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THE EXISTENCE THEOREM FOR THE GENERAL RELATIVISTIC CAUCHY PROBLEM ON THE LIGHT-CONE

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Abstract

We prove existence of solutions of the vacuum Einstein equations with initial data induced by a smooth metric on a light-cone.

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1. Introduction

Systematic ways of constructing general solutions of the vacuum Einstein equations are provided by Cauchy problems of various flavours. One such classical problem consists of prescribing initial data on a light-cone. The formal aspects of this Cauchy problem are well understood by now [5, 9, 11, 19]. However, because of the singularity at the vertex, there arise significant difficulties when attempting to prove an existence theorem for general initial data, and only special cases have been established in the literature so far [4, 5]. It is the purpose of this work to fill this gap and prove an existence theorem for an exhaustive class of initial data, in the sense that every smooth light-cone in every smooth vacuum space–time arises from our construction.

Ricci-flat metrics are flat in space–time dimensions $n + 1 = 2$ and 3, which renders trivial the associated local Cauchy problem on the light-cone. Therefore we restrict attention to $n \geqslant 3$.

We will prove existence of a space–time with initial data on a light-cone C_O, with vertex O, using the wave-map gauge scheme of [5, 11, 19]. For this one needs to prove that the initial fields $g_{\mu\nu}|_{C_O}$ arise from some smooth metric, so that the Cagnac–Dossa theorem applies [13]. In this scheme the fields $g_{\mu\nu}|_{C_O}$ are constructed by solving a set of wave-gauge constraint equations starting from geometric initial data $(\tilde g, \kappa)$ (for notation, see below), which results in a tensor field $g_{\mu\nu}|_{C_O}$ on C_O with seemingly intractable behaviour at the vertex. The problem addressed, and solved, in this work is to show that $g_{\mu\nu}|_{C_O}$ is indeed the restriction to C_O of some smooth metric, which leads to our first main result:

THEOREM 1.1. *Let $\mathscr{M}$ be an n-dimensional manifold, $n \geqslant 3$, and let $O \in \mathscr{M}$. Consider a symmetric tensor $\tilde g$ induced by a smooth Lorentzian metric C on its null cone C_O centred at O. Then there exists a smooth metric g defined in a neighbourhood of O, solution of the vacuum Einstein equations to the future of O, such that C_O is the light-cone of g and $\tilde g$ is the restriction of g to TC_O.*

The proof of Theorem 1.1 is outlined in Section 2.

The neighbourhood, say $\mathscr{O}$, of O constructed in the proof of the theorem is a priori very small; in particular, it does not necessarily enclose C_O. Once $\mathscr{O}$ has been obtained from Theorem 1.1, one can appeal to [18] to obtain a vacuum metric defined in a (small) neighbourhood of the smooth part of C_O.

In Rendall's approach to the characteristic initial value problem [19] one requires an affine parameterisation of the geodesic generators of C_O: $\kappa \equiv 0$. In coordinates adapted to the light-cone, one prescribes a tensor field

$$\gamma_{AB}(r, x^C)\, dx^A\, dx^B$$

which determines $\tilde g$ after multiplication by a conformal factor. In such a setting we prove:

THEOREM 1.2. *Let $\gamma_{AB}(r, x^C)\, dx^A\, dx^B$ be induced by a smooth Lorentzian metric C on its light-cone centred at O in adapted coordinates, where r is a C-affine parameter. Then there exists a smooth metric g defined in a neighbourhood of O, solution of the vacuum Einstein equations to the future of O, such that*

$$g_{AB}|_{C_O} = \Omega^2 \gamma_{AB},$$

for some positive function Ω which is the restriction to C_O of a smooth function on space–time, where r is a g-affine parameter.

We show in Section 3 how to deduce Theorem 1.2 from Theorem 6.1 below.

There exists yet another scheme for prescribing the initial data on C_O, where the fields $g_{\mu\nu}|_{C_O}$ are given a priori, and the 'wave-map gauge constraint functions' $\Box_g y^\mu|_{C_O}$ are calculated from the data [9]. To establish existence of solutions

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within this scheme, given $g_{\mu\nu}|_{C_O}$ one needs to prove that a certain vector field $\mathring{W}^\mu$ arises as the restriction to the light-cone of a smooth vector field. We prove that this is the case when the fields $g_{\mu\nu}|_{C_O}$ arise from the restriction to the light-cone of a smooth metric, leading to the following theorem.

THEOREM 1.3. *Given any smooth Lorentzian metric C, there exists a smooth metric g, defined on a neighbourhood of O and solving the vacuum Einstein equations to the future of O, such that*

$$g_{\mu\nu}|_{C_O} = C_{\mu\nu}|_{C_O}.$$

The proof of Theorem 1.3 is the contents of Section 7.

There is a useful lesson to learn from this work, that *an effective way of solving constrained problems on a light-cone proceeds by constructing Taylor series that provide approximate solutions of the characteristic constraint equations*. Another key observation, without which this work would not have been completed, is that *a Taylor series solving all the equations is not needed* for the argument. Our approach opens the avenue for proving light-cone existence results, with data on a light-cone at finite distance or at past infinity, for Yang–Mills equations, Einstein equations with matter fields, or any other constrained nonlinear sets of equations. We note that, even for linear constrained systems, such as Maxwell or Dirac equations [17], the question of existence of *smooth* solutions of the Cauchy problem on the light-cone has not been solved so far, and the method here provides a good candidate for settling the problem. The idea of using Taylor expansions has meanwhile been implemented in [10, 15] to prove existence of solutions of the vacuum Einstein equations with scattering data on the future light-cone of past timelike infinity.

2. Outline of the argument

Throughout, we use the conventions and notations from [5]. In particular, the coordinates x are linked to the coordinates y, which define $\mathbb{R}^{n+1}$ as a C^∞ manifold, by the relations

$$y^0 = x^1 - x^0, \quad y^i = r\Theta^i(x^A) \quad \text{with} \sum_{i=1}^{n} \Theta^i(x^A)^2 = 1, \tag{2.1}$$

where the x^A are local coordinates on the sphere S^{n-1}, or angular polar coordinates. In the adapted coordinates $\{x^\mu\}$, on the light-cone the metric is written as

$$\overline{g} = \overline{g}_{00}(dx^0)^2 + 2\nu_A\, dx^0\, dx^A + 2\nu_0\, dx^0\, dx^1 + 2\overline{g}_{AB}\, dx^A\, dx^B.$$

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We underline components of tensors in coordinates y and do not underline those in coordinates x; we overline the restrictions to ('traces on') C_O. Thus $g_{\mu\nu}$ denotes the components of the metric in the x-coordinate system, $\underline{g_{\mu\nu}}$ or $g_{\underline{\mu\nu}}$ denotes the components of the metric in the y-coordinate system, $\overline{g}_{\mu\nu}$ denotes the restriction to the light-cone of the components of the metric in the x-coordinate system, and so on. We use the wave-map gauge with a Minkowskian target metric; in the notation of [5], $\widehat{g} \equiv \eta$. We assume that we are given a smooth metric C, for which we introduce normal coordinates y^μ.

As discussed in [9], there are many ways in which C can be used for the construction of a solution. We give existence proofs for three such schemes here. Theorem 1.1 is the most elegant geometrically, and the most natural from our perspective. We prove Theorem 1.2 because this is a setup which is most widely used in the literature. Theorem 1.3 is natural for many applications, such as numerical treatment, or possible generalisations to matter fields. The core of the three proofs is the construction of approximate solutions of the characteristic constraint equations in Section 5. This has to be complemented by further arguments, which differ according to which scheme is used, possibly in different orders. For this reason it seemed simplest to organise this paper by following the hierarchical nature of the Einstein equations. This does not necessarily coincide with the order of the arguments within each distinct setup.

In Theorem 1.1 one assumes that the angular part C_{AB} of the Lorentzian metric $C_{\mu\nu}$ provides the initial data tensor field $\tilde{g} := \overline{g}_{AB}\, dx^A\, dx^B$ directly,

$$\overline{g}_{AB} := \overline{C}_{AB}.$$

Then the parallel-transport coefficient κ, defined through the equation

$$\nabla_{\partial_r}\partial_r = \kappa\partial_r,$$

is determined, at least in a neighbourhood of the vertex O, from $\overline{C}_{AB}$ by algebraically solving the *Raychaudhuri equation*:

$$\partial_1\tau - \kappa\tau + \frac{\tau^2}{n-1} + |\sigma|^2 = 0. \tag{2.2}$$

Here τ is the 'divergence scalar' of C_O (cf. (3.13) below) and σ its 'shear tensor' (defined at the beginning of Section 3.1). The remaining metric functions $\overline{g}_{\mu\nu}$ on the light-cone can then be obtained by solving the wave-map gauge constraint equations of [5] in adapted coordinates x. A coordinate transformation, which is singular at the vertex of C_O because the adapted coordinates are singular there, provides the y-coordinate components $\underline{\overline{g}_{\mu\nu}}$ of the metric. As already pointed out, the difficulty is to show cone-smoothness of the metric functions $\underline{\overline{g}_{\mu\nu}}$ near the tip

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of the light-cone. (We say that a function f on C_O is *cone-smooth* if there exists a smooth function ϕ on space–time such that f is the restriction of ϕ to C_O.)

For this, the first step is to obtain sharp control of the behaviour at the vertex of the fields $|\sigma|^2$, τ, κ, and $\nu_0 \equiv \overline{g}_{01}$. This is carried out in Sections 3.1, 3.2, 3.3 and 4. While the proofs there are easy in retrospect, the simple solutions that we provide might not be completely obvious.

The already-mentioned next key step, established in Section 5, is the proof of existence of a smooth space–time metric $\check{C}_{\mu\nu}$, in the wave-map gauge, which solves all the wave-map gauge constraint equations up to an error term. For smooth $C_{\mu\nu}$, the error term decays to infinite order at the origin, with $\overline{\check{C}}_{AB} = \overline{g}_{AB}$, and with the corresponding function $\check{\nu}_0 := \overline{\check{C}}_{01}$ associated with $\check{C}$ differing from $\nu_0 := \overline{g}_{01}$ by an error term which again decays to infinite order at the origin. (A function f is said to decay to infinite order near $r = 0$, we then write $f = O_\infty(r^\infty)$, if for all $N \in \mathbb{N}$ we have $|f| \leqslant C_N r^N$ for small r for some constant C_N; similarly for all derivatives of f.) For $C_{\mu\nu}$ with finite differentiability, say C^k, the error terms above can be made to decay to order $O(|\vec{w}|^{k-m_1})$, for some $m_1 \in \mathbb{N}$ independent of the differentiability index k and of the dimension.

The second constraint equation is then rewritten as an equation for

$$\nu_A - \overline{\check{C}}_{0A} \quad \text{where } \nu_A := \overline{g}_{0A},$$

the solution of which is shown to decay to infinite order at the origin in the smooth case, or to order $O(|\vec{y}|^{k-m_1-m_2})$, for some $m_2 \in \mathbb{N}$ which again does not depend upon k or n in the C^k case. The final constraint is rewritten as an equation for

$$\overline{g}_{00} - \overline{\check{C}}_{00},$$

the solution of which is shown to have similar decay properties at the origin. The decay properties of the differences of metric functions allow one to show that the y^μ-coordinate components $\overline{g_{\mu\nu}}$ of $\overline{g}$ can be smoothly extended off the light-cone in the C^∞ case, or $C^{k-m_1-m_2-m_3}$-extended in the C^k case, for some m_3 independent of k and n. This allows us to use the Cagnac–Dossa theorems [2, 12] to solve the wave-gauge reduced Einstein equations. When k is large enough one can then appeal to the results in [5] to obtain Theorem 1.1 or, in fact, its somewhat more general version, Theorem 6.1 below.

In Theorem 1.2 the metric functions C_{AB} provide a conformal class,

$$\overline{g}_{AB} := \Omega^2 \overline{C}_{AB},$$

in which case one needs to solve the Raychaudhuri equation, understood as a second-order ordinary differential equation (ODE) for the conformal factor Ω.

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The reduction of this problem to that of Theorem 1.1 relies on the analysis of τ and $|\sigma|^2$ carried out in Sections 3.1 and 3.2. Using the Raychaudhuri equation ('first constraint'), in Section 3.3 we determine the divergence τ and the conformal factor Ω relating $\overline{g}_{AB}$ and the initial data $\gamma_{AB} \equiv \overline{C}_{AB}$, and analyse their properties at the vertex. This part of the argument is rather similar to that in [6] where, however, restrictive hypotheses have been made on the initial data.

In Theorem 1.3 all the metric functions are prescribed directly on C_O using the metric C,

$$\overline{g}_{\mu\nu} := \overline{C}_{\mu\nu}.$$

As a first, and key, step of the proof we construct a metric $\check{C}_{\mu\nu}$ such that $\overline{\check{C}}_{\mu\nu} = \overline{C}_{\mu\nu}$, the Ricci tensor of which, when contracted with a null tangent to the light-cone, decays to infinite order on C_O near the vertex along the light-cone. The equations $\overline{S}_{\mu\nu}\ell^\nu = 0$, where S is the Einstein tensor and ℓ is tangent to the generators of the light-cone, become now equations for a wave-gauge vector $\overline{H}^\mu$. One then needs to show cone-smoothness of the metric functions $\underline{\overline{H}^\mu}$ near the tip of the light-cone. Comparing a suitably defined gauge vector $\overline{H}^\mu$, as calculated for the desired vacuum metric, with the harmonicity vector $\overline{\check{H}}^\mu$, as calculated for the metric $\check{C}_{\mu\nu}$, allows us to show that $\overline{H}^\mu$ extends smoothly. One completes the proof by known arguments.

3. From a conformal class γ to $\tilde{g}$

Consider a tensor field γ which is induced on C_O by a smooth metric C in a space–time neighbourhood of C_O, i.e., $\gamma_{AB} = \overline{C}_{AB}$, where C_{AB} are the components with indices AB in the coordinates x of a metric C. The smoothness of C is insured by the hypothesis of smoothness of its components $\underline{C_{\alpha\beta}}$ in the y-coordinates. (This property is clearly a necessary condition for the desired vacuum metric to satisfy the requirements of our theorem.) Then $\overline{g}_{AB} = \Omega^2\gamma_{AB}$ will be the components in the coordinates x of indices AB of the trace of a smooth metric if and only if the conformal factor Ω is the trace of a smooth positive function.

