Canad. Math. Bull. Vol. 17 (5), 1975

AN EXISTENCE THEOREM FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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1. **Introduction.** Boundary value problems for ordinary differential equations have long been the subject of extensive research activity. In particular, questions concerning the existence and uniqueness of solutions for these problems have received much attention, and algebraic fixed-point theorems have served as important tools in such investigations. For example Picard [8] based his pioneering work in this area on the use of successive approximation techniques, and recently his classical methods have been refined and extended to more general nonlinear problems (see [1] and [4]). The standard procedure for applying these techniques requires that the boundary value problem under consideration first be converted into an equivalent integral equation through the choice of a suitable Green's function. The resulting theory is consequently limited to problems for which such a formulation is possible.

Another approach that has been used lately is to work directly with the operators that result from the boundary value problems. No formulation as an equivalent integral equation is necessary, and hence very general boundary value problems can be treated. Although a large number of useful fixed-point theorems are available in the literature (see [3] and [7] for surveys), so far only the Newton iterative technique has been used in this connection. In [10] the so-called "modified" Newton's method is applied to the problem, and in [5] a generalization of the Kantorovich theorem on the convergence of Newton's method is utilized. In both cases existence and uniqueness criteria are established for solutions of nonlinear differential equations subject to broad classes of both linear and nonlinear boundary conditions.

This paper presents a new technique for establishing the existence of solutions for boundary value problems. We begin by reformulating the boundary value problem under examination as an operator equation between normed linear spaces. However, rather than use a Green's function to obtain an integral equation, we employ a method that allows very general boundary value problems to be considered. The desired existence criteria for solutions are then obtained by applying a well-known form of the contraction mapping principle.

* This research was supported in part by the National Research Council of Canada under Grant A-8766.

Received by the editors June 20, 1973 and, in revised form, October 4, 1973.

Let $C(I)$ denote the linear space of continuous functions from the compact interval $I=[a, b]$ into n-dimensional real arithmetic space R_n, and let $C'(I)$ be the subspace of continuously differentiable functions on I. We shall consider the boundary value problem for a first-order system of n ordinary differential equations on I given by

$$x'+F(x, t) = 0, \qquad f(x) = 0. \tag{1.1}$$

The function f is a mapping from a subset of $C(I)$ into R_m, where m and n are not necessarily equal. The problem (1.1) will be referred to as a *nonlinear boundary value problem.*

2. **The equivalent problem.** Boundary value problem (1.1) will now be formulated as a nonlinear operator equation. We begin by making $C(I)$ into a Banach space with the uniform norm defined by

$$\|x\| = \max_{t\in I} \|x(t)\|, \qquad x \in C(I).$$

The linear space $C'(I)$ will be regarded as a subspace of $C(I)$ with the same norm. We shall also need to consider the product space $Y=C(I)\times R_m$, which is a Banach space for the norm

$$\|[\psi, v]\| = \max\{\|\psi\|, \|v\|\}, \qquad [\psi, v] \in Y.$$

In any normed linear space the open ball with centre at x and radius r is denoted by $S(x, r)$ and the corresponding closed ball by $\bar{S}(x, r)$. The identity mapping on a linear space will be denoted by E.

We shall assume that the function F in (1.1) is twice continuously differentiable on $U\times I$ where U is a nonempty open subset of R_n. The domain of the operator f is assumed to be a nonempty open subset D of $C(I)$, and we require that $x(t) \in U$, $t \in I$, for every choice of $x \in D$. Hence we can define an operator $T: D\to C(I)$ by

$$T(x)(t) = F(x(t), t), \qquad a \leq t \leq b.$$

Let A be a $n\times n$ matrix with continuous entries on $I=[a, b]$, and let L be a linear operator from $C(I)$ to R_m. Define a linear operator M from $C'(I)$ into Y by

$$Mx = [x'+Ax, Lx]. \tag{2.1}$$

Suppose M^+ is any operator from Y into $C'(I)$ such that $MM^+=E$, the identity mapping on Y. We then have the following basic result, in which A and L are assumed to have the properties just stated.

