$\langle I,I\rangle$	$\langle I,\{x\}\rangle$	$\langle I,\{y\}\rangle$	$\langle I,\emptyset\rangle$	$\langle\{x\},I\rangle$	$\langle\{x\},\{y\}\rangle$	$\langle\{y\},I\rangle$	$\langle\emptyset,I\rangle$

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$\to$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, \{x\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle \{x\}, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$
$\langle \{x\}, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$
$\langle \{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, I\rangle$	$\langle I, I\rangle$	$\langle \{x\}, I\rangle$
$\langle \emptyset, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$

$\prec$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle I, \{x\}\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \{y\}, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, I\rangle$	$\langle I, I\rangle$	$\langle \{x\}, I\rangle$
$\langle I, \emptyset\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle \{x\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle \{x\}, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, I\rangle$	$\langle I, I\rangle$	$\langle \{x\}, I\rangle$
$\langle \{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$
$\langle \emptyset, I\rangle$	$\langle I, I\rangle$	$\langle I, \{x\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \emptyset\rangle$	$\langle \{x\}, I\rangle$	$\langle \{x\}, \{y\}\rangle$	$\langle \{y\}, I\rangle$	$\langle \emptyset, I\rangle$

Hence the set of values are closed under the operations. Now if the toggling negation $\sim$ is definable, then $\langle v^+(p), v^-(p)\rangle = \langle \{x\}, \{y\}\rangle$ implies $v^+(\sim p) = \{y\}$ and $v^-(\sim p) = \{x\}$, but $\langle \{y\}, \{x\}\rangle$ does not belong to the collection of pairs. Therefore $\sim$ cannot be defined in $\mathcal{L}_{\{\top,\bot\}}$. □

COROLLARY 6.2. *The toggling negation is not definable in* $\mathcal{L}_{\{\top,\bot,\Rightarrow,\prec\}}$.

Proof. If it is definable, then it is definable in classical models in $\mathcal{L}_{\{\top,\bot\}}$, by Remark 6.1. Yet Proposition 6.3 shows that this is impossible. □

Let us next consider languages including $\mathbf{n}$. We have already seen that $\mathcal{L}_{\mathbf{n}}$ does not define half-toggling connectives; so *a fortiori* it does not define the toggling connective. As for $\mathcal{L}_{\{\mathbf{n},\Rightarrow,\prec\}}$, the situation is rather different from that of $\mathcal{L}_{\{\top,\bot,\Rightarrow,\prec\}}$. Recall that half-toggling connectives are definable already in the latter language. We may then apply a technique of combining the support of truth condition of one connective and the support of falsity condition of another connective, in order to define the toggling negation (cf. e.g. [3, 37]).

PROPOSITION 6.4. *The toggling negation is definable in* $\mathcal{L}_{\{\mathbf{n},\Rightarrow,\prec\}}$.

Proof. We claim that the formula $(A \Rightarrow \mathbf{b}) \vee ((\mathbf{b} \prec A) \wedge \mathbf{n})$ defines the toggling negation. Indeed:

$$\dfrac{\dfrac{\overline{\overline{\mathbf{b}}}}{A\Rightarrow\mathbf{b}} \qquad \dfrac{\dfrac{\overline{\overline{\mathbf{b}}}\qquad\overline{A}}{(\mathbf{b}\mathrel{=\!\!\!\prec} A)}}{(\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n}}}{(A\Rightarrow\mathbf{b})\vee((\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n})} \qquad \dfrac{\dfrac{\overline{\overline{(A\Rightarrow\mathbf{b})\vee((\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n})}}}{(\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n}} \qquad \dfrac{[\![\mathbf{b}\mathrel{=\!\!\!\prec} A]\!] \qquad \overline{\mathbf{b}}}{A} \qquad \dfrac{[\![\mathbf{n}]\!]}{A}}{A}$$

$$\dfrac{\dfrac{\overline{\mathbf{b}} \qquad \overline{\overline{A}}}{A\Rightarrow\mathbf{b}}}{(A\Rightarrow\mathbf{b})\vee((\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n})} \qquad \dfrac{\overline{(A\Rightarrow\mathbf{b})\vee((\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n})} \qquad \dfrac{[\![(A\Rightarrow\mathbf{b})]\!] \qquad \overline{\overline{\mathbf{b}}}}{A} \qquad \dfrac{\dfrac{[(\mathbf{b}\mathrel{=\!\!\!\prec} A)\wedge\mathbf{n}]}{\mathbf{n}}}{A}}{A} .$$

□

Hence if one wishes to keep clear of the toggling negation (in the present setting), then it is necessary and sufficient to give up either the neither constant or the strong implication/co-implication pair.