Consider a metric C such that $C(O) = \eta$, the Minkowski metric. If C is C^k with values η at O then its components in the coordinates y admit at O an expansion, where the c are numbers, and the error terms $o_k(|y|^k)$ (see the beginning of appendix A for the definition of the symbol $o_k(|y|^k)$) are C^k functions of the y, of the form

$$\underline{C_{\alpha\beta}} = \underline{\eta_{\alpha\beta}} + \sum_{p=1}^{k} \frac{1}{p!} c_{\alpha\beta,\alpha_1\cdots\alpha_p} y^{\alpha_1}\cdots y^{\alpha_p} + o_k(|y|^k)$$

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(compare Lemma A.1). If $\partial_\alpha \underline{C_{\beta\gamma}}(O) = 0$, the expansion starts at $p = 2$. This is satisfied in particular if the y are normal coordinates for C with origin O. In the coordinates x it holds that

$$C_{AB} \equiv r^2 \underline{C_{ij}} \frac{\partial \Theta^i}{\partial x^A} \frac{\partial \Theta^j}{\partial x^B},$$

where

$$y^0 = r, \quad y^i = r\Theta^i, \quad \sum_{i=1}^{n} (\Theta^i)^2 = 1.$$

On C_O this leads to an expansion of the form, with the c and the d numbers determined by the $\underline{C_{ij}}$:

$$\begin{aligned} \gamma_{AB} &\equiv \overline{C}_{AB} \\ &= r^2 \Bigg\{ s_{AB} + \frac{\partial \Theta^i}{\partial x^A} \frac{\partial \Theta^j}{\partial x^B} \sum_{p=1}^{k} (c_{ij,h_1\ldots h_p} y^{h_1} \cdots y^{h_p} \\ &\quad + r\, d_{ij,h_1\ldots h_{p-1}} y^{h_1} \cdots y^{h_{p-1}}) \Bigg\} + o_k(r^{k+2}) \\ &=: r^2 \Bigg\{ s_{AB} + \frac{\partial \Theta^i}{\partial x^A} \frac{\partial \Theta^j}{\partial x^B} (c_{ij} + r\, d_{ij}) \Bigg\}. \end{aligned} \tag{3.1}$$

We note that an exhaustive intrinsic description of such tensors γ_{AB} in space–time dimension four can be found in [7]. (We take this opportunity to point out that the ‘only if’ part of Theorem 1.2 of [7] is not sufficiently justified. However, the ‘if’ part is correctly proved, and this is enough to infer Theorem 1.1 of [7], which is the main result there.)

3.1. The functions $|\sigma|^2$ and τ.

The function $|\sigma|^2$ which appears in vacuum as a source of the Einstein wave-map gauge constraints is defined on C_O by

$$|\sigma|^2 := \sigma_A{}^B \sigma_B{}^A,$$

where $\sigma_A{}^C$ is the traceless part of $\frac{1}{2}\gamma^{BC}\partial_1\gamma_{AB}$. We assume that there exists a smooth metric C such that

$$\gamma_{AB} = \overline{C}_{AB},$$

and we start by studying the differentiability properties of possible extensions of $|\sigma|^2$ off the light-cone.

More precisely, let y^μ denote normal coordinates for C centred at O. Set

$$L := y^\mu \frac{\partial}{\partial y^\mu}, \qquad X_{\mu\nu} := \frac{1}{2}\mathcal{L}_L C_{\mu\nu} - C_{\mu\nu}. \tag{3.2}$$

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Let the coordinates x^μ be defined as in (2.1):

$$y^0 = x^1 - x^0, \quad y^i = x^1 \Theta^i(x^A) \quad \text{with} \sum_{i=1}^n (\Theta^i(x^A))^2 = 1. \tag{3.3}$$

We write interchangeably x^1 and r. As already mentioned, we underline the components of the metric associated with the coordinate system y^μ; for example,

$$\underline{C_{\mu\nu}} := C(\partial_{y^\mu}, \partial_{y^\nu}), \quad C_{\mu\nu} := C(\partial_{x^\mu}, \partial_{x^\nu}),$$

and so on. Recall that in normal coordinates it holds that [20] (see [7, Appendix B] for a reference which is easier to access)

$$\underline{C_{\mu\nu}} y^\mu = \underline{\eta_{\mu\nu}} y^\mu. \tag{3.4}$$

One has the identity

$$L \equiv x^0 \partial_{x^0} + x^1 \partial_{x^1}, \tag{3.5}$$

which implies that on the light-cone we have $\overline{L} = x^1 \partial_1$ and

$$\overline{\mathcal{L}_L C_{\mu\nu}} = x^1 \partial_1 \overline{C}_{\mu\nu} + \overline{\delta^0_\mu C_{0\nu} + \delta^0_\nu C_{0\mu} + \delta^1_\mu C_{1\nu} + \delta^1_\nu C_{1\mu}}.$$

In particular,

$$\overline{\mathcal{L}_L C_{AB}} = x^1 \partial_1 \overline{C}_{AB} = x^1 \partial_1 \gamma_{AB}.$$

It follows from definition (3.3) that (3.4) is equivalent to

$$\overline{C}_{01} = 1, \quad \overline{C}_{i1} = 0,$$

which is further equivalent to

$$\overline{C}^{01} = 1, \quad \overline{C}^{00} = \overline{C}^{0A} = 0, \quad \overline{C}^{AB} = \gamma^{AB}, \tag{3.6}$$

where C^{AB} are the contravariant components with indices AB in the coordinates x of the metric C. (The last equation, (3.6), is, of course, a consequence of the remaining ones.) The tensor X defined in (3.2) obeys the key property

$$\overline{X_{\mu 1}} = 0.$$

This allows us to rewrite

$$\frac{1}{2}\gamma^{BC}\partial_1 \gamma_{AB} = \frac{1}{2}\overline{C^{BC}\partial_1 C_{AB}} = \frac{1}{r}(\delta^C_A + \overline{C^{BC} X_{AB}}) = \frac{1}{r}(\delta^C_A + \overline{Z^C}_A), \tag{3.7}$$

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where we have introduced the smooth space–time tensor

$$Z^{\nu}{}_{\mu} := C^{\nu\lambda} X_{\lambda\mu}. \tag{3.8}$$

Hence $\sigma_A{}^B$ can be constructed from the restriction to the cone of the traceless part of $Z^B{}_A$. We can then calculate the norm $|\sigma|^2$ using Z, as follows. We have

$$\overline{\mathrm{tr} Z} = \overline{C^{\mu\nu} X_{\mu\nu}} = \gamma^{AB} \overline{X_{AB}}, \tag{3.9}$$
$$\overline{|Z|^2} := \overline{\mathrm{tr} Z^2} = \overline{C^{\mu\alpha} C^{\nu\beta} X_{\mu\nu} X_{\alpha\beta}} = \gamma^{AB} \gamma^{CD} \overline{X_{AC} X_{BD}}, \tag{3.10}$$

which implies that the norm $|\sigma|^2$ equals $(x^1)^{-2} \equiv r^{-2}$ times the restriction of a smooth function in space–time to the light-cone:

$$|\sigma|^2 \quad \equiv \quad \frac{1}{r^2} \left(\overline{|Z|^2 - \frac{1}{n-1} (\mathrm{tr} Z)^2} \right), \tag{3.11}$$

as desired. Incidentally, this equals $(1/r^2)\overline{|Z^{\mathrm{TF}}|^2}$, where Z^{TF} is the trace-free part of Z.

Note that $\underline{X_{\mu\nu}} = O(r^2)$ along C_O, which shows that for smooth metrics C in normal coordinates the function $|\sigma|^2$ is $O(r^2)$ and has an expansion for any k, up to a factor r^{-2}:

$$|\sigma|^2 \equiv \frac{1}{r^2} \left(\sum_{p=4}^{k} \sigma_{i_1 \dots i_p} y^{i_1} \cdots y^{i_p} + r \sigma'_{i_1 \dots i_{p-1}} y^{i_1} \cdots y^{i_{p-1}} + o_k(r^k) \right). \tag{3.12}$$

This can also be written as

$$|\sigma|^2 \equiv \sum_{p=4}^{k} \sigma_p r^{p-2} + o_k(r^{k-2}),$$

with

$$\sigma_p := \sigma_{i_1 \dots i_p} \Theta^{i_1} \cdots \Theta^{i_p} + \sigma'_{i_1 \dots i_{p-1}} \Theta^{i_1} \cdots \Theta^{i_{p-1}}.$$

When C is used to prescribe γ, the function τ is obtained by integration of one of the wave-map gauge characteristic constraint equations; we return to this in Section 3.3. On the other hand, if C is used to prescribe $\tilde{g} := \overline{g}_{AB}\, dx^A\, dx^B$ directly as $\tilde{g} = \overline{C}_{AB}\, dx^A\, dx^B$, then the *divergence* τ of the light-cone,

$$\tau := \tfrac{1}{2} \overline{g}^{AB} \partial_r \overline{g}_{AB} \tag{3.13}$$

(often denoted by θ in the literature; cf., for example, [**16**]), is calculated directly from $\overline{C}_{AB}$. In that last case, it follows from (3.7)–(3.9) that

$$\tau \equiv \frac{1}{2} \gamma^{AB} \partial_1 \gamma_{AB} = \frac{1}{r} (n - 1 + \gamma^{AB} X_{AB}) = \frac{1}{r} (n - 1 + \overline{\mathrm{tr} Z}). \tag{3.14}$$

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Hence, in such a context the function $r\tau$ is the restriction to the light-cone of a smooth space–time function, with

$$r\tau - (n-1) = O(r^2). \tag{3.15}$$

Summarising, we have proved the following.

PROPOSITION 3.1. *Suppose that σ and τ arise from the light-cone of a C^k metric with affinely parameterised generators. Then $r\tau$ and $r^2|\sigma|^2$ are C^{k-1}-cone-smooth. Further, $|\sigma|^2 = O(r^2)$, and (3.15) holds.*

3.2. Boundary conditions on κ. As discussed in detail in [**5**, **9**], one of the important objects appearing in the formulation of the characteristic initial value problem is the following connection coefficient κ:

$$\nabla_{\partial_r}\partial_r = \kappa\partial_r. \tag{3.16}$$

A rather natural *gauge-choice* is to assume that the generators of the light-cones are affinely parameterised, which translates to the condition $\kappa = 0$. However, it might be more convenient in some situations not to impose this restriction. The question then arises, what is a natural class of functions κ for the problem at hand.

To motivate our hypotheses, suppose, momentarily, that the tensor field $\tilde{g} = \overline{g}_{AB}\, dx^A\, dx^B$ arises from a *smooth vacuum* metric g, using a smooth coordinate system in which the light-cone takes the usual form $\{y^0 = |\vec{y}|\}$, but the coordinates y^μ are not necessarily normal, and so κ is not necessarily zero. In this case τ still behaves as $(n-1)/r$ near and away from $r = 0$; hence is nowhere vanishing for r sufficiently small. We can then algebraically solve for κ from (2.2):

$$\kappa = \frac{1}{\tau}\left(\partial_1\tau + \frac{\tau^2}{n-1} + |\sigma|^2\right). \tag{3.17}$$

We rewrite this in the following way:

$$r\kappa = \frac{1}{(r\tau)}\left(r\partial_1(r\tau) - (r\tau) + \frac{(r\tau)^2}{n-1} + r^2|\sigma|^2\right). \tag{3.18}$$

Now we have seen that, in normal coordinates, $r\tau$ and $r^2|\sigma|^2$ are restrictions to the light-cone of smooth functions, with $r\tau \to_{r\to 0} n-1$. Since τ and σ are intrinsic objects on C_O, it is natural to suppose that these properties will remain true in the new coordinates. But then the right-hand side of (3.18) is cone-smooth. This motivates the condition that $r\kappa$ is the restriction to the light-cone of a smooth function on space–time; equivalently,

$$r\kappa \text{ is cone-smooth.} \tag{3.19}$$

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We also find that

$$\kappa = O(r), \tag{3.20}$$

whenever $|\sigma|^2 = O(r^2)$ together with (3.15) holds.

In any case, the fact that (3.19)–(3.20) are adequate assumptions on κ in the setup of Theorem 1.1 is provided by the following proposition.

PROPOSITION 3.2. *Suppose that τ and σ arise from an affinely parameterised light-cone of a C^k metric C. Then*

$$\kappa \text{ is } O(r) \text{ and } r\kappa \text{ is } C^{k-2}\text{-cone-smooth.}$$

Note that κ here refers to the acceleration parameter of the vacuum metric g with the same fields τ and σ, and not that of C (for which the acceleration parameter κ is zero by hypothesis).

Proof. In normal coordinates y^μ for C we have

$$\partial_1 \tau + \frac{\tau^2}{n-1} + |\sigma|^2 + \overline{T}_{11} = 0, \tag{3.21}$$

where $r\tau$ and $r^2\sigma^2$ arise by restriction of smooth functions on space–time, and where

$$T_{11} = \underline{T_{00}} + 2\underline{T_{0i}}\frac{y^i}{|\vec{y}|} + \underline{T_{ij}}\frac{y^i}{|\vec{y}|}\frac{y^j}{|\vec{y}|}.$$

If we use $\tilde{g} := \overline{C}_{AB}\, dx^A\, dx^B$ as initial data for a vacuum gravitational field, comparing (3.21) with the Raychaudhuri equation (2.2) we will have

$$\kappa\tau = -\overline{\underline{T_{00}} + 2\underline{T_{0i}}\frac{y^i}{|\vec{y}|} + \underline{T_{ij}}\frac{y^i}{|\vec{y}|}\frac{y^j}{|\vec{y}|}}.$$

Equivalently,

$$r\kappa = -\frac{1}{r\tau}\overline{(\underline{T_{00}}r^2 + 2\underline{T_{0i}}y^i t + \underline{T_{ij}}y^i y^j)},$$

which shows that the resulting function κ satisfies (3.19)–(3.20). □

From now on, consistently with the above, we will assume that the parallel transport coefficient κ satisfies (3.19)–(3.20).

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3.3. Integration of τ and of the conformal factor. As already mentioned in the introduction, in the approach of Rendall [19] the tensor field $\tilde{g} = \overline{g}_{AB}\, dx^A\, dx^B$ is taken of the form

$$\overline{g}_{AB} = \Omega^2 \gamma_{AB},$$

where the tensor field γ_{AB} is a priori given. Using (3.28) below, (2.2) becomes then an equation for the conformal factor Ω. Our objective is to show that the function Ω so obtained is cone-smooth when γ_{AB} arises from a smooth metric C: $\gamma_{AB} = \overline{C}_{AB}$.

Indeed, in the remainder of this section we will show that there exists a smooth positive function on space–time, say χ, so that Ω is the restriction to the light-cone of χ. Setting $\check{C}_{AB} = \chi^2 C_{AB}$, we then obtain $\overline{g}_{AB} = \overline{\check{C}}_{AB}$, where $\check{C}$ is a smooth tensor field on space–time. This reduces the study of Rendall's approach to our treatment in Sections 4–6 below.

To prove existence of χ we follow the approach in [4], with some simplifications, and making more precise the results there, as possible in the current context.

To carry out the analysis it is convenient to introduce

$$y := \frac{n-1}{\tau},$$

where τ is the divergence of C_O given by (3.13). In terms of y, the vacuum Raychaudhuri equation (2.2) reads

$$y' = 1 + \kappa y + \frac{1}{n-1}|\sigma|^2 y^2. \tag{3.22}$$

We assume that $|\sigma|^2$ is of the form (3.12); this will be true when the metric C inducing γ is C^{k+1}, and thus for any k when C is smooth. We further assume that κ satisfies (3.19)–(3.20), with $r\kappa$ being the restriction to C_O of a function of C^k differentiability class. Lemma A.1, appendix A, shows that κ has an expansion

$$\kappa = \frac{1}{r}\sum_{p=2}^{k} \kappa_{p-1} r^p + o_k(r^k), \tag{3.23}$$

with

$$\kappa_{p-1} \equiv \kappa_{i_1 \ldots\, i_p} \Theta^{i_1} \cdots \Theta^{i_p} + \kappa'_{i_1 \ldots\, i_{p-1}} \Theta^{i_1} \cdots \Theta^{i_{p-1}}, \tag{3.24}$$

for some collection of numbers $\kappa_{i_1 \ldots\, i_p}$ and $\kappa'_{i_1 \ldots\, i_{p-1}}$.