THEOREM 1. *Let R and S be the operators on D given by*

$$R(x) = Ax-T(x),$$
$$S(x) = Lx-f(x).$$

Assume that the operator M in (2.1) *has a right inverse M^+, and let the operator $Q: D\to C'(I)$ be given by*

$$Q(x) = M^+[R(x), S(x)]. \tag{2.2}$$

If x^* is a fixed point of Q, then x^* is a solution of the boundary value problem (1.1).

Proof. Let x^* be a fixed point of Q. Hence $Q(x^*)=x^*$ and it follows that $x^* \in D \cap C'(I)$. Thus we can apply the operator M to both sides of the expression

$$x^* = M^+[R(x^*), S(x^*)]$$

to obtain

$$Mx^* = [R(x^*), S(x^*)].$$

Therefore, using (2.1) and the definitions of the operations on Y, we have

$$[x^{*\prime}+T(x^*), f(x^*)] = 0. \tag{2.3}$$

From the form of (2.3) it follows immediately that x^* is a solution of (1.1). This completes the proof.

Theorem 1 makes it possible to find solutions of nonlinear boundary value problems by seeking fixed points for operators of the form (2.2). In particular we can use a contraction mapping method to solve $Q(x)=x$. We shall apply the contraction mapping principle in the following form (for a proof, see [4, pp. 11–20]).

THEOREM 2. *Let D be an open subset of a Banach space X. Suppose P is a twice Fréchet differentiable operator from D into X. For a given point $x_0 \in D$, assume there exist constants $\eta>0$, $K>0$, and $\delta \in [0, 1)$ such that*

(i) $\|x_0-P(x_0)\| \leq \eta$;
(ii) $\|P'(x_0)\| \leq \delta$;
(iii) $\|P''(x)\| \leq K$ *for all* $x \in \bar{S}(x_0, r_0)$

where

$$r_0 = \frac{1-(1-2h)^{1/2}}{h} \frac{\eta}{1-\delta}$$

and

$$h = \frac{K\eta}{(1-\delta)^2} < \frac{1}{2}, \quad \bar{S}(x_0, r_0) \subset D.$$

Then the contraction mapping sequence $\{x_n\}$ for P beginning at x_0, namely

$$x_{n+1} = P(x_n), \qquad n = 0, 1, 2, \ldots,$$

is defined, remains in $\bar{S}(x_0, r_0)$, and converges to a fixed point x^* of P in $\bar{S}(x_0, r_0)$. The rate of convergence is given by

$$\|x^*-x_n\| \leq r_0[1-(1-\delta)(1-2h)^{1/2}]^n, \qquad n = 0, 1, 2, \ldots.$$

3. **Right inverses.** Suppose the operator $M:C'(I)\to Y$ is defined as in (2.1). Let Φ be any fundamental matrix on I for the linear system

$$x'+Ax = 0$$

and define the linear operator N on R_n by

$$N\xi = L(\Phi\xi). \tag{3.1}$$

https://doi.org/10.4153/CMB-1974-126-3 Published online by Cambridge University Press

Since we have $N: R_n \to R_m$, it follows that there is a $m \times n$ real matrix which represents the operator N. The following theorem deals with the existence of right inverses for M.

THEOREM 3. *Let B be a $m \times n$ matrix representation of the operator N in* (3.1). *Suppose there exists a $n \times m$ matrix B^+ such that $BB^+ = E_m$ (the $m \times m$ identity matrix). Then for any given $x_0 \in C'(I)$ there exists an operator $M^+: Y \to C'(I)$ with the following properties:*

$$MM^+ = E \tag{3.2}$$

and

$$(M^+M)(x_0) = x_0 \tag{3.3}$$

Proof. Right inverses for M can be found by dealing with linear boundary value problems of the form

$$x' + Ax = \psi \tag{3.4}$$

$$Lx = v \tag{3.5}$$

where $[\psi, v] \in Y$. If we let $Hx = x' + Ax$, it then follows that the inverse image of any $\psi \in C(I)$ under H is the set of solutions of (3.4) and is represented by the following linear variety in $C'(I)$:

$$\left\{ \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)\, ds \right\} + \mathcal{N}(H),$$

where $\mathcal{N}(H)$ denotes the null space of H. Thus H maps $C'(I)$ onto $C(I)$ and equation (3.4) has solutions for every choice of $\psi \in C(I)$. Moreover, the null space of H is isomorphic to R_n under the isomorphism given by

$$\xi \leftrightarrow \Phi\xi, \qquad \xi \in R_n. \tag{3.6}$$

Because the operator H is onto, it has right inverses, and one such right inverse is given by

$$(H^+\psi)(t) = \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)\, ds, \qquad a \le t \le b.$$