6.3. Indirectly half-toggling connectives. It is possible to come up with a notion of toggling that is less direct than half-toggling. We shall call a unary connective $\sim_*$ *indirectly half-toggling* if it satisfies either:

$$\mathcal{M}, x \models^+ \sim_* A \text{ iff } \forall y \geq x \exists z \geq y\ \mathcal{M}, z \models^- A, \text{ or}$$
$$\mathcal{M}, x \models^- \sim_* A \text{ iff } \forall y \geq x \exists z \geq y\ \mathcal{M}, z \models^+ A.$$

As before, we will call the former the positive and the latter the negative form of indirect half-toggling. This notion clearly coincides with that of a half-toggling connective in a classical model, which provides a potential reason to be sceptical of the acceptability of these connectives for those who take a more strict stance towards the separation of proof and disproof.

It is straightforward to check that positive/negative indirectly half-toggling connectives are definable in $\mathcal{L}_{\{\top,\bot\}}$ via the formulas in Remark 6.1, namely $(\bot \mathrel{-\!\!\!\prec} A) \rightarrow \bot$ and $\top \mathrel{-\!\!\!\prec} (A \rightarrow \top)$. In comparison, once $\top$ and $\bot$ are discarded, we can avoid such formulas, again even in the presence of $\Rightarrow$ and $\mathrel{=\!\!\!\prec}$.

PROPOSITION 6.5. *An indirectly half-toggling connective is not definable in* $\mathcal{L}_{\{\Rightarrow,\mathrel{=\!\!\!\prec}\}}$.

Proof. Let $I := \{x, y, z\}$, $\leq := \{\langle x, x\rangle, \langle x, y\rangle, \langle y, y\rangle, \langle z, z\rangle\}$. Let $\mathcal{M} := \langle I, \leq, v^+, v^-\rangle$ be s.t. $\langle v^+(p), v^-(p)\rangle \in \{\langle I, I\rangle, \langle I, \{y\}\rangle, \langle\{y\}, I\rangle\}$ for all $p \in$ Prop. Then we obtain the following tables for $\langle v^+(A), v^-(A)\rangle$ for each formula A in $\mathcal{L}_{\{\Rightarrow,\mathrel{=\!\!\!\prec}\}}$.

$\wedge$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle\{y\}, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle\{y\}, I\rangle$	$\langle\{y\}, I\rangle$	$\langle\{y\}.I\rangle$	$\langle\{y\}, I\rangle$

$\vee$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$
$\langle\{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$

$\rightarrow$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle\{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$

$\mathrel{-\!\!\!\prec}$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle\{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$

$\Rightarrow$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle\{y\}, I\rangle$
$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$	$\langle I, I\rangle$	$\langle\{y\}, I\rangle$
$\langle\{y\}, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$

$\mathrel{=\!\!\!\prec}$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle\{y\}, I\rangle$
$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$
$\langle I, \{y\}\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$	$\langle I, I\rangle$
$\langle\{y\}, I\rangle$	$\langle I, \{y\}\rangle$	$\langle I, \{y\}\rangle$	$\langle I, I\rangle$

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Now if $\sim_*$ s.t. $\mathcal{M}, w \models^+ \sim_* A$ if and only if $\forall w' \geq w \exists w'' \geq w'(\mathcal{M}, w'' \models^- A)$ is definable, then for $v^-(A) = \{y\}$ we would need $v^+(\sim_* A) = \{x, y\}$, but this is not possible. Similarly, if $\sim_*$ s.t. $\mathcal{M}, w \models^- \sim_* A$ if and only if $\forall w' \geq w \exists w'' \geq w'(\mathcal{M}, w'' \models^+ A)$ is definable, then for $v^+(A) = \{y\}$ we would need $v^-(\sim_* A) = \{x, y\}$, but this is again impossible. □