Using known arguments (compare [1, 8] and [14, Lemma 8.2]), it follows from (3.22) that there exist functions $y_i \in C^\infty(S^{n-1})$ such that

$$\begin{aligned} y &= \sum_{i=1}^{k+2} y_i r^i + o_k(r^{k+2}) \\ &= r + \frac{\kappa_1}{2} r^2 + \sum_{i=3}^{k+2} y_i r^i + o_k(r^{k+2}) \end{aligned} \tag{3.25}$$

(with the first nonzero term in the sum being equal to $\sigma_4 r^5/5$ when $\kappa = 0$), where the y_{p-1} take the form

$$y_{p-1} \equiv y_{i_1 \ldots i_p} \Theta^{i_1} \cdots \Theta^{i_p} + y'_{i_1 \ldots i_{p-1}} \Theta^{i_1} \cdots \Theta^{i_{p-1}}, \tag{3.26}$$

for some collection of numbers $y_{i_1 \ldots i_p}$ and $y'_{i_1 \ldots i_{p-1}}$. Lemma A.1 shows that, for all $k \in \mathbb{N} \cup \{\infty\}$, the function y/r is the restriction to C_O of a C^k function on space–time equal to one at the origin.

Let δy be defined as

$$y = r(1 + \delta y);$$

thus δy is the restriction to C_O of a C^k function on space–time vanishing at the origin. Hence

$$\tau = \frac{n-1}{y} = \frac{n-1}{r(1+\delta y)} = \frac{n-1}{r}\left(1 - \frac{\delta y}{1+\delta y}\right),$$

which shows that $r\tau$ is the restriction to C_O of a C^k function on space–time equal to $n-1$ at the origin.

Let us write

$$\overline{g}_{AB} = e^{\omega} \gamma_{AB}. \tag{3.27}$$

We then have

$$\tau = \partial_1 \log \sqrt{\det \gamma} + \frac{n-1}{2} \partial_1 \omega, \tag{3.28}$$

with $\omega|_{r=0} = 0$. Integrating this equation for ω, Lemma B.1 allows us to assert the following.

PROPOSITION 3.3. *Let $k \in \mathbb{N} \cup \{\infty\}$. Suppose that the metric γ_{AB} arises by restriction to C_O of a C^{k+1} metric in normal coordinates, and that we are given a function $r\kappa$ which is the restriction to C_O of a C^k function vanishing at the origin to order two. Then the conformal factor Ω^2, relating $\overline{g}_{AB}$ and γ_{AB}, obtained by solving the vacuum Raychaudhuri equation,*

$$\partial_1 \tau - \kappa \tau + \frac{\tau^2}{n-1} + |\sigma|^2 = 0, \tag{3.29}$$

is the restriction to C_O of a C^k function which equals one at the vertex. □

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4. Integration of ν_0

From [5], in vacuum and in wave-map gauge the following equation has to hold:

$$\partial_1 \nu^0 = -\left(\frac{\tau}{2} + \kappa\right)\nu^0 + \frac{1}{2}\overline{g}^{AB} r s_{AB}. \tag{4.1}$$

We want to show that the function ν_0, solution of (4.1), is the restriction to the light-cone of a smooth function on space–time. For this we rewrite (4.1) as

$$r\partial_1 \nu^0 = -\left(\frac{r\tau}{2} + r\kappa\right)\nu^0 + \frac{1}{2}\overline{g}^{AB} r^2 s_{AB}. \tag{4.2}$$

Let the conformal factor Ω be defined by

$$\overline{g}_{AB} = \Omega^2 \gamma_{AB}, \tag{4.3}$$

with $\Omega = 1 + O(r^2)$, and let φ be defined as

$$\varphi := \left(\frac{\det \tilde{g}}{\det s_{n-1}}\right)^{1/(2n-2)} = \Omega\left(\frac{\det \gamma}{\det s_{n-1}}\right)^{1/(2n-2)}, \tag{4.4}$$

with $\varphi = r + O(r^3)$; recall that

$$\tau = (n-1)\,\partial_1 \log\varphi \quad \text{equivalently} \quad \partial_1\varphi = \frac{\tau}{n-1}\varphi. \tag{4.5}$$

We assume first that $\kappa = 0$. Using φ we can rewrite (4.1) in the form

$$\partial_1(\nu^0 \varphi^{(n-1)/2}) = \frac{\varphi^{(n-1)/2}}{2}\overline{g}^{AB} r s_{AB}, \tag{4.6}$$

and hence, since $\nu^0\varphi^{(n-1)/2} \to_{r\to 0} 0$,

$$\nu^0(r, x^A) = \frac{\varphi^{-(n-1)/2}(r, x^A)}{2}\int_0^r (\varphi^{(n-1)/2}\overline{g}^{AB} r s_{AB})(s, x^A)\, ds. \tag{4.7}$$

From (3.6) one has

$$\overline{C^{\mu\nu}\eta_{\mu\nu}} = \gamma^{AB} r^2 s_{AB} + 2. \tag{4.8}$$

Using (4.8) one is led to

$$\nu^0(r, x^A) = \frac{\varphi^{-(n-1)/2}(r, x^A)}{2}\int_0^r (\varphi^{(n-1)/2}\Omega^{-2}(\overline{C^{\mu\nu}\eta_{\mu\nu}} - 2))(s, x^A)\, s^{-1}\, ds. \tag{4.9}$$

It now follows from Lemma B.1, appendix B, that ν^0 is the restriction to the light-cone of a smooth function on space–time. One also finds that $\nu^0 \to 1$ as r

approaches zero. A closer inspection of the series expansion [4] of the integrand shows cancellations which lead to

$$\nu^0 = 1 + O(r^4). \tag{4.10}$$

When $\kappa \neq 0$ we let

$$H(r, x^A) = \int_0^r \kappa(s, x^A)\, ds, \tag{4.11}$$

and then (4.9) is replaced by

$$\begin{aligned}\nu^0(r, x^A) = &\frac{(e^{-H}\varphi^{-(n-1)/2})(r, x^A)}{2} \\ &\times \int_0^r (\varphi^{(n-1)/2}\Omega^{-2}(\overline{C^{\mu\nu}\eta_{\mu\nu}} - 2)e^{H(s,x^A)})(s, x^A)\, s^{-1}\, ds,\end{aligned} \tag{4.12}$$

with identical conclusion.

Summarising, we have the following.

PROPOSITION 4.1. *Under the hypotheses of Proposition 3.3, the solution ν_0 of (4.1) is the restriction to C_O of a C^k function which equals one at the vertex.* □

We show in appendix C that for any smooth metric C such that $C_{1A} = C_{11} = 0$, and for any cone-smooth function ν_0, there exists another smooth metric $\tilde{C}$ satisfying $\overline{C}_{AB} = \overline{\tilde{C}}_{AB}$, $\overline{\tilde{C}}_{1A} = \overline{\tilde{C}}_{11} = 0$, and $\overline{\tilde{C}}_{01} = \nu_0$. This is not used in our indirect proof below, but could be used towards a direct proof of our main results in this paper, if such a proof is found.

5. Approximate polynomial solutions

As the next step, we construct a smooth metric which is an approximate solution of the constraint equations.

Throughout this section the x^μ are Cartesian coordinates on $\mathbb{R}^{n+1}$ in which the metric coefficients are smooth, and the light-cone is given by the equation $\eta_{\mu\nu}x^\mu x^\nu = 0$, where $\eta_{\mu\nu}$ is a diagonal matrix with entries $(-1, 1, \ldots, 1)$ on the diagonal. This should not be confused with the coordinates adapted to the light-cone, denoted by x^μ in the remaining sections of this paper. One can think of the coordinates x^μ of this section as the coordinates y^μ of Section 2, except that we are not assuming that the x^μ here are normal for the metric C.

5.1. The scalar wave equation. In our construction of the solutions of the Einstein equations we will need existence and uniqueness of approximate

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polynomial solutions for the scalar wave equation. This can be reduced to an analysis of polynomial solutions of the wave equation in Minkowski space–time. The aim of this section is to establish these results.

Let $\Box_\eta$ denote the Minkowskian wave operator,

$$\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu .$$

We start with the following observation.

LEMMA 5.1. *Let $k \in \mathbf{N}$. For any homogeneous polynomial P of degree k there exists a unique homogeneous polynomial W of degree $k+2$ such that $\Box_\eta W = P$ and $W|_{C_O} = 0$.*

Proof. Any such P can be uniquely written as

$$P = C_{\alpha_1\ldots\alpha_k} x^{\alpha_1} \cdots x^{\alpha_k}, \tag{5.1}$$

where $C_{\alpha_1\ldots\alpha_k}$ is symmetric under permutations, $C_{\alpha_1\ldots\alpha_k} = C_{(\alpha_1\ldots\alpha_k)}$: indeed, the $C_{\alpha_1\ldots\alpha_k}$ can be calculated by differentiating k times the polynomial P, and hence are unique.

We seek a solution of the form

$$W = A_{(\alpha_1\ldots\alpha_k}\eta_{\alpha_{k+1}\alpha_{k+2})} x^{\alpha_1} \cdots x^{\alpha_{k+2}},$$

where $A_{\alpha_1\ldots\alpha_k}$ is also symmetric in all indices. All such polynomials W vanish on the light-cone, as desired.

We start by noting that the map

$$A_{\alpha_1\ldots\alpha_k} \mapsto W = A_{(\alpha_1\ldots\alpha_k}\eta_{\alpha_{k+1}\alpha_{k+2})} x^{\alpha_1} \cdots x^{\alpha_{k+2}}, \tag{5.2}$$

which is surjective by definition, is also injective. Indeed, this statement is equivalent to the fact that the only solution of the equation

$$A_{(\alpha_1\ldots\alpha_k}\eta_{\alpha_{k+1}\alpha_{k+2})} = 0, \tag{5.3}$$

is zero. To see this, let $k+2 = 2m+\epsilon$, with $\epsilon \in \{0,1\}$. Contracting (5.3) with $\eta^{\alpha_1\alpha_2}\ldots\eta^{\alpha_{2m-1}\alpha_{2m}}$ we find that

$$0 = \begin{cases} A^{\alpha_1}{}_{\alpha_1}\ldots{}^{\alpha_m}{}_{\alpha_m}, & k = 2m; \\ A^{\alpha_1}{}_{\alpha_1}\ldots{}^{\alpha_m}{}_{\alpha_m\alpha}, & k = 2m+1. \end{cases}$$

If k equals zero or one we are done. Otherwise one can contract now (5.3) with $\eta^{\alpha_1\alpha_2}\ldots\eta^{\alpha_{2m-3}\alpha_{2m-2}}$, and using the previous equation obtain

$$0 = \begin{cases} A^{\alpha_1}{}_{\alpha_1}\ldots{}^{\alpha_{m-2}}{}_{\alpha_{m-1}\beta\gamma}, & k = 2m; \\ A^{\alpha_1}{}_{\alpha_1}\ldots{}^{\alpha_{m-2}}{}_{\alpha_{m-1}\beta\gamma\delta}, & k = 2m+1. \end{cases}$$

Continuing this way, after a finite number of steps we obtain the vanishing of $A_{\alpha_1\ldots\alpha_k}$, as desired.

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Consider, now, the linear map which to the tensor $A_{\alpha_1\ldots\alpha_k}$ assigns the tensor $C_{\alpha_1\ldots\alpha_k}$, obtained in the obvious way from what has been said so far:

$$A_{\alpha_1\ldots\alpha_k} \longleftrightarrow W \mapsto \Box_\eta W \longleftrightarrow C_{\alpha_1\ldots\alpha_k}.$$

This map is injective: indeed, let $\Box_\eta W_2 = P = \Box W_1$; then $\Box_\eta(W_1 - W_2) = 0$, with $W_1 - W_2 = 0$ on the light-cone, and hence $W_1 - W_2 = 0$ by uniqueness of solutions of the characteristic Cauchy problem on the light-cone. Surjectivity follows now by elementary finite-dimensional algebra. $\square$

For further reference we note that for $k \geqslant 2$ one finds, in space–time dimension $n+1$, that

$$\begin{aligned}
\Box W &= (k+2)\eta^{\mu\nu}\partial_\nu(A_{(\alpha_1\ldots\alpha_k}\eta_{\alpha_{k+1}\mu)}x^{\alpha_1}\cdots x^{\alpha_{k+1}})\\
&= (k+2)(k+1)\eta^{\mu\nu}A_{(\alpha_1\ldots\alpha_k}\eta_{\mu\nu)}x^{\alpha_1}\cdots x^{\alpha_k}\\
&= \frac{(k+2)(k+1)}{(k+2)!}\eta^{\mu\nu}(2k!\,A_{\alpha_1\ldots\alpha_k}\eta_{\mu\nu} + 4\times k\times k!\,A_{\mu(\alpha_1\ldots\alpha_{k-1}}\eta_{\alpha_k)\nu}\\
&\quad + k\times(k-1)\times k!\,A_{\mu\nu(\alpha_1\ldots\alpha_{k-2}}\eta_{\alpha_{k-1}\alpha_k)})x^{\alpha_1}\cdots x^{\alpha_k}\\
&= \eta^{\mu\nu}(2A_{\alpha_1\ldots\alpha_k}\eta_{\mu\nu} + 4kA_{\mu(\alpha_1\ldots\alpha_{k-1}}\eta_{\alpha_k)\nu}\\
&\quad + k(k-1)A_{\mu\nu(\alpha_1\ldots\alpha_{k-2}}\eta_{\alpha_{k-1}\alpha_k)})x^{\alpha_1}\cdots x^{\alpha_k}\\
&= (2(n+2k+1)A_{\alpha_1\ldots\alpha_k} + k(k-1)A^\mu{}_{\mu(\alpha_1\ldots\alpha_{k-2}}\eta_{\alpha_{k-1}\alpha_k)})x^{\alpha_1}\cdots x^{\alpha_k}.
\end{aligned} \tag{5.4}$$

So Lemma 5.1 is equivalent to the statement that the equations

$$2(n+2k+1)A_{\alpha_1\ldots\alpha_k} + k(k-1)A^\mu{}_{\mu(\alpha_1\ldots\alpha_{k-2}}\eta_{\alpha_{k-1}\alpha_k)} = C_{\alpha_1\ldots\alpha_k} \tag{5.5}$$

have a unique totally symmetric solution $A_{\alpha_1\ldots\alpha_k}$ for any totally symmetric $C_{\alpha_1\ldots\alpha_k}$.

A similar but simpler calculation shows that the formula (5.4) remains valid for $k=0$ and 1, and so for example we obtain

$$W = \begin{cases} \dfrac{C}{2(n+1)}\eta_{\alpha\beta}x^\alpha x^\beta, & k=0;\\[2ex] \dfrac{1}{2(n+3)}C_{(\gamma}\eta_{\alpha\beta)}x^\alpha x^\beta x^\gamma, & k=1.\end{cases}$$

As an obvious corollary of Lemma 5.1 one finds the following.

COROLLARY 5.2. *Let $k \in \mathbf{N}$. For any polynomial P of degree k there exists a unique polynomial W of degree $k+2$ such that $\Box_\eta W = P$ and $W|_{C_O} = 0$.* $\square$

Let $\Box_g$ be the Laplace–Beltrami operator of a metric g. We will need the following result, the proof of which gives a taste of the induction needed for the corresponding result for the Einstein equations.