Hence, using (3.6), we see that all solutions of (3.4) can be represented by

$$x = \Phi\xi + H^+\psi, \qquad \xi \in R_n. \tag{3.7}$$

Therefore x will be a solution of (3.4), (3.5) if $\xi \in R_n$ is a solution of

$$B\xi = v - LH^+\psi. \tag{3.8}$$

By assumption B has a right inverse B^+, and thus

$$\xi = \xi_0 + B^+(v - LH^+\psi)$$

is a solution of (3.8) for each $\xi_0 \in \mathcal{N}(N)$. By (3.7) it follows that, for every such

ξ_0, an operator M^+ satisfying (3.2) is given by

$$M^+[\psi, v] = \Phi(B^+v+\xi_0-B^+LH^+\psi)+H^+\psi.$$

We now seek a right inverse which also satisfies (3.3). Using (2.1) and an integration by parts, we obtain

$$(M^+M)(x_0) = \Phi\xi_0+x_0-\Phi\Phi^{-1}(a)x_0(a)+\Phi B^+B\Phi^{-1}(a)x_0(a).$$

Therefore if we choose

$$\xi_0 = \Phi^{-1}(a)x_0(a)-B^+B\Phi^{-1}(a)x_0(a),$$

it then follows that $B\xi_0=0$ and (3.3) holds. This establishes the theorem.

4. **The existence theorem.** It is necessary to require that certain relationships hold among the norms on the various spaces of matrices in this problem. To be specific, let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be respectively the linear spaces of $j\times k$, $k\times l$, and $j\times l$ real matrices with corresponding norms $\|.\|_1$, $\|.\|_2$, and $\|.\|_3$. Then the norms are said to be *compatible* if for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have $\|AB\|_3\leq\|A\|_1\,\|B\|_2$. If we make the natural identification of elements of R_n with $n\times 1$ matrices, then this notion of compatibility is a generalization of the concept defined in [6, p. 427]. We shall require that the arithmetic spaces R_n and R_m be given norms which are compatible with the norms introduced on the other spaces of matrices in the forthcoming development.

Since the function F in (1.1) is assumed to be twice continuously differentiable on $U\times$I, it follows that the resulting operator T is twice continuously Fréchet differentiable on D. For any given $x_0 \in D$, the functional value of $T'(x_0)$ at any $x \in C(I)$ can be represented by

$$(T'(x_0)x)(t) = G(t)x(t), \qquad a \leq t \leq b,$$

where G is a $n\times n$ matrix of continuous functions and the indicated multiplication is ordinary multiplication of a matrix by a vector (see [9, p. 95]).

Suppose that the operator f is twice Fréchet differentiable on the open subset D of $C(I)$ and let $D'=D \cap C'(I)$. For a given $x_0 \in D'$, define the linear operator $M: C'(I)\to Y$ by

$$Mx = [x'+T'(x_0)x, f'(x_0)x],$$

and let Φ be any fundamental matrix on I for

$$x'+T'(x_0)x = 0.$$

Define the linear operator N on R_n by $N\xi=f'(x_0)(\Phi\xi)$, and let B be a $m\times n$ matrix representation of N. We shall assume that B has a right inverse B^+. Furthermore, letting

$$\xi_0 = \Phi^{-1}(a)x_0(a)-B^+B\Phi^{-1}(a)x_0(a), \tag{4.1}$$

we also define the operator $M^+: Y \to C'(I)$ by

$$M^+[\psi, v] = H_1^+[\psi, v] + \Phi\xi_0$$

for all $[\psi, v] \in Y$ where

$$H_1^+[\psi, v] = \Phi B^+ v - \Phi B^+ f'(x_0) H^+ \psi + H^+ \psi,$$

$$(H^+\psi)(t) = \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)\, ds, \qquad a \le t \le b.$$

We are now ready to establish the main result concerning the existence of solutions for (1.1).