REMARK 6.2. *Connectives satisfying one of the next conditions can be used to define positive/negative indirectly half-toggling connectives (when both of them exist), respectively by* $\dagger_+ \dagger_- A$ *and* $\dagger_- \dagger_+ A$.

$$\mathcal{M}, x \models^+ \dagger_+ A \text{ iff } \forall y \geq x\ \mathcal{M}, y \not\models^- A, \text{ or } \mathcal{M}, x \models^- \dagger_- A \text{ iff } \forall y \geq x\ \mathcal{M}, y \not\models^+ A.$$

The converse is generally not the case, as can be shown by an argument similar to the proof of Theorem 4.1. However it is not difficult to see that they are definable from an indirectly half-toggling connective once we are in $\mathcal{L}_{\{\top,\bot\}}$. *So the definability of a pair of indirectly half-toggling connectives and the pair* $\dagger_+, \dagger_-$ *turns out to be equivalent, when it comes to the languages under consideration.*

Proof-theoretically, $\dagger_+$ *and* $\dagger_-$ *have an advantage over an indirectly half-toggling connective in that they have a simpler presentation in terms of the admissibility of pairs of rules: respectively,*

$$\dfrac{\dfrac{\begin{matrix}(\Delta;\Gamma,[\![A]\!])\\ \vdots\end{matrix}}{\dagger_+ A}}{\dagger_+ A}\ ,\quad \dfrac{\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\end{matrix}}{\dagger_+ A} \qquad \dfrac{\begin{matrix}(\Delta';\Gamma')\\ \vdots\end{matrix}}{A}}{B} \qquad \text{and} \qquad \dfrac{\dfrac{\begin{matrix}([A],\Delta;\Gamma)\\ \vdots\end{matrix}}{\dagger_- A}}{\dagger_- A}\ ,\quad \dfrac{\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\end{matrix}}{\dagger_- A} \qquad \dfrac{\begin{matrix}(\Delta';\Gamma')\\ \vdots\end{matrix}}{A}}{B}\ .$$

Then we may observe a behaviour similar to positive/negative half-toggling, but with respect to $\dagger_+ A \to A$ *or* $A \prec \dagger_- A$. *For the former:*

$$\dfrac{\dfrac{[\dagger_+ A] \qquad \dfrac{\dfrac{}{\dagger_+ A \to A} \qquad [\dagger_+ A]}{A}}{\dagger_+(p \prec p)}}{\sim_* A} \qquad \dfrac{\dfrac{\dfrac{\sim_* A \qquad [\dagger_+ A]}{\dagger_+(p \prec p)} \qquad \dfrac{[\![p]\!]}{p \prec p}}{A}}{\dagger_+ A \to A}\ ,$$

(where $\sim_* A := \dagger_+ A \to \dagger_+(p \prec p)$). For the latter:

$$\dfrac{\dfrac{[\![\dagger_- A]\!] \qquad \dfrac{A \prec \dagger_- A \qquad [\![\dagger_- A]\!]}{A}}{\dagger_-(p \to p)}}{\sim_* A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\sim_* A} \qquad [\![\dagger_- A]\!]}{\dagger_-(p \prec p)} \qquad \dfrac{[p]}{p \to p}}{A}}{A \prec \dagger_- A}\ ,$$

(*where* $\sim_* A := \dagger_-(p \to p) \prec \dagger_- A$). *In particular, if one of the following:*

$$\dfrac{\dfrac{}{\dagger_+ A \to A}}{A} \text{ (for the former case) or } \dfrac{A \prec \dagger_- A}{A} \text{ (for the latter case),}$$

is satisfied, then $\sim_* A$ *becomes positively/negatively half-toggling in each case.*

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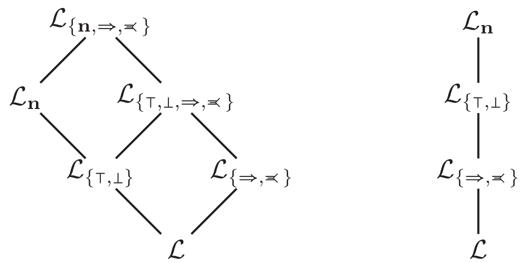
Table 6. *Hierarchy in constructive (left) and classical (right) settings*