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PROPOSITION 5.3. *Let g be a smooth Lorentzian metric, and let there be given a coordinate system near p such that $x^\mu(p)=0$. For any smooth function ψ there exists a unique polynomial ϕ_{k+2} of degree $k+2$ such that*

$$\Box_g\phi_{k+2}-\psi=O(|x|^{k+1}),\quad \phi_{k+2}|_{C_O}=0. \tag{5.6}$$

If $\psi=O(|x|^\ell)$, then $\phi_{k+2}=O(|x|^{\ell+2})$. The result remains true for $k=\infty$, in the sense that there exists a smooth function ϕ_∞ vanishing at the light-cone such that $\Box_g\phi_\infty-\psi$ vanishes to arbitrary order at the origin, similarly for derivatives of arbitrarily high order of $\Box_g\phi_\infty-\psi$.

Proof. By a linear change of coordinates we can without loss of generality assume that $g(0)=\eta$.

We will use induction upon k.

For $k=0$, existence is obtained by setting $\phi_2=(\psi(0)/2(n+1))\eta_{\alpha\beta}x^\alpha x^\beta$. To prove uniqueness, consider the difference of two such polynomials solving (5.6); call it W. Introduce a new coordinate system where x^i is replaced by ϵx^i; one obtains

$$\partial_i\left(\sqrt{\det g(\epsilon x)}g^{ij}(\epsilon x)\partial_j W(x)\right)=O(\epsilon|x|). \tag{5.7}$$

Passing to the limit $\epsilon\to 0$ we find that

$$\Box_\eta W=0,$$

and, since W vanishes on the light-cone, the vanishing of W follows from, for example, Corollary 5.2.

Suppose, next, that the result has been established for some k; thus there exists a polynomial solution ϕ_{k+2} to (5.6).

Taylor expanding ψ, we can write

$$\psi=\psi_k+\delta\psi_{k+1}+O(|x|^{k+2}),$$

where ψ_k is a polynomial of order k, and $\delta\psi_{k+1}$ is a homogeneous polynomial of order $k+1$. Similarly Taylor expanding $\Box_g\phi_{k+2}$, we can write

$$\Box_g\phi_{k+2}-\psi_k=\chi_{k+1}+O(|x|^{k+2}), \tag{5.8}$$

where χ_{k+1} is a homogeneous polynomial of order $k+1$.

Let $\delta\phi_{k+3}$ be the solution given by Lemma 5.1 of the equation

$$\Box_\eta\delta\phi_{k+3}=\delta\psi_{k+1}-\chi_{k+1}.$$

This implies that

$$\Box_g\delta\phi_{k+3}=\delta\psi_{k+1}-\chi_{k+1}+O(|x|^{k+2}). \tag{5.9}$$

Set

$$\phi_{k+3} = \phi_{k+2} + \delta\phi_{k+3}.$$

Adding (5.8) and (5.9) we obtain

$$\Box_g \phi_{k+3} - \psi_k - \delta\psi_{k+1} = O(|x|^{k+2}), \tag{5.10}$$

which implies (5.6) with k replaced by $k+1$, providing existence of the solution.

Uniqueness follows by a scaling argument similar to the one leading to (5.7), where the equation for the difference W of two such polynomials becomes instead

$$\partial_i(\sqrt{\det g(\epsilon x)}g^{ij}(\epsilon x)\partial_j W(x)) = O(\epsilon|x|^{k+2}). \tag{5.11}$$

When $k = \infty$, the function φ_∞ is obtained from the above sequence of polynomials by Borel summation; see Lemma A.2 below. Uniqueness of φ_∞ up to an $O(|x|^\infty)$-function follows from what has been said, using the fact that the difference W of any two such solutions satisfies (5.11) with an integer k as large as desired. □

We also have a uniqueness result.

PROPOSITION 5.4. *Let g be a smooth Lorentzian metric, and let ϕ be a smooth function such that, for some $\ell \in$ N,*

$$\Box_g \phi = O(|x|^\ell), \quad \phi|_{C_O} = 0. \tag{5.12}$$

Then

$$\phi = O(|x|^{\ell+2}).$$

Proof. Let ϕ_{k+2} be the first nonvanishing homogeneous polynomial of degree $k+2$ in the Taylor expansion of ϕ, and suppose that $k < \ell$. Then ϕ_{k+2} vanishes on C_O, and a Taylor expansion of the left-hand side of (5.12) shows that $\Box_\eta \phi_{k+2} = 0$; hence $\phi_{k+2} = 0$ by Corollary 5.2, a contradiction. □

5.2. The Ricci tensor. We continue with a perturbation lemma; namely, we wish to deform a given smooth metric g to a new smooth metric $\widehat{g}$, with the property that some components of the Ricci tensor of $\widehat{g}$ tend to zero with decay rate ℓ along the light-cone C_O near its tip, with ℓ as large as desired, and such that the new metric coincides with the old one on C_O.

The metric g in the current section should be thought of as the metric C in the remaining parts of the paper. Similarly to Section 5.1, the symbol x^μ is *not* used to denote the coordinates adapted to the light-cone, as is the case in the main body of the paper: these are regular space–time coordinates near the vertex in which the light-cone is given by the Minkowskian equation $\eta_{\mu\nu}x^\mu x^\nu = 0$.

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LEMMA 5.5. *Let g be a smooth Lorentzian metric with the light-cone C_O of O described by the equation $C_O = \{x^\alpha : \eta_{\mu\nu}x^\mu x^\nu = 0\}$, where, as elsewhere, $\eta_{\alpha\beta}$ denotes the Minkowski metric. We assume moreover that*

$$g_{\alpha\beta} - \eta_{\alpha\beta} = O(|x|^2), \quad \partial_\sigma g_{\alpha\beta} = O(|x|). \tag{5.13}$$

For any $\ell \in \mathbb{N} \cup \{\infty\}$ there exists a smooth metric $\widehat{g}$ defined for $|x|$ small enough, which coincides with g on C_O,

$$\overline{g}_{\mu\nu} = \overline{\widehat{g}}_{\mu\nu}, \tag{5.14}$$

and such that

$$\widehat{R}_{\mu\nu} = O(|x|^{\min(\ell,2)}), \quad \widehat{R}_{\mu\nu}x^\nu = O(|x|^{\ell+1}) + \overset{(\ell)}{P}_\mu \eta_{\alpha\beta}x^\alpha x^\beta, \tag{5.15}$$

for small $|x|$, for some smooth functions $\overset{(\ell)}{P}_\nu$, where $\widehat{R}_{\mu\nu}$ denotes the Ricci tensor of the metric $\widehat{g}$, and $\widehat{R}$ its Ricci scalar.

Proof. The Ricci tensor of g can be written as

$$R_{\alpha\beta} = -\tfrac{1}{2}\Box_g g_{\alpha\beta} + \tfrac{1}{2}(g_{\alpha\lambda}\partial_\beta \Gamma^\lambda + g_{\beta\lambda}\partial_\alpha \Gamma^\lambda) + q_{\alpha\beta}(g, \partial g). \tag{5.16}$$

Here, it is usual to take $\Box_g$ to be the Laplace operator acting on functions:

$$\Box_g f = |\det g|^{-1/2}\partial_\mu(|\det g|^{1/2} g^{\mu\nu}\partial_\nu f). \tag{5.17}$$

Further, q is a quadratic form in the first derivatives ∂g of g with coefficients polynomial in g and its contravariant associate, and the Γ^λ are defined as

$$\Gamma^\alpha := g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu}. \tag{5.18}$$

However, instead of (5.17) one can take $g^{\mu\nu}\partial_\mu\partial_\nu$, with a different q in (5.16); this implies that it suffices to do the estimates below using $g^{\mu\nu}\partial_\mu\partial_\nu$.

We assume first that $\ell < \infty$. The proof will be done by induction upon ℓ.

To clarify notation, $\overset{(\ell)}{g}$ will denote a metric satisfying (5.15). We set $\overset{(0)}{g} = g$, consistently with this requirement. In particular, setting $\overset{(0)}{P}_\mu = 0$, the result is true for $\ell = 0$. For $\ell \geqslant 1$ the metric $\overset{(\ell)}{g}$ will be of the form

$$\overset{(\ell)}{g}_{\alpha\beta} = \overset{(\ell-1)}{g}_{\alpha\beta} + \overset{(\ell+1)}{\delta g}_{\alpha\beta}, \tag{5.19}$$

where the correction term $\overset{(\ell+1)}{\delta g}_{\alpha\beta}$ will be $O(|x|^{\ell+1})$ near $x = 0$. Thus, the index ℓ over g denotes the induction step, while the index ℓ over δg denotes the decay rate for small x. We let $\overset{(\ell)}{R}_{\alpha\beta}$ denote the Ricci tensor of $\overset{(\ell)}{g}$.

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The first step is to achieve the result with $\ell = 1$. In this case the first equality in (5.15) is the important one, since the second automatically holds with $\overset{(1)}{P}_\nu = 0$. It follows from the calculations that we are about to do that the result is achieved by setting

$$\widehat{g}_{\mu\nu} = \overset{(1)}{g} := g_{\mu\nu} + \eta_{\alpha\beta}x^\alpha x^\beta A_{\mu\nu}, \tag{5.20}$$

where $A_{\mu\nu}$ is given by (5.34). The formula (5.20) defines a Lorentzian metric for $|x|$ small enough, and maintains (5.13).

Similarly, for the result with $\ell = 2$ only the first equality in (5.15) needs to be established; the second one with $\overset{(2)}{P}_\nu = 0$ automatically follows.

In all subsequent steps one wishes to establish the second equality in (5.15), making sure that the first one remains true at each induction step.

So, assuming that the result is true for some $\ell \geqslant 0$, we write

$$\overset{(\ell+1)}{g}{}_{\alpha\beta} = \overset{(\ell)}{g}{}_{\alpha\beta} + \overset{(\ell+2)}{\delta g}{}_{\alpha\beta}, \tag{5.21}$$

where $\overset{(\ell+2)}{\delta g}$ takes the form

$$\overset{(\ell+2)}{\delta g}{}_{\alpha\beta} := A_{\alpha\beta(\gamma_1\ldots\gamma_\ell}\eta_{\gamma_{\ell+1}\gamma_{\ell+2})}x^{\gamma_1}\cdots x^{\gamma_{\ell+2}}, \tag{5.22}$$

and hence vanishes on C_O. We consider one by one the terms that occur in (5.16) with g there replaced by $\overset{(\ell+1)}{g}$. We assume that (5.15) holds with $\widehat{R}_{\alpha\beta}$ replaced by $\overset{(\ell)}{R}_{\alpha\beta}$, and we want to choose $\overset{(\ell+2)}{\delta g}{}_{\alpha\beta}$ to achieve the corresponding properties of the Ricci tensor of $\overset{(\ell+1)}{g}$.

The quadratic terms are simplest to analyse:

$$q_{\alpha\beta}(\overset{(\ell+1)}{g}, \partial\overset{(\ell+1)}{g}) = q_{\alpha\beta}(\overset{(\ell)}{g}, \partial\overset{(\ell)}{g}) + O(|x|^{\ell+2}). \tag{5.23}$$

Indeed, q is a sum of terms of the form

$$p(\overset{(\ell+1)}{g})\partial\overset{(\ell+1)}{g}\partial\overset{(\ell+1)}{g},$$

for a rational function p of $\overset{(\ell+1)}{g}$, which thus read (keeping in mind that $\partial\overset{(\ell)}{g} = O(|x|)$ for all ℓ)

$$\begin{aligned} &p(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g})\partial(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g})\partial(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g}) \\ &\quad = \underbrace{(p(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g}) - p(\overset{(\ell)}{g}))}_{O(|x|^{\ell+2})}\underbrace{\partial(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g})}_{O(|x|)}\underbrace{\partial(\overset{(\ell)}{g} + \overset{(\ell+2)}{\delta g})}_{O(|x|)} + p(\overset{(\ell)}{g})\partial\overset{(\ell)}{g}\partial\overset{(\ell)}{g} \end{aligned}$$

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$$
+\,2\,\underbrace{p(\overset{(\ell)}{g})}_{O(1)}\,\underbrace{\partial\overset{(\ell)}{g}}_{O(|x|)}\,\underbrace{\partial\overset{(\ell+2)}{\delta g}}_{O(|x|^{\ell+1})}\,+O(|x|^{2\ell+2})
$$

$$
= p(\overset{(\ell)}{g})\partial\overset{(\ell)}{g}\partial\overset{(\ell)}{g} + O(|x|^{\ell+2}).
$$

Now,

$$
-\tfrac{1}{2}\Box_{\overset{(\ell+1)}{g}}\overset{(\ell)}{g}_{\alpha\beta} = -\tfrac{1}{2}\Box_{\overset{(\ell)}{g}}\overset{(\ell)}{g}_{\alpha\beta} + O(|x|^{\ell+2}), \tag{5.24}
$$

$$
-\tfrac{1}{2}\Box_{\overset{(\ell+1)}{g}}\overset{(\ell+1)}{g}_{\alpha\beta} = -\tfrac{1}{2}\Box_{\overset{(\ell)}{g}}\overset{(\ell)}{g}_{\alpha\beta} - \tfrac{1}{2}\Box_{\overset{(\ell)}{g}}\overset{(\ell+2)}{\delta g}_{\alpha\beta} + O(|x|^{\ell+2}), \tag{5.25}
$$

where by (5.4) we also have

$$
\begin{aligned}
-\tfrac{1}{2}\Box_{\overset{(\ell)}{g}}\overset{(\ell+2)}{\delta g}_{\alpha\beta} &= -\tfrac{1}{2}\Box_{\eta}\overset{(\ell+2)}{\delta g}_{\alpha\beta} + O(|x|^{\ell+2}) \\
&= -\tfrac{1}{2}(2(n+2\ell+1)A_{\alpha\beta\alpha_1\ldots\alpha_\ell} \\
&\quad + \ell(\ell-1)A_{\alpha\beta}{}^{\mu}{}_{\mu(\alpha_1\ldots\alpha_{\ell-2}}\eta_{\alpha_{\ell-1}\alpha_\ell)})x^{\alpha_1}\cdots x^{\alpha_\ell} + O(|x|^{\ell+2}).
\end{aligned} \tag{5.26}
$$

Next,

$$
\begin{aligned}
\overset{(\ell+1)}{g}_{\alpha\lambda}\partial_\beta\overset{(\ell+1)}{\Gamma}{}^{\lambda} &\equiv \overset{(\ell+1)}{g}_{\alpha\lambda}\partial_\beta(\overset{(\ell+1)}{g}{}^{\mu\nu}\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu}) = \overset{(\ell)}{g}_{\alpha\lambda}\partial_\beta(\overset{(\ell+1)}{g}{}^{\mu\nu}\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu}) + O(|x|^{\ell+2}) \\
&= \overset{(\ell)}{g}_{\alpha\lambda}\partial_\beta(\overset{(\ell)}{g}{}^{\mu\nu}\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu}) + O(|x|^{\ell+2}) \\
&= \overset{(\ell)}{g}_{\alpha\lambda}\partial_\beta\overset{(\ell)}{\Gamma}{}^{\lambda} + \underbrace{\eta_{\alpha\lambda}\eta^{\mu\nu}\partial_\beta(\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu} - \overset{(\ell)}{\Gamma}{}^{\lambda}_{\mu\nu})} + O(|x|^{\ell+2}),
\end{aligned} \tag{5.27}
$$

where, to estimate the error term in the last line, we have used

$$
\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu} - \overset{(\ell)}{\Gamma}{}^{\lambda}_{\mu\nu} = O(|x|^{\ell+1}), \quad \partial_\beta(\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu} - \overset{(\ell)}{\Gamma}{}^{\lambda}_{\mu\nu}) = O(|x|^{\ell}).
$$