THEOREM 4. *Suppose there exist non-negative constants* α, β *and* κ *such that*

(i) $\| [x_0' + T(x_0), f(x_0)] \| \le \alpha$;
(ii) $K_1 \|B^+\| + K_1 K_2 (b-a)(1 + K_1 \|B^+\| \, \|f'(x_0)\|) \le \beta$

where

$$K_1 = \max_{t \in I} \|\Phi(t)\|, \qquad K_2 = \max_{t \in I} \|\Phi^{-1}(t)\|;$$

(iii) $\| [T''(x), f''(x)] \| \le \kappa$ *for all* $x \in \bar{S}(x_0, r_0)$

where

$$r_0 = \frac{1-(1-2h)^{1/2}}{\beta\kappa}, \; h = \alpha\beta^2\kappa < \tfrac{1}{2}, \; \bar{S}(x_0, r_0) \subset D.$$

Then the contraction mapping sequence $\{x_n\}$ *for the operator*

$$Q(x) = M^+[T'(x_0)x - T(x), f'(x_0)x - f(x)]$$

starting at x_0, *namely*

$$x_{n+1} = Q(x_n), \qquad n = 0, 1, 2, \ldots,$$

is defined, remains in $\bar{S}(x_0, r_0)$, *and converges to a solution* x^* *of the boundary value problem* (1.1). *The rate of convergence is given by*

$$\|x^* - x_n\| \le r_0[1-(1-2h)^{1/2}]^n$$

for $n = 0, 1, 2, \ldots$.

Proof. By an argument strictly analogous to that given in Theorem 3, it follows from the definitions of Φ, ξ_0, B^+, and H^+ that M^+ is a right inverse for M. Thus by Theorem 1 it suffices to show that the operator Q defined on D satisfies the hypotheses of Theorem 2.

We first consider the operator H_1^+, which is clearly linear from Y into $C'(I)$. To show that H_1^+ is bounded, consider

$$\|H_1^+\| = \sup_{\substack{[\psi,v] \in Y \\ [\psi,v] \ne [0,0]}} \left\{ \frac{\|H_1^+[\psi, v]\|}{\|[\psi, v]\|} \right\}.$$

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Using the boundedness of $f'(x_0)$ and the compatibility of the matrix norms, we obtain for each fixed $t \in I$ and every choice of $[\psi, v] \in Y$ that

$$\|H_1^+[\psi, v](t)\| \leq K_1 \|B^+\| \|v\| + (b-a)K_1^2K_2 \|B^+\| \|f'(x_0)\| \|\psi\| + K_1K_2(b-a) \|\psi\|.$$

Hence by hypothesis (ii) it follows that

$$\|H_1^+\| \leq \beta. \tag{4.2}$$

By the definition of the operator Q we have

$$\|x_0 - Q(x_0)\| = \|x_0 - M^+[T'(x_0)x_0 - T(x_0), f'(x_0)x_0 - f(x_0)]\|.$$

Thus by (4.1) and Theorem 3 we see that the operator M^+ satisfies both $MM^+ = E$ and $(M^+M)(x_0) = x_0$. Hence

$$\|x_0 - Q(x_0)\| = \|H_1^+[x_0' + T(x_0), f(x_0)]\| \leq \|H_1^+\| \|[x_0' + T(x_0), f(x_0)]\|, \tag{4.3}$$

which by (4.2) and hypothesis (i) implies that $\|x_0 - Q(x_0)\| \leq \alpha\beta$. Since $Q'(x_0) = 0$, we may choose $\delta = 0$ in Theorem 2, and therefore hypothesis (iii) now guarantees that the operator Q satisfies all the conditions of Theorem 2. Hence an appeal to Theorem 1 finishes the proof.

5. **An application.** The boundary value problem (1.1) is very comprehensive in that it includes nonlinear differential equations subject to many different classes of boundary conditions. It encompasses as special cases familiar boundary conditions such as two-point and multi-point conditions together with less common problems such as those involving integral conditions and conditions at an infinite set of points in I. As a consequence the existence criteria for solutions given in Theorem 4 can only be general in nature. For specific classes of problems and particular examples, much more precise conditions can readily be formulated. For instance consider the determination of the number β in hypothesis (ii) of Theorem 4. The choice of the initial point x_0 will determine a value for β since, by (4.2), β is a bound for the operator H_1^+. However, the value for β obtained from the computation indicated in hypothesis (ii) is often only a crude bound for the operator. With specific examples an alternative approach is to calculate H_1^+ explicitly and then obtain an estimate for the operator norm by a direct computation. Such a procedure generally yields a smaller value for the operator norm.