6.4. The indirectly toggling connective. It is of course possible to think of a connective which relates to indirectly half-toggling connectives in the same way the toggling negation relates to half-toggling connectives: i.e., the connective defined by:

$$\mathcal{M}, x \models^+ \sim_* A \text{ iff } \forall y \geq x \exists z \geq y\ \mathcal{M}, z \models^- A, \text{ and}$$
$$\mathcal{M}, x \models^- \sim_* A \text{ iff } \forall y \geq x \exists z \geq y\ \mathcal{M}, z \models^+ A.$$

We shall call this connective the *indirectly toggling* connective, for which we have the next observation.

Proposition 6.6. *The next statements hold.*

1. *The indirectly toggling connective is definable in* $\mathcal{L}_{\mathbf{n}}$.
2. *The indirectly toggling connective is not definable in* $\mathcal{L}_{\{\top,\bot,\Rightarrow,\Yleft\}}$.

Proof. 1. This follows, as in Proposition 6.4, from the availability of both **b** and **n** in the language. Recalling that both positive and negative indirectly half-toggling connectives are definable in $\mathcal{L}_{\{\top,\bot\}}$, let us denote them by $\sim_+$ and $\sim_-$. Then $(\sim_+ A \wedge \mathbf{b}) \vee (\sim_- A \wedge \mathbf{n})$ defines the indirectly toggling connective.

2. Suppose that the indirectly toggling connective is definable in $\mathcal{L}_{\{\top,\bot,\Rightarrow,\Yleft\}}$. Then in particular it is definable in all classical models, wherein the notions of indirect toggling and the plain toggling become undistinguished. We know from Corollary 6.2, however, that the toggling connective is not definable with respect to the class of classical models in $\mathcal{L}_{\{\top,\bot,\Rightarrow,\Yleft\}}$: a contradiction. □

6.5. Summary. We end up with Table 6, which exhibits the hierarchy of languages for models/classical models. Table 7 displays more in detail possibilities of defining types of toggling connectives under various languages.

An emerging picture is that a bilateralist who wishes to include only $\top$, $\bot$ or only $\Rightarrow$, $\Yleft$ can adopt a criterion that half-toggling/indirectly half-toggling is acceptable while other types of toggling are not. Alternatively, one can be permissive and allow both half- and indirectly half-toggling connectives in order to include all of $\top$, $\bot$ and $\Rightarrow$, $\Yleft$. Or, one can accept the indirectly toggling connective along with the neither constant. Even with such liberal attitudes, however, one must at least make a choice between the neither constant and strong implication/co-implication, so as to avoid the toggling negation. Finally, one can remain strict and reject all four kinds of toggling, which supports the position of **B2C** that does not accept any of **n**, $\top$, $\bot$ or $\Rightarrow$, $\Yleft$.

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Table 7. *Definability of toggling connectives*

Definable?	$\mathcal{L}$	$\mathcal{L}_{\{\Rightarrow,\prec\}}$	$\mathcal{L}_{\{\top,\bot\}}$	$\mathcal{L}_{\{\top,\bot,\Rightarrow,\prec\}}$	$\mathcal{L}_{\mathbf{n}}$	$\mathcal{L}_{\{\mathbf{n},\Rightarrow,\prec\}}$
toggling	No	No	No	No	No	Yes
indirectly toggling	No	No	No	No	Yes	Yes
half-toggling	No	Yes	No	Yes	No	Yes
indirectly half-toggling	No	No	Yes	Yes	Yes	Yes

Definable in classical models?	$\mathcal{L}$	$\mathcal{L}_{\{\Rightarrow,\prec\}}$	$\mathcal{L}_{\{\top,\bot\}}$	$\mathcal{L}_{\mathbf{n}}$
toggling	No	No	No	Yes
indirectly toggling	No	No	No	Yes
half-toggling	No	Yes	Yes	Yes
indirectly half-toggling	No	Yes	Yes	Yes

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DEPARTMENT OF PHILOSOPHY I
RUHR UNIVERSITY BOCHUM
44801 BOCHUM
GERMANY
E-mail: heinrich.wansing@rub.de, satoru.niki@rub.de
sergei.drobyshevich@ruhr-uni-bochum.de