The underbraced expression in (5.27) can be analysed as follows:

$$
\begin{aligned}
\eta^{\mu\nu}(\overset{(\ell+1)}{\Gamma}{}^{\lambda}_{\mu\nu} - \overset{(\ell)}{\Gamma}{}^{\lambda}_{\mu\nu}) &= \tfrac{1}{2}\eta^{\mu\nu}(\overset{(\ell+1)}{g}{}^{\lambda\sigma}(2\partial_\nu\overset{(\ell+1)}{g}_{\mu\sigma} - \partial_\sigma\overset{(\ell+1)}{g}_{\mu\nu}) \\
&\quad - \overset{(\ell)}{g}{}^{\lambda\sigma}(2\partial_\nu\overset{(\ell)}{g}_{\mu\sigma} - \partial_\sigma\overset{(\ell)}{g}_{\mu\nu})) \\
&= \eta^{\mu\nu}\eta^{\lambda\sigma}(\partial_\nu\overset{(\ell+2)}{\delta g}_{\mu\sigma} - \tfrac{1}{2}\partial_\sigma\overset{(\ell+2)}{\delta g}_{\mu\nu}) + O(|x|^{\ell+3}).
\end{aligned}
$$

The underbraced term in (5.27) thus reads

$$
\eta^{\mu\nu}(\partial_\beta\partial_\nu\overset{(\ell+2)}{\delta g}_{\mu\alpha} - \tfrac{1}{2}\partial_\beta\partial_\alpha\overset{(\ell+2)}{\delta g}_{\mu\nu}) + O(|x|^{\ell+2}).
$$

It follows that the sum $\overset{(\ell+1)}{g}{}_{\alpha\lambda}\partial_\beta \overset{(\ell+1)}{\Gamma}{}^{\lambda} + \overset{(\ell+1)}{g}{}_{\beta\lambda}\partial_\alpha \overset{(\ell+1)}{\Gamma}{}^{\lambda}$ gives a contribution to the Ricci tensor $\overset{(\ell+1)}{R}{}_{\alpha\beta}$ of $\overset{(\ell+1)}{g}$ equal to

$$\tfrac{1}{2}\eta^{\mu\nu}(\partial_\beta\partial_\nu \overset{(\ell+2)}{\delta g}{}_{\mu\alpha} + \partial_\alpha\partial_\nu \overset{(\ell+2)}{\delta g}{}_{\mu\beta} - \partial_\beta\partial_\alpha \overset{(\ell+2)}{\delta g}{}_{\mu\nu}) + O(|x|^{\ell+2}). \tag{5.28}$$

All this leads to the formula

$$\begin{aligned}\overset{(\ell+1)}{R}{}_{\alpha\beta} = &-\tfrac{1}{2}\Box_\eta \overset{(\ell+2)}{\delta g}{}_{\alpha\beta} + \tfrac{1}{2}\eta^{\mu\nu}(\partial_\beta\partial_\nu \overset{(\ell+2)}{\delta g}{}_{\mu\alpha} + \partial_\alpha\partial_\nu \overset{(\ell+2)}{\delta g}{}_{\mu\beta} - \partial_\beta\partial_\alpha \overset{(\ell+2)}{\delta g}{}_{\mu\nu})\\ &+ \overset{(\ell)}{R}{}_{\alpha\beta} + O(|x|^{\ell+2}).\end{aligned} \tag{5.29}$$

Inserting (5.22) with $\ell = 0$ into (5.28), one obtains at O

$$2A_{\alpha\beta} - \eta^{\mu\nu}A_{\mu\nu}\eta_{\alpha\beta}. \tag{5.30}$$

Next, the polynomial part of (5.28) with $\ell = 1$ reads

$$3\eta^{\mu\nu}((A_{\alpha\mu(\beta} + A_{\beta\mu(\alpha})\eta_{\nu\gamma)} - A_{\mu\nu(\alpha}\eta_{\beta\gamma)})x^\gamma. \tag{5.31}$$

For $\ell \geqslant 2$ the corresponding calculations require more work: we have

$$\begin{aligned}\partial_\beta\partial_\nu \overset{(\ell+2)}{\delta g}{}_{\mu\alpha} = &(\ell(\ell-1)A_{\mu\alpha\beta\nu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} + 2\ell A_{\mu\alpha\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\nu}\\ &+ 2\ell A_{\mu\alpha\nu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} + 2A_{\mu\alpha\gamma_1\ldots\gamma_{\ell-1}\gamma_\ell}\eta_{\beta\nu})x^{\gamma_1}\cdots x^{\gamma_\ell},\\ \partial_\beta\partial_\alpha \overset{(\ell+2)}{\delta g}{}_{\mu\nu} = &(\ell(\ell-1)A_{\mu\nu\beta\alpha(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} + 2\ell A_{\mu\nu\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}\\ &+ 2\ell A_{\mu\nu\alpha(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} + 2A_{\mu\nu\gamma_1\ldots\gamma_{\ell-1}\gamma_\ell}\eta_{\beta\alpha})x^{\gamma_1}\cdots x^{\gamma_\ell},\end{aligned}$$

which results in a polynomial part of (5.28) equal to

$$\begin{aligned}(&\tfrac{1}{2}\ell(\ell-1)A^\mu{}_{\alpha\beta\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} + \tfrac{1}{2}\ell(\ell-1)A^\mu{}_{\beta\alpha\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)}\\ &+ \ell A_{\alpha(\gamma_1\ldots\gamma_\ell)\beta} + \ell A_{\beta(\gamma_1\ldots\gamma_\ell)\alpha} + \ell A^\mu{}_{\alpha\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} + \ell A^\mu{}_{\beta\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}\\ &+ 2A_{\beta\alpha\gamma_1\ldots\gamma_\ell} - \tfrac{1}{2}\ell(\ell-1)A^\mu{}_{\mu\beta\alpha(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} - \ell A^\mu{}_{\mu\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}\\ &- \ell A^\mu{}_{\mu\alpha(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} - A^\mu{}_{\mu\gamma_1\ldots\gamma_\ell}\eta_{\beta\alpha})x^{\gamma_1}\cdots x^{\gamma_\ell}.\end{aligned} \tag{5.32}$$

Recall that we wish to choose $\overset{(\ell+2)}{\delta g}$ so that the Ricci tensor of $\overset{(\ell+1)}{g}$ satisfies (5.15) with ℓ replaced by $\ell+1$ there. In view of (5.26) with $\ell = 0$ and (5.30), to establish (5.15) with $\ell = 1$ we need to show existence of solutions to the set of equations

$$-(n-1)A_{\alpha\beta} - \eta^{\mu\nu}A_{\mu\nu}\eta_{\alpha\beta} = -R_{\alpha\beta}(O), \tag{5.33}$$

with symmetric tensors $A_{\alpha\beta}$ and $R_{\alpha\beta}(O)$. The solution is

$$A_{\alpha\beta} = \frac{1}{n-1}\left(R_{\alpha\beta}(O) - \frac{1}{2n}\eta^{\mu\nu}R_{\mu\nu}(O)\eta_{\alpha\beta}\right). \tag{5.34}$$

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Having thus established the result with $\ell = 1$, we expand the Ricci tensor $\overset{(0)}{R}$ of $\overset{(0)}{g}$ in Taylor series to order one,

$$\overset{(0)}{R}_{\alpha\beta} = C_{\alpha\beta\gamma} x^\gamma + O(|x|^2).$$

In view of the equations derived so far, we will obtain

$$\overset{(\ell)}{R}_{\alpha\beta} = O(|x|^2) \tag{5.35}$$

with $\ell = 1$ if we can solve the set of equations

$$-(n+3)A_{\alpha\beta\gamma} + 3\eta^{\mu\nu}((A_{\alpha\mu(\beta} + A_{\beta\mu(\alpha})\eta_{\nu\gamma)} - A_{\mu\nu(\alpha}\eta_{\beta\gamma)}) = -C_{\alpha\beta\gamma}, \tag{5.36}$$

keeping in mind that A and C are symmetric in the first two indices. Moreover, because of the contracted Bianchi identity, C satisfies

$$C^\alpha{}_{\alpha\beta} = 2C^\alpha{}_{\beta\alpha}. \tag{5.37}$$

Now, either directly from (5.32), or by expanding, (5.36) can be rewritten as

$$\begin{aligned} &-(n+1)A_{\alpha\beta\gamma} + A_{\alpha\gamma\beta} + A_{\beta\gamma\alpha} + A_{\alpha\mu}{}^{\mu}\eta_{\beta\gamma} + A_{\beta\mu}{}^{\mu}\eta_{\alpha\gamma} - 3A^\mu{}_{\mu(\alpha}\eta_{\beta\gamma)} \\ &\quad = -C_{\alpha\beta\gamma}. \end{aligned} \tag{5.38}$$

As a consistency check with the contracted Bianchi identity, we take a trace in α and β of (5.38) to obtain

$$-2(n+2)A^\alpha{}_{\alpha\gamma} + 4A_{\gamma\alpha}{}^{\alpha} = -C^\alpha{}_{\alpha\gamma},$$

while a trace in α and γ yields

$$-(n+2)A^\alpha{}_{\alpha\beta} + 2A_{\gamma\alpha}{}^{\alpha} = -C^\alpha{}_{\beta\alpha},$$

as required by (5.37).

To invert equation (5.38) we express $A_{\alpha\beta\gamma}$ as a linear combination of all possible linear terms which we can form from $C_{\alpha\beta\gamma}$ with the correct symmetry, with unknown coefficients which need to be determined. Replacing that expression in (5.38) gives a linear system for the coefficients, which we can solve. The result is

$$\begin{aligned} A_{\alpha\beta\gamma} = {}& \frac{1}{(n+2)(n-1)}(nC_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} + C_{\beta\gamma\alpha} - C^\mu{}_{\gamma\mu}\eta_{\alpha\beta}) \\ &+ c(C^\mu{}_{\beta\mu}\eta_{\alpha\gamma} + C^\mu{}_{\alpha\mu}\eta_{\beta\gamma}), \end{aligned} \tag{5.39}$$

where c is an arbitrary constant. Choosing, for example, $c = 0$, establishes our claim with $\ell = 1$.

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A similar, but rather more involved, analysis applies for $\ell \geqslant 2$; note that (5.35) remains true under the current changes of the metric for all $\ell \geqslant 2$.

We Taylor-expand $\overset{(\ell)}{R}_{\alpha\beta}$ to order ℓ. Note that so far all error terms were of order $O(|x|^{\ell+2})$, but this Taylor expansion leaves behind an error term $O(|x|^{\ell+1})$. Denote by

$$C_{\alpha\beta\gamma_1\ldots\gamma_\ell} x^{\gamma_1} \cdots x^{\gamma_\ell}$$

the homogeneous polynomial of order ℓ in that Taylor expansion. In view of (5.26) and (5.32), the homogeneous polynomial of order ℓ in the Taylor expansion of $\overset{(\ell+1)}{R}_{\alpha\beta}$ is

$$\begin{aligned}
&(-\tfrac{1}{2}(2(n+2\ell+1)A_{\alpha\beta\gamma_1\ldots\gamma_\ell} + \ell(\ell-1)A_{\alpha\beta}{}^{\mu}{}_{\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)}) \\
&\quad + \tfrac{1}{2}\ell(\ell-1)A^{\mu}{}_{\alpha\beta\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} + \tfrac{1}{2}\ell(\ell-1)A^{\mu}{}_{\beta\alpha\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} \\
&\quad + \ell A_{\alpha(\gamma_1\ldots\gamma_\ell)\beta} + \ell A_{\beta(\gamma_1\ldots\gamma_\ell)\alpha} + \ell A^{\mu}{}_{\alpha\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} + \ell A^{\mu}{}_{\beta\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha} \\
&\quad + 2A_{\beta\alpha\gamma_1\ldots\gamma_{\ell-1}\gamma_\ell} - \tfrac{1}{2}\ell(\ell-1)A^{\mu}{}_{\mu\beta\alpha(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)} - \ell A^{\mu}{}_{\mu\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha} \\
&\quad - \ell A^{\mu}{}_{\mu\alpha(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta} - A^{\mu}{}_{\mu\gamma_1\ldots\gamma_\ell}\eta_{\beta\alpha} + C_{\alpha\beta\gamma_1\ldots\gamma_\ell})x^{\gamma_1}\cdots x^{\gamma_\ell}.
\end{aligned} \tag{5.40}$$

Multiplying by x^β, and disregarding momentarily all terms involving the Minkowski metric η, we obtain

$$\begin{aligned}
&(-(n+2\ell-1)A_{\alpha\beta\gamma_1\ldots\gamma_\ell} + \ell A_{\beta(\gamma_1\ldots\gamma_\ell)\alpha} + \ell A_{\alpha(\gamma_1\ldots\gamma_\ell)\beta} \\
&\quad + C_{\alpha\beta\gamma_1\ldots\gamma_\ell})x^\beta x^{\gamma_1}\cdots x^{\gamma_\ell}.
\end{aligned} \tag{5.41}$$

Set

$$E_{\alpha\beta\gamma_1\ldots\gamma_\ell} := C_{\alpha(\beta\gamma_1\ldots\gamma_\ell)}; \tag{5.42}$$

thus $E_{\alpha\beta\gamma_1\ldots\gamma_\ell}$ is totally symmetric in the last $\ell+1$ indices. Let us write

$$A_{\alpha\beta\gamma_1\ldots\gamma_\ell} = \underbrace{aC_{\alpha\beta\gamma_1\ldots\gamma_\ell} + b(C_{\alpha(\gamma_1\ldots\gamma_\ell)\beta} + C_{\beta(\gamma_1\ldots\gamma_\ell)\alpha})}_{=:\widehat{A}_{\alpha\beta\gamma_1\ldots\gamma_\ell}} + B_{\alpha\beta\gamma_1\ldots\gamma_\ell} \tag{5.43}$$

where, for reasons that will become apparent shortly, we will choose the constants a and b to cancel the following linear combination of the $E_{\alpha\beta\gamma_1\ldots\gamma_\ell}$ terms in (5.41):

$$-(n+\ell-1)\widehat{A}_{\alpha(\beta\gamma_1\ldots\gamma_\ell)} + \ell\widehat{A}_{(\beta\gamma_1\ldots\gamma_\ell)\alpha} + E_{\alpha\beta\gamma_1\ldots\gamma_\ell} = 0. \tag{5.44}$$

To check that this is possible, we calculate

$$\begin{aligned}
\widehat{A}_{\alpha(\beta\gamma_1\ldots\gamma_\ell)} &= (a+b)E_{\alpha\beta\gamma_1\ldots\gamma_\ell} + bC_{(\beta\gamma_1\ldots\gamma_\ell)\alpha}, \\
\ell\widehat{A}_{\beta\gamma_1\ldots\gamma_\ell\alpha} &= a\ell C_{\beta\gamma_1\ldots\gamma_\ell\alpha} + b\ell(C_{\beta(\gamma_2\ldots\gamma_\ell\alpha)\gamma_1} + C_{\gamma_1(\gamma_2\ldots\gamma_\ell\alpha)\beta}), \\
&= a\ell C_{\beta\gamma_1\ldots\gamma_\ell\alpha} + b(C_{\alpha\beta(\gamma_2\ldots\gamma_\ell)\gamma_1} + (\ell-1)C_{\beta(\gamma_2\ldots\gamma_\ell)\gamma_1\alpha} \\
&\qquad + C_{\alpha\gamma_1(\gamma_2\ldots\gamma_\ell)\beta} + (\ell-1)C_{\gamma_1(\gamma_2\ldots\gamma_\ell)\beta\alpha}), \\
\ell\widehat{A}_{(\beta\gamma_1\ldots\gamma_\ell)\alpha} &= 2bE_{\beta\gamma_1\ldots\gamma_\ell\alpha} + (a\ell + 2b(\ell-1))C_{(\beta\gamma_1\gamma_2\ldots\gamma_\ell)\alpha}.
\end{aligned}$$