The proof of Theorem 4 offers important information concerning the application of the above technique in particular cases. As with all local iterative techniques, the quality of the initial approximation x_0 is the key factor which determines whether or not this method can be applied to a specific boundary value problem. It is therefore essential to have a practical means for obtaining initial approximations to fixed points of the operator Q. Such initial guesses would be almost impossible to obtain directly since the form of Q itself depends upon the particular choice of initial point x_0. However, from (4.3) we see that in general a "good" initial approximation to a solution of the nonlinear boundary value problem (1.1) will correspond to a suitable first guess for a fixed point of Q. The only restriction is that the norm of the operator H_1^+ must be reasonably small. Since it is quite natural and easy to work directly with problem (1.1), this procedure gives a practical means for obtaining a starting point for the iterations and for defining precisely the iteration function Q.

As an illustration of these points we conclude by treating a problem that has received considerable study in its linear form but for which little nonlinear theory exists (see [2, pp. 149, 163–165]). Consider the nonlinear boundary value problem given by

$$\frac{dx}{dt}+F(x, t) = 0, \quad \int_{a_1}^{b_1} A(t)x(t)\,dt = c. \tag{5.1}$$

We assume that the function F has the properties outlined in section 2, that A is a $m\times n$ matrix with continuous entries on $[a_1, b_1]$, and that $c \in R_m$ and $[a_1, b_1]\subset I$. In this case we have

$$B = \int_{a_1}^{b_1} A(t)\Phi(t)\,dt$$

and $f''(x)=0$ for every $x \in D$. Because the initial approximation x_0 determines the values of α and β in Theorem 4, it follows that the calculation of suitable choices for r_0 and κ depends only upon the properties of the operator T:

$$\|[T''(x), 0]\| \leq \kappa \quad \text{for all} \quad x \in \bar{S}(x_0, r_0).$$

We now deal with a specific example related to the class of problems described by (5.1). The following boundary value problem on $I=[0, 0.4]$ is chosen to illustrate the wide range of nonlinearities that can be treated by using the technique of Theorem 4:

$$\ddot{x}-(\tan \dot{x})^3x^2 = \tfrac{1}{2}\sin(t^4) \tag{5.2}$$

$$4\dot{x}(0)+\int_0^{1/4} x(t)\,dt = 0.2. \tag{5.3}$$

The linear space R_2 is given the norm

$$\|v\| = \max\{|v_1|, |v_2|\}$$

and the linear space $\mathscr{A}$ of 1×2 real matrices is normed by

$$\|A\| = |a_1|+|a_2|\,.$$

We choose

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

as the initial approximation, which yields $\alpha=0.05$ as a suitable value in hypothesis (i). If we choose Φ to be the principal matrix solution of $x'+T'(x_0)x=0$ on I, then the matrix B is given by

$$B = (0 \quad 4)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\int_0^{1/4} (1 \quad 0)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} dt = \begin{pmatrix} \frac{1}{4} & \frac{129}{32} \end{pmatrix}.$$

Clearly B has right inverses, and we choose

$$B^+ = \begin{pmatrix} \frac{32}{137} \\ \frac{32}{137} \end{pmatrix}.$$

By calculating H_1^+ explicitly in this case, we find that $\beta=1.14$ satisfies the requirement of hypothesis (ii). Another direct computation establishes that

$$\|[T''(x), 0]\| < 1.615 \quad \text{for all} \quad x \in \bar{S}(x_0, r),$$

where $r=0.15$. Therefore a choice of $\kappa=1.615$ yields

$$h = \alpha\beta^2\kappa < 0.105 \tag{5.4}$$

in hypothesis (iii). Since we obtain $r_0<0.061$ for any value of h in the range indicated by (5.4), it follows that all the conditions of Theorem 4 can be satisfied. Hence we have established that the nonlinear boundary value problem (5.2), (5.3) has a solution x^* which lies in $\bar{S}(x_0, 0.061)$ and is the limit of the iterative sequence defined in Theorem 4.

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