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We thus find that (5.44) is equivalent to

$$[-(n+\ell-1)(a+b)+2b+1]E_{\alpha\beta\gamma_1\ldots\gamma_\ell}+\underbrace{[a\ell-(n+1-\ell)b]}C_{(\beta\gamma_1\ldots\gamma_\ell)\alpha} \\ =0. \tag{5.45}$$

We choose a to make the underbraced term vanish,

$$\ell a=(n+1-\ell)b,$$

and then determine b by requiring the vanishing of (5.45):

$$(n-1)(n+\ell+1)b=\ell.$$

Therefore the coefficients are

$$a=\frac{n+1-\ell}{(n-1)(n+1+\ell)},\quad b=\frac{\ell}{(n-1)(n+1+\ell)}.$$

Inserting (5.43) in (5.40), $\overset{(\ell+1)}{R}{}_{\alpha\beta}x^\beta$ now takes the form

$$\begin{aligned}
&\overset{(\ell)}{P}{}_\alpha\eta_{\beta\gamma}x^\beta x^\gamma|_\ell+(-\tfrac{1}{2}(2(n+2\ell+1)B_{\alpha\beta\gamma_1\ldots\gamma_\ell}\\
&\quad+\ell(\ell-1)B_{\alpha\beta}{}^{\mu}{}_{\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)})+\tfrac{1}{2}\ell(\ell-1)B^{\mu}{}_{\alpha\beta\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)}\\
&\quad+\tfrac{1}{2}\ell(\ell-1)B^{\mu}{}_{\beta\alpha\mu(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)}+\ell B_{\alpha(\gamma_1\ldots\gamma_\ell)\beta}\\
&\quad+\ell B_{\beta(\gamma_1\ldots\gamma_\ell)\alpha}+\ell B^{\mu}{}_{\alpha\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta}+\ell B^{\mu}{}_{\beta\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}\\
&\quad+2B_{\beta\alpha\gamma_1\ldots\gamma_{\ell-1}\gamma_\ell}-\tfrac{1}{2}\ell(\ell-1)B^{\mu}{}_{\mu\beta\alpha(\gamma_1\ldots\gamma_{\ell-2}}\eta_{\gamma_{\ell-1}\gamma_\ell)}\\
&\quad-\ell B^{\mu}{}_{\mu\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}-\ell B^{\mu}{}_{\mu\alpha(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\beta}-B^{\mu}{}_{\mu\gamma_1\ldots\gamma_\ell}\eta_{\beta\alpha}+\eta_{\alpha(\beta}\check{C}_{\gamma_1\ldots\gamma_\ell)}\\
&\quad+\eta_{(\beta\gamma_1}\check{C}_{\gamma_2\ldots\gamma_\ell)\alpha})x^\beta x^{\gamma_1}\cdots x^{\gamma_\ell}+O(|x|^{\ell+2}),
\end{aligned}\tag{5.46}$$

for some tensors $\check{C}_{\gamma_1\ldots\gamma_\ell}$ and $\check{C}_{\gamma_1\ldots\gamma_{\ell-1}\alpha}$; we have denoted by $\overset{(\ell)}{P}{}_\alpha\eta_{\beta\gamma}x^\beta x^\gamma|_\ell$ the polynomial of order ℓ in the Taylor series of $\overset{(\ell)}{P}{}_\alpha\eta_{\beta\gamma}x^\beta x^\gamma$. Without loss of generality we can assume that $\check{C}_{\gamma_1\ldots\gamma_\ell}$ is completely symmetric.

Many terms in (5.46) are proportional to $\eta_{\mu\nu}x^\mu x^\nu$, and thus are of the desired form. However, in the homogeneous part of (5.46) of order $\ell+1$ there remain some terms proportional to $x_\alpha := \eta_{\alpha\beta}x^\beta$ which are *not* multiplied by a factor $\eta_{\mu\nu}x^\mu x^\nu$, and which need to be set to zero. We start by removing from (5.46) those terms which obviously vanish on the light-cone; what remains is

$$\begin{aligned}
&(-(n+2\ell+1)B_{\alpha\beta\gamma_1\ldots\gamma_\ell}+\ell B_{\alpha(\gamma_1\ldots\gamma_\ell)\beta}+\ell B_{\beta(\gamma_1\ldots\gamma_\ell)\alpha}\\
&\quad+\ell B^{\mu}{}_{\beta\mu(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}+2B_{\beta\alpha\gamma_1\ldots\gamma_{\ell-1}\gamma_\ell}-\ell B^{\mu}{}_{\mu\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha}\\
&\quad-B^{\mu}{}_{\mu\gamma_1\ldots\gamma_\ell}\eta_{\beta\alpha}+\eta_{\alpha(\beta}\check{C}_{\gamma_1\ldots\gamma_\ell)})x^\beta x^{\gamma_1}\cdots x^{\gamma_\ell}+O(|x|^{\ell+2}).
\end{aligned}\tag{5.47}$$

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To continue, the tensor $B_{\alpha\beta\gamma_1\ldots\gamma_\ell}$ in (5.43) is taken of the form

$$B_{\alpha\beta\gamma_1\ldots\gamma_\ell} = \eta_{\alpha\beta} B_{\gamma_1\ldots\gamma_\ell}, \tag{5.48}$$

where $B_{\gamma_1\ldots\gamma_\ell}$ is symmetric in all indices. The formula (5.47) becomes, up to terms which vanish on the light-cone,

$$\begin{aligned} &(-(n+2\ell-1)\eta_{\alpha\beta}B_{\gamma_1\ldots\gamma_\ell} + \ell\eta_{\alpha(\gamma_1}B_{\ldots\gamma_\ell)\beta} + \ell B_{\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha} \\ &\quad - \ell(n-1)B_{\beta(\gamma_1\ldots\gamma_{\ell-1}}\eta_{\gamma_\ell)\alpha} - (n-1)B_{\gamma_1\ldots\gamma_\ell}\eta_{\beta\alpha} \\ &\quad + \eta_{\alpha(\beta}\check{C}_{\gamma_1\ldots\gamma_\ell)})x^\beta x^{\gamma_1}\cdots x^{\gamma_\ell} + O(|x|^{\ell+2}). \end{aligned} \tag{5.49}$$

Equivalently,

$$\eta_{\alpha(\beta}(-(n+1)(\ell+2)B_{\gamma_1\ldots\gamma_\ell)} + \check{C}_{\gamma_1\ldots\gamma_\ell)})x^\beta x^{\gamma_1}\cdots x^{\gamma_\ell} + O(|x|^{\ell+2}). \tag{5.50}$$

Setting

$$B_{\gamma_1\ldots\gamma_\ell} = \frac{1}{(n+1)(\ell+2)}\check{C}_{\gamma_1\ldots\gamma_\ell},$$

the polynomial in (5.50) vanishes. This finishes the induction, and proves the result for all $\ell \in \mathbf{N}$.

When $\ell = \infty$, the result is obtained by Borel-summing (see Lemma A.2) the sequence of corrections $\overset{(\ell+2)}{\delta g}$ constructed above. □

For the purposes of Theorem 6.1 below it is convenient to have the conclusion of Lemma 5.5 in coordinates which are harmonic for the metric $\widehat{g}$. Note that the transition to such coordinates as in the next lemma will *not* change $\tilde{g}$, though it will in general change the remaining metric functions on C_O.

LEMMA 5.6. *Under the hypotheses of Lemma 5.5, for any $\ell \in \mathbb{N}\cup\{\infty\}$ there exists a smooth metric $\widehat{g}$ defined for $|x|$ small enough, such that the tensor field $\tilde{\widehat{g}} = \hat{g}_{AB}|_{C_O}\, dx^A\, dx^B$ induced by $\widehat{g}$ on C_O coincides with $\tilde{g}$, such that with (5.15) holding for small $|x|$, and the coordinates in which (5.15) holds can be chosen to be harmonic for the metric $\widehat{g}$, coinciding with the original ones on the light-cone.*

Proof. We define $\overset{(\ell)}{x}{}^\mu$ as being normal-wave coordinates for a metric $\overset{(\ell)}{g}$ defined using a modification, explained below, of the proof of Lemma 5.5: by definition, these are coordinates which satisfy the wave equation in the metric $\overset{(\ell)}{g}$, with $\overset{(\ell)}{x}{}^\mu$ coinciding with the original normal coordinates x^μ on the light-cone.

Although some components of the metric tensor on C_O will change when passing to the new coordinates, the AB components will not. We need to marginally modify the construction of Lemma 5.5 so that the introduction of

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harmonic coordinates does not affect the remaining conclusions of that lemma, as follows.

We start with an observation. Suppose that a function $f + \delta f$ solves the wave equation for a metric h. Given any other metric g, we then have

$$\begin{aligned} 0 &= \Box_h(f+\delta f) = h^{\mu\nu}\partial_\mu\partial_\nu(f+\delta f) - h^{\mu\nu}\Gamma(h)^\lambda{}_{\mu\nu}\,\partial_\lambda(f+\delta f) \\ &= (h^{\mu\nu}-g^{\mu\nu})\partial_\mu\partial_\nu(f+\delta f) + (g^{\mu\nu}-h^{\mu\nu})\Gamma(h)^\lambda{}_{\mu\nu}\,\partial_\lambda(f+\delta f) \\ &\quad + g^{\mu\nu}(\Gamma(g)^\lambda{}_{\mu\nu} - \Gamma(h)^\lambda{}_{\mu\nu})\,\partial_\lambda(f+\delta f) + \underbrace{\Box_g(f+\delta f)}_{=\Box_g\delta f \text{ if } \Box_g f = 0}. \end{aligned} \tag{5.51}$$

We consider (5.51) with $f = x^\mu$, where x^μ denotes normal coordinates for the metric g, and with $h := \overset{(0)}{g} \equiv g$, $\delta f = \overset{(0)}{x}{}^\mu - x^\mu$. We then have $\partial(f+\delta f) = O(1)$, $\partial\partial(f+\delta f) = O(1)$, $g^{\mu\nu} - h^{\mu\nu} = O(|x|^2)$, $\Gamma(g)^\lambda{}_{\mu\nu} = O(|x|)$, $\Gamma(h)^\lambda{}_{\mu\nu} = O(|x|)$, $\Box_g f = O(|x|)$, and so (5.51) implies that

$$\Box_g(\overset{(0)}{x}{}^\mu - x^\mu) = O(|x|).$$

Proposition 5.4 gives

$$\overset{(0)}{x}{}^\mu - x^\mu = O(|x|^3). \tag{5.52}$$

From the tensorial transformation law of the Ricci tensor, we conclude that after the coordinate change $x^\mu \to \overset{(0)}{x}{}^\mu$, the equation

$$\overset{(0)}{R}_{\alpha\beta}(O) = 0$$

will still hold in the new coordinates. Then, in the proof of Lemma 5.5 we make this coordinate change after having constructed the metric $\overset{(0)}{g}$ there. The construction of the metric $\overset{(1)}{g}$ in that proof is thus done using the coordinates $\overset{(0)}{x}{}^\mu$.

To continue, we write

$$\overset{(\ell+1)}{x}{}^\mu = \overset{(\ell)}{x}{}^\mu + \delta\overset{(\ell+3)}{x}{}^\mu,$$

where the notation anticipates the fact, which we are about to prove, that the coordinates $\overset{(\ell+1)}{x}{}^\mu$ differ from the coordinates $\overset{(\ell)}{x}{}^\mu$ by terms which are $O(|x|^{\ell+3})$. We consider (5.51) with $f = \overset{(\ell)}{x}{}^\mu$, $g = \overset{(\ell)}{g}$, $h = \overset{(\ell+1)}{g}$, and $\delta f = \delta\overset{(\ell+3)}{x}{}^\mu$. We again have $\partial(f+\delta f) = O(1)$, $\partial\partial(f+\delta f) = O(1)$, $\Gamma(g)^\lambda{}_{\mu\nu} = O(|x|)$, $\Gamma(h)^\lambda{}_{\mu\nu} = O(|x|)$, but now $g^{\mu\nu} - h^{\mu\nu} = O(|x|^{\ell+2})$, $\Gamma(g)^\lambda{}_{\mu\nu} - \Gamma(h)^\lambda{}_{\mu\nu} = O(|x|^{\ell+1})$, and $\Box_g f = 0$. It then follows from (5.51) that

$$\Box_g\delta\overset{(\ell+3)}{x}{}^\mu = O(|x|^{\ell+1}),$$

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and Proposition 5.4 gives

$$\delta \overset{(\ell+3)}{x}{}^{\mu} = O(|x|^{\ell+3}), \tag{5.53}$$

as anticipated by the notation.

This shows that, in the proof of Lemma 5.5, after having constructed the metric $\overset{(\ell+1)}{g}$, a coordinate change

$$\overset{(\ell)}{x} \to \overset{(\ell+1)}{x},$$

will preserve (5.15) (with x^{μ} there equal to $\overset{(\ell)}{x}{}^{\mu}$), and for $\ell < \infty$ the proof is completed.

If $\ell = \infty$, the construction above provides a sequence of Taylor coefficients of the metric which are needed so that both (5.15) and the harmonicity vector vanish to any order. Using Borel summation, we obtain a metric for which both $R_{\mu\nu}x^{\mu}$ and the wave-gauge vector vanish at the vertex of the light-cone to infinite order along C_O. Denoting by y^{μ} the normal-wave coordinates for this metric, by Proposition 5.4 we have

$$y^{\mu} - x^{\mu} = O(|x|^{\infty}).$$

Transforming the metric to the y-coordinates, the result follows. □

6. The remaining constraints: the $(\kappa, \tilde{g})$ scheme

In this section we consider the scheme of [5], where one seeks a metric which realises the initial data $(\kappa, \tilde{g})$ satisfying the first constraint equation (3.29). We further assume that $\tilde{g}$ is induced on C_O by a smooth metric C. The analysis of Section 3 shows how the unconstrained scheme, where κ and the conformal class $[\tilde{g}]$ are prescribed, is reduced to the current one, by rescaling C by a conformal factor, and again calling C the resulting metric.

Let $\check{C}$ be the metric obtained by applying Lemma 5.6 of Section 5.2 to the metric C, so that the Ricci tensor $\check{R}_{\mu\nu}$ of $\check{C}_{\mu\nu}$ satisfies

$$\underline{\check{R}_{\alpha\mu}}\check{y}^{\alpha}|_{C_O} = O_{\ell}(r^{\ell}), \tag{6.1}$$

for any ℓ when C is smooth. This equation holds in coordinates near O, which we denote by $\check{y}^{\mu}$, such that $\check{y}^{\mu} = y^{\mu}$ on the light-cone and such that

$$\Box_{\check{C}}\check{y}^{\mu} = 0. \tag{6.2}$$

The symbols $\underline{\check{C}_{\mu\nu}}$ will refer to the coefficients of the metric $\check{C}$ in these coordinates. Then the coordinates $\check{x}^{\mu}$, constructed as in (3.3) using the $\check{y}^{\mu}$ instead of the y^{μ}, coincide on C_O with the x^μ. The tensor field $\overline{C}_{AB}\,dx^A\,dx^B$ is intrinsic to C_O, and thus coincides with $\overline{\check{C}}_{AB}d\check{x}^Ad\check{x}^B$. Hence, in the checked coordinates $\check{x}^\mu$ we still have

$$\check{C}_{AB}(\check{r}=r,\check{x}^A=x^A)=C_{AB}(r,x^A)=:g_{AB}(r,x^A).$$

Let $\check{H}^\mu$ be the wave-map gauge vector associated with the metric $\check{C}$,

$$\check{H}^\mu := \underbrace{\check{C}^{\alpha\beta}(\check{\Gamma}^\mu_{\alpha\beta}}_{=:\check{\Gamma}^\mu}-\hat{\Gamma}^\mu_{\alpha\beta}) =: \check{\Gamma}^\mu-\check{W}^\mu, \tag{6.3}$$

where the $\hat{\Gamma}^\mu_{\alpha\beta}$ are the Christoffel symbols of the flat metric

$$\hat{g}\equiv\eta=-(d\check{y}^0)^2+(d\check{y}^1)^2+\cdots+(d\check{y}^n)^2=-d\check{u}^2+2d\check{u}d\check{r}+\check{r}^2s_{AB}d\check{x}^Ad\check{x}^B.$$

It follows from (6.2) that all the components $\underline{\check{H}}^\mu$ vanish; hence we have $\check{H}^\mu=0$ in any coordinates.

Summarising,

$$\overline{\check{C}}_{AB}=\overline{C}_{AB}=\overline{g}_{AB}\quad \text{at } r=\check{r},\, x^A=\check{x}^A \text{ and } \overline{\check{H}}^\mu=0. \tag{6.4}$$

Let us denote by $\check{\tau}$, $\check{\sigma}$, and so on, the fields τ and σ associated with the metric $\check{C}$; for example,

$$\check{\chi}_{AB}:=\tfrac{1}{2}\partial_{\check{r}}\check{C}_{AB}. \tag{6.5}$$

From (6.4) we find in particular that

$$\check{\sigma}_{AB}=\sigma_{AB}\quad\text{and}\quad\check{\tau}=\tau\quad\text{at } r=\check{r},\,x^A=\check{x}^A. \tag{6.6}$$

Set

$$\check{\kappa}:=\overline{\check{\Gamma}}{}^1_{11}.$$

Let $\ell^\mu=\check{x}^\mu/\check{r}$. From [5, Equation (6.11)] we have

$$\begin{aligned}O_\ell(\check{r}^{\ell-1})=\overline{\check{R}}_{\mu\nu}\ell^\mu\ell^\nu &= -\partial_1\check{\tau}+\overline{\check{\Gamma}}{}^1_{11}\tau-\check{\chi}_A{}^B\check{\chi}_B{}^A\\ &= -\partial_1\tau+\check{\kappa}\tau-\tfrac{\tau^2}{n-1}-|\sigma|^2.\end{aligned} \tag{6.7}$$

Keeping in mind the equation satisfied by τ,

$$\partial_1\tau-\kappa\tau+\frac{\tau^2}{n-1}+|\sigma|^2=0, \tag{6.8}$$

and using the fact that τ behaves as $(n-1)/r$ for small r, we conclude, at $r=\check{r}$, that

$$\tau(\check{\kappa}-\kappa)=O_\ell(r^{\ell-1})\Longrightarrow\check{\kappa}-\kappa=O_\ell(r^\ell). \tag{6.9}$$

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To continue, recall the identities [5, Appendix A]

$$
\begin{aligned}
\overline{\check{\Gamma}}{}^1_{11} &= \check{\nu}^0\partial_1\check{\nu}_0 - \tfrac{1}{2}\check{\nu}^0\overline{\partial_0\check{g}_{11}}, && (6.10)\\
\check{\nu}_0\overline{\check{\Gamma}}{}^0 &= \check{\nu}^0\overline{\partial_0\check{g}_{11}} - \tfrac{1}{2}\overline{\check{g}}^{AB}\partial_1\overline{\check{g}}_{AB} = \check{\nu}^0\overline{\partial_0\check{g}_{11}} - \check{\tau}, && (6.11)\\
\overline{\check{W}}{}^0 &= -\check{r}\overline{\check{g}}^{AB}s_{AB}; && (6.12)
\end{aligned}
$$

hence, since $\check{H}^\mu \equiv \check{\Gamma}^\mu - \check{W}^\mu = 0$,

$$
\begin{aligned}
\check{\kappa} \equiv \overline{\check{\Gamma}}{}^1_{11} &= \check{\nu}^0\partial_1\check{\nu}_0 - \tfrac{1}{2}(\check{\nu}_0\overline{\check{\Gamma}}{}^0 + \check{\tau})\\
&= \check{\nu}^0\partial_1\check{\nu}_0 - \tfrac{1}{2}(-\check{r}\check{\nu}_0\overline{\check{g}}^{AB}s_{AB} + \check{\tau}).
\end{aligned} \tag{6.13}
$$

Keeping in mind that $\check{\nu}^0 = 1/\check{\nu}_0$, we obtain

$$
\partial_1\check{\nu}_0 = (\check{\kappa} + \tfrac{1}{2}(-(\check{\nu}^0)^2\check{r}\overline{\check{g}}^{AB}s_{AB} + \check{\tau}))\check{\nu}_0; \tag{6.14}
$$

equivalently,

$$
\begin{aligned}
\partial_1\check{\nu}^0 &= -\big(\check{\kappa} + \tfrac{\check{\tau}}{2}\big)\check{\nu}^0 + \tfrac{1}{2}\check{r}\overline{\check{g}}^{AB}s_{AB}\\
&= -\big(\kappa + \tfrac{\tau}{2} + O_\ell(r^\ell)\big)\check{\nu}^0 + \tfrac{1}{2}r\overline{g}^{AB}s_{AB}.
\end{aligned} \tag{6.15}
$$

Comparing with the equation satisfied by ν^0,

$$
\partial_1\nu^0 = -\left(\frac{\tau}{2} + \kappa\right)\nu^0 + \frac{1}{2}\overline{g}^{AB}rs_{AB}, \tag{6.16}
$$

and using the fact that $\check{\nu}^0$ is smooth, and hence $\check{\nu}^0 = O_\ell(1)$ for any ℓ, we find that

$$
\partial_1(\nu^0 - \check{\nu}^0) = -\left(\frac{\tau}{2} + \kappa\right)(\nu^0 - \check{\nu}^0) + O_\ell(r^\ell). \tag{6.17}
$$

Integrating, we conclude that

$$
\nu_0 = \check{\nu}_0 + O_\ell(r^\ell). \tag{6.18}
$$

6.1. Integration of the second constraint. With a Minkowski target the vacuum wave-map gauge $\mathcal{C}_A$ constraint reduces to [5]

$$
\mathcal{C}_A \equiv -\tfrac{1}{2}(\partial_1\xi_A + \tau\xi_A) + \tilde{\nabla}_B\chi_A{}^B - \partial_A\tau = 0, \tag{6.19}
$$

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where $\tilde{\nabla}$ is the covariant derivative operator of the metric $\overline{g}_{AB}\,dx^A\,dx^B$, and where ξ_A is defined as

$$\xi_A = -2\nu^0\partial_1\nu_A + 4\nu^0\nu_B\chi_A{}^B + f_A, \tag{6.20}$$

with [5, Section 8.1]

$$f_A = -\left(r\overline{g}^{CD}s_{CD} + \frac{2\nu^0}{r}\right)\nu_A + \overline{g}_{AB}\overline{g}^{CD}(S^B_{CD} - \tilde{\Gamma}^B_{CD}), \tag{6.21}$$

and where the $\tilde{\Gamma}^B_{CD}$ are the Christoffel symbols of the metric $\overline{g}_{AB}\,dx^A\,dx^B$. On the other hand, for the metric $\check{C}$ we have the identity

$$-\frac{1}{2}(\partial_1\check{\xi}_A + \check{\tau}\check{\xi}_A) + \tilde{\nabla}_B\check{\chi}_A{}^B - \partial_A\check{\tau} = \frac{\partial y^i}{\partial x^A}\overline{\check{R}_{i\nu}\ell^\nu} = O_\ell(\check{r}^{\ell-1}), \tag{6.22}$$

where $\check{\xi}_A$ is

$$\begin{aligned}\check{\xi}_A &= -2\check{\nu}^0\partial_1\check{\nu}_A + 4\check{\nu}^0\check{\nu}_B\check{\chi}_A{}^B + \check{f}_A\\ &= -2\nu^0\partial_1\check{\nu}_A + 4\nu^0\check{\nu}_B\chi_A{}^B + \check{f}_A + O_\ell(\check{r}^\ell).\end{aligned} \tag{6.23}$$

In the second line above we have used the calculations in [4], which show that

$$\check{\nu}_A = O_\ell(\check{r}^3).$$

Further,

$$\begin{aligned}\check{f}_A &= -\left(\check{r}\overline{\check{g}}^{CD}s_{CD} + \frac{2\check{\nu}^0}{\check{r}}\right)\check{\nu}_A + \overline{\check{g}}_{AB}\overline{\check{g}}^{CD}(S^B_{CD} - \tilde{\check{\Gamma}}^B_{CD})\\ &= -\left(r\overline{g}^{CD}s_{CD} + \frac{2\nu^0}{r} + O_\ell(r^{\ell-1})\right)\check{\nu}_A\\ &\quad + \overline{g}_{AB}\overline{g}^{CD}(S^B_{CD} - \tilde{\Gamma}^B_{CD}),\end{aligned} \tag{6.24}$$

at $r = \check{r}$.

Set

$$\delta\nu_A := \nu_A - \check{\nu}_A, \quad \delta\xi_A := \xi_A - \check{\xi}_A.$$

Subtracting (6.19) from (6.22), one obtains

$$-\tfrac{1}{2}(\partial_1\delta\xi_A + \tau\delta\xi_A) = O_\ell(r^{\ell-1}). \tag{6.25}$$

Integrating, one finds that

$$\delta\xi_A(r, x^A) = O_\ell(r^\ell). \tag{6.26}$$

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Subtracting (6.23) from (6.20), we obtain

$$-2\partial_1\delta\nu_A - \left(r\overline{g}^{CD}s_{CD}\nu^0 + \frac{2}{r}\right)\delta\nu_A + 4\chi_A{}^B\delta\nu_B = O_\ell(r^\ell). \tag{6.27}$$

Integrating again, Proposition B.5 in appendix B gives

$$\nu_A = \check{\nu}_A + O_\ell(r^{\ell+1}). \tag{6.28}$$

6.2. Integration of the third constraint. We pass now to the '$\mathcal{C}_0$ constraint operator' of [5]. It arises from an identity, which for the $\check{C}$-metric takes the form

$$\begin{aligned} 0 = &(\check{\nu}^0)^2[2\partial_1^2(\overline{\check{C}}_{00} - \overline{g}^{AB}\check{\nu}_A\check{\nu}_B) - (\tau + 4\overline{W}^1)\partial_1(\overline{\check{C}}_{00} - \overline{g}^{AB}\check{\nu}_A\check{\nu}_B)\\ &+ (-\partial_1(\tau + 2\overline{W}^1) + \overline{W}^1(\tau + 2\overline{W}^1))(\overline{\check{C}}_{00} - \overline{g}^{AB}\check{\nu}_A\check{\nu}_B)]\\ &-2(\partial_1\overline{W}^1 + \tau\overline{W}^1) - \tilde{R} + \tfrac{1}{2}\overline{g}^{AB}\check{\xi}_A\check{\xi}_B - \overline{\check{C}}^{AB}\tilde{\nabla}_A\check{\xi}_B\\ &-\overline{\check{S}}_{11}\overline{\check{C}}^{11} - 2\overline{\check{S}}_{1A}\overline{\check{C}}^{1A} - 2\overline{\check{S}}_{01}\overline{\check{C}}^{01}, \end{aligned} \tag{6.29}$$

where $\check{S}$ is the Einstein tensor of the metric $\check{C}$; here, for simplicity, we have omitted to put hats on those fields which coincide with their unhatted equivalents, for example $\widehat{\tau} = \tau$, and so on. For the vacuum metric $g_{\mu\nu}$ that we seek to construct, this provides instead a constraint-type equation for $\overline{g}_{00}$:

$$\begin{aligned} 0 = &(\nu^0)^2[2\partial_1^2(\overline{g}_{00} - \overline{g}^{AB}\nu_A\nu_B) - (\tau + 4\overline{W}^1)\partial_1(\overline{g}_{00} - \overline{g}^{AB}\nu_A\nu_B)\\ &+ (-\partial_1(\tau + 2\overline{W}^1) + \overline{W}^1(\tau + 2\overline{W}^1))(\overline{g}_{00} - \overline{g}^{AB}\nu_A\nu_B)]\\ &-2(\partial_1\overline{W}^1 + \tau\overline{W}^1) - \tilde{R} + \tfrac{1}{2}\overline{g}^{AB}\check{\xi}_A\check{\xi}_B - \overline{g}^{AB}\tilde{\nabla}_A\check{\xi}_B. \end{aligned} \tag{6.30}$$

Subtracting (6.29) from (6.30), we obtain an ODE for $\overline{\check{C}_{00} - g_{00}}$ which, as before, leads to

$$\overline{g}_{00} = \overline{\check{C}}_{00} + O_\ell(r^\ell).$$

To establish this, the reader might find it convenient to argue in two steps, by first considering the first-order ODE satisfied by the difference between $\partial_1(\overline{g}_{00} - \overline{g}^{AB}\nu_A\nu_B)$ and $\partial_1(\overline{C}_{00} - \overline{g}^{AB}\nu_A\nu_B)$.

6.3. End of the proof. Let $C_{\mu\nu}$ be a smooth metric, and let κ be a function on C_O such that κ/r extends to a smooth function on space–time.

From what has been said, there exist smooth space–time functions $\delta\check{C}_{0A}$, $\delta\check{C}_{01}$ and $\delta\check{C}_{00}$ vanishing to infinite order at the origin such that

$$\overline{\delta\check{C}_{0A}} = -\overline{\check{C}}_{0A} + \nu_A, \quad \overline{\delta\check{C}_{00}} = -\overline{\check{C}}_{00} + \overline{g}_{00} \quad \overline{\delta\check{C}_{01}} = -\overline{\check{C}}_{01} + \nu_0.$$

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Then the tensor field $\delta\check{C}$ defined as

$$\delta\check{C} := 2\delta\check{C}_{01}du\,dr + 2\delta\check{C}_{0A}du\,dx^A + \delta\check{C}_{00}du^2$$

has smooth components $\underline{\delta\check{C}_{\mu\nu}}$, and satisfies

$$\overline{\delta\check{C}_{AB}} = 0 = \overline{\delta\check{C}_{A1}} = \overline{\delta\check{C}_{11}}.$$

It follows that the tensor

$$\underline{\check{C}_{\mu\nu} + \delta\check{C}_{\mu\nu}}$$

has smooth components, and satisfies the Raychaudhuri constraint equation (3.29) with prescribed function κ, as well as the remaining wave-map gauge constraint equations. The existence theorem of [12] shows existence of a smooth metric $g_{\mu\nu}$, defined in a neighbourhood of the vertex O, which satisfies the vacuum Einstein equations to the future of O, such that

$$\underline{g_{\mu\nu}}|_{C_O} = \underline{\check{C}_{\mu\nu} + \delta\check{C}_{\mu\nu}}.$$

It then follows from the analysis in [5] that $H^\mu \equiv 0$ (compare the argument at the end of Section 7), and that $g_{\mu\nu}$ solves the Einstein vacuum equations to the future of O, with

$$\overline{\Gamma}{}^1_{11} = \kappa.$$

We have therefore proved the following.

THEOREM 6.1. *Consider a pair* $(\kappa, \tilde{g})$*, where* $\tilde{g}$ *is a symmetric tensor field induced by a smooth Lorentzian metric C on its null cone* C_O *with vertex at* O*, and where* $r\kappa$ *is the restriction to* C_O *of a smooth function on space–time vanishing to second order at* O*. Suppose moreover that* $(\kappa, \tilde{g})$ *satisfy the Raychaudhuri equation*

$$\partial_1\tau - \kappa\tau + \frac{\tau^2}{n-1} + |\sigma|^2 = 0, \tag{6.31}$$

where τ *is the divergence of* C_O *and* σ *its shear. Then there exists a smooth metric* g*, defined in neighbourhood* $\mathscr{O}$ *of* O *and solving the vacuum Einstein equations in* $I^+(O) \cap \mathscr{O}$*, such that* C_O *is the light-cone of* g*,* $\tilde{g}$ *is the tensor field induced by* g *on* $C_O \setminus \{O\}$*, and* κ *determines parallel transport along the generators of* C_O*: in adapted coordinates,*

$$\nabla_{\partial_r}\partial_r = \kappa\partial_r.$$

We note that (6.31) is a necessary condition for g to be vacuum, so Theorem 6.1 is in fact an if-and-only-if statement.

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7. The $\overline{g}_{\mu\nu}$ scheme

In this section we prove Theorem 1.3, namely existence of solutions of the vacuum Cauchy problem on the light-cone in the scheme of [9], where all the metric functions are prescribed by restricting a smooth metric C to its light-cone.

As in our previous treatment, we use a 'generalised wave-map gauge' with target metric $\hat{g}$ being the Minkowski metric $\eta = -(dy^0)^2 + (dy^1)^2 + \cdots + (dy^n)^2$. As gravitational initial data, we choose a smooth tensor field C. The coordinates y are chosen so that the future light-cone C_O of C with vertex at O coincides with the Minkowskian light-cone $y^0 = |\vec{y}|$. We then use the metric components $\overline{C_{\mu\nu}} = C_{\mu\nu}|_{C_O}$ as initial data for g:

$$\overline{g}_{\mu\nu} := \overline{C}_{\mu\nu}.$$

It follows from Lemma 5.5 that there exists a metric $\check{C}$ such that

$$\overline{g}_{\mu\nu} = \overline{C}_{\mu\nu} = \overline{\check{C}}_{\mu\nu}, \tag{7.1}$$

with the Ricci tensor $\check{R}_{\mu\nu}$ of the metric $\check{C}$ satisfying the conclusions of that lemma: for small $r \equiv |\vec{y}|$,

$$\check{R}_{\mu\nu} = O(r^2), \quad \check{R}_{\mu\nu} y^\nu|_{C_O} = O_\infty(r^\infty). \tag{7.2}$$

To obtain a well-posed system of evolution equations for the metric g we will impose a *generalised wave-map gauge condition*,

$$H^\lambda = 0,$$

with the *harmonicity vector* H^μ defined as

$$H^\lambda := \underbrace{g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}}_{=:\Gamma^\lambda} - W^\lambda \quad \text{with } W^\lambda := \underbrace{g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}}_{=:\hat{W}^\lambda} + \mathring{W}^\lambda, \tag{7.3}$$

where the $\hat{\Gamma}^\lambda_{\alpha\beta}$ are the Christoffel symbols of the metric $\eta \equiv \hat{g}$. Roughly speaking, we calculate $\overline{\Gamma}^\lambda - \overline{\hat{W}}^\lambda$ from the initial data, and use the result as the definition of $\overline{\mathring{W}}^\lambda$; this will ensure the vanishing of $\overline{H}^\mu$. The details are somewhat less straightforward, as $\overline{H}^\lambda - \overline{\hat{W}}^\lambda$ involves some transverse derivatives of the metric which are not part of the initial data; this is taken care of as in [9]. One then needs to prove that $\overline{\mathring{W}}^\mu$ is the restriction to the light-cone of a smooth vector field in space–time, and this is the focus of the work here.

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Recall that the vector field $\check{H}^\mu$ has been defined in (6.3) as

$$\check{H}^\lambda := \underbrace{\check{C}^{\alpha\beta}(\check{\Gamma}^\lambda_{\alpha\beta}}_{=:\check{\Gamma}^\lambda} - \hat{\Gamma}^\lambda_{\alpha\beta}), \tag{7.4}$$

where the $\check{\Gamma}^\lambda_{\alpha\beta}$ are the Christoffel symbols of the metric $\check{C}$. This is clearly a smooth vector field in space–time. We will show that the components of $\overline{\mathring{W}}^\mu$ differ from those of $\overline{\check{H}}^\mu$ by terms which are $O_\infty(r^\infty)$. It easily follows from Lemma A.1, appendix A, that a vector field, defined along C_O, with (u, r, x^A)-components that are $O_\infty(r^\infty)$ extends to a smooth vector field on space–time, which will establish the desired property of $\overline{\mathring{W}}^\mu$.

We pass now to the details of the above. There exists a neighbourhood of O on which τ has no zeros. There we solve the first constraint by setting

$$\kappa = \frac{\partial_1 \tau + \frac{1}{n-1}\tau^2 + |\sigma|^2}{\tau}. \tag{7.5}$$

The argument leading to (6.9) applies, and gives

$$\check{\kappa} - \kappa = O_\infty(r^\infty). \tag{7.6}$$

Following [9], we choose $\overline{\mathring{W}}^0$ to be

$$\overline{\mathring{W}}^0 = -\overline{\hat{W}}^0 - \nu^0(2\kappa + \tau) - 2\partial_1 \nu^0; \tag{7.7}$$

equivalently, using the unchecked versions of (6.10)–(6.12),

$$\overline{\Gamma}^1_{11} = \kappa - \tfrac{1}{2}\nu_0 \overline{H}^0. \tag{7.8}$$

The last equation is further equivalent to (compare the unchecked version of (6.7))

$$\overline{R}_{11} = -\tfrac{1}{2}\nu_0 \overline{H}^0 \tau. \tag{7.9}$$

Comparing the definition (7.4) of $\check{H}$ with (7.7), using (7.1) and (7.6) we find that

$$\overline{\mathring{W}}^0 = \overline{\check{H}}^0 + 2\nu^0(\check{\kappa} - \kappa) = \overline{\check{H}}^0 + O_\infty(r^\infty). \tag{7.10}$$

The next constraint equation follows from $\overline{R}_{2A} = 0$. We note the identity [5]

$$(\partial_r + \tau)\overline{\Gamma}^1_{1A} + \tilde{\nabla}_B \sigma_A^B - \frac{n-2}{n-1}\partial_A \tau - \partial_A \overline{\Gamma}^1_{11} = \overline{R}_{2A}, \tag{7.11}$$

where $\tilde{\nabla}$ is the covariant derivative associated to the Riemannian metric g_{AB}.

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We let ξ_A to be the unique solution, which vanishes at the tip of the light-cone, of the equation obtained by replacing $\overline{\Gamma}^1_{1A}$ in (7.11) by $-\xi_A/2$, $\overline{\Gamma}^1_{11}$ by κ, and setting the right-hand side to zero,

$$-\frac{1}{2}(\partial_r+\tau)\xi_A+\tilde{\nabla}_B\sigma_A^B-\frac{n-2}{n-1}\partial_A\tau-\partial_A\kappa=0, \tag{7.12}$$

as in (6.19). We choose $\overline{\mathring{W}^A}$ to be

$$\begin{aligned}\overline{\mathring{W}^A}:=\overline{g}^{AB}[\xi_B+2\nu^0(\partial_r\nu_B-2\nu_C\sigma_B{}^C-\nu_B\tau)-\nu_B(\overline{\mathring{W}^0}+\overline{\hat{W}^0})]\\+\overline{g}^{CD}\tilde{\Gamma}^A_{CD}-\overline{\hat{W}^A};\end{aligned} \tag{7.13}$$

equivalently,

$$\begin{aligned}\xi_A=-2\nu^0\partial_r\nu_A+4\nu^0\nu_B\sigma_A{}^B+2\nu^0\nu_A\tau+\nu_A(\overline{\mathring{W}^0}+\overline{\hat{W}^0})\\+\overline{g}_{AB}(\overline{\mathring{W}^B}+\overline{\hat{W}^B})-\overline{g}_{AB}\overline{g}^{CD}\tilde{\Gamma}^B_{CD}.\end{aligned} \tag{7.14}$$

This has been chosen so that, using the formulae in [5, Appendix A and Section 9],

$$\overline{R}_{1A}=-\tfrac{1}{2}(\partial_r+\tau)(\overline{g}_{AB}\overline{H}^B+\nu_A\overline{H}^0)+\tfrac{1}{2}\partial_A(\nu_0\overline{H}^0). \tag{7.15}$$

Moreover, one finds that (cf. [5, Equation (10.35)])

$$\xi_A=-2\overline{\Gamma}^1_{1A}-\overline{g}_{AB}\overline{H}^B-\nu_A\overline{H}^0. \tag{7.16}$$

We let $\check{\xi}_A$ be $-2\overline{\check{\Gamma}}^1_{1A}$. The check-equivalent of (7.11) reads

$$-\frac{1}{2}(\partial_r+\tau)\check{\xi}_A+\tilde{\nabla}_B\sigma_A^B-\frac{n-2}{n-1}\partial_A\tau-\partial_A\check{\kappa}=O_\infty(r^\infty). \tag{7.17}$$

Comparing with (7.12) defining ξ_A, we find that

$$-\tfrac{1}{2}(\partial_r+\tau)(\xi_A-\check{\xi}_A)=O_\infty(r^\infty). \tag{7.18}$$

Integration establishes that

$$\xi_A=\check{\xi}_A+O_\infty(r^\infty). \tag{7.19}$$

The field $\check{H}^A$, defined in (7.4) and written out in detail using [5, Appendix A], takes the form

$$\begin{aligned}\overline{\check{H}}^A=\overline{g}^{AB}[\check{\xi}_B+2\nu^0(\partial_r\nu_B-2\nu_C\sigma_B{}^C-\nu_B\tau)-\nu_B\overline{\check{\Gamma}}^0]\\+\overline{g}^{CD}\tilde{\Gamma}^A_{CD}-\overline{\hat{W}^A}.\end{aligned} \tag{7.20}$$

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Comparing with (7.13), and using (7.10) and (7.19), we conclude that

$$\begin{aligned}\overline{\mathring{W}}^{A} &= \overline{\check{H}}^{A} + \overline{g}^{AB}[\xi_B - \check{\xi}_B - \nu_B(\overline{\mathring{W}}^0 + \overline{\hat{W}}^0 - \overline{\check{\Gamma}}^0)] \\ &= \overline{\check{H}}^{A} + O_\infty(r^\infty).\end{aligned} \tag{7.21}$$

Let $S_{\mu\nu}$ denote the Einstein tensor of g. We continue with the equation $\overline{S}_{01} = 0$; equivalently, $\overline{g}^{AB}\overline{R}_{AB} = 0$. Using the identities (10.33) and a corrected version of [5, Equation (10.36)] we find the identity

$$\begin{aligned}\overline{g}^{AB}\overline{R}_{AB} \equiv{}& (\partial_r + \overline{\Gamma}^1_{11} + \tau)(2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11}) \\ & -2\overline{g}^{AB}\overline{\Gamma}^1_{1A}\overline{\Gamma}^1_{1B} - 2\overline{g}^{AB}\tilde{\nabla}_A\overline{\Gamma}^1_{1B} + \tilde{R}.\end{aligned} \tag{7.22}$$

(On the far-right-hand side of (10.36) in [5] a term $\tau\overline{g}^{11}/2$ is missing.) This motivates the equation

$$(\partial_r + \kappa + \tau)\zeta + (\tilde{\nabla}_A - \tfrac{1}{2}\xi_A)\xi^A + \tilde{R} = 0, \tag{7.23}$$

with $\xi^A := \overline{g}^{AB}\xi_B$, and where the quantity ζ will be the restriction of

$$2(\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\overline{g}^{11})$$

to C_O, once the final vacuum metric g has been constructed. We integrate (7.23), viewed as a first-order ODE for ζ, as

$$\begin{aligned}\zeta ={}& -\frac{e^{-\int_1^r(\kappa+\tau-((n-1)/\tilde{r}))\,d\tilde{r}}}{r^{n-1}}\int_0^r \tilde{r}^{n-1}e^{\int_1^{\tilde{r}}(\kappa+\tau-((n-1)/\tilde{\tilde{r}}))\,d\tilde{\tilde{r}}} \\ & \times\left(\tilde{R} + \overline{g}^{AB}\tilde{\nabla}_A\xi_B - \frac{1}{2}\overline{g}^{AB}\xi_A\xi_B\right)d\tilde{r} \\ ={}& -(n-1)r^{-1} + O(1).\end{aligned}$$

We choose

$$\overline{\mathring{W}}^{1} := \tfrac{1}{2}\zeta - (\partial_r + \kappa + \tfrac{1}{2}\tau)\overline{g}^{11} - \overline{\hat{W}}^1; \tag{7.24}$$

equivalently,

$$\zeta = 2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \overline{g}^{11}(\tau + \nu_0\overline{H}^0) - 2\overline{H}^1. \tag{7.25}$$

With the choice (7.24) we have

$$\begin{aligned}\overline{g}^{AB}\overline{R}_{AB} ={}& (\partial_r + \kappa + \tau - \tfrac{1}{2}\overline{g}_{12}\overline{H}^0)(2\overline{H}^1 - \overline{g}_{12}\nu^0\overline{H}^0) - \tfrac{1}{2}\overline{g}_{12}\overline{H}^0\zeta \\ & + (\tilde{\nabla}_A - \xi_A - \tfrac{1}{2}\overline{g}_{AB}\overline{H}^B - \tfrac{1}{2}\nu_A\overline{H}^0)(\overline{H}^A + \nu_C\overline{g}^{AC}\overline{H}^0).\end{aligned} \tag{7.26}$$

Let $\check\zeta$ be the check-counterpart of ζ,

$$\check\zeta := 2(\overline{g}^{AB}\overline{\check\Gamma}{}^1_{AB} + \tau\nu^0). \tag{7.27}$$

Integrating the check-version of (7.22), we obtain

$$\begin{aligned}\check\zeta &= -\frac{e^{-\int_1^r(\check\kappa+\tau-((n-1)/\tilde r))\,d\tilde r}}{r^{n-1}}\int_0^r \tilde r^{n-1}e^{\int_1^{\tilde r}(\check\kappa+\tau-((n-1)/\tilde{\tilde r}))\,d\tilde{\tilde r}}\\ &\quad\times\left(\tilde R + \overline{g}^{AB}\tilde\nabla_A\check\xi_B - \tfrac12\overline{g}^{AB}\check\xi_A\check\xi_B - \overline{g}^{AB}\overline{\check R}_{AB}\right)d\tilde r\\ &= \zeta + O_\infty(r^\infty).\end{aligned} \tag{7.28}$$

From (7.27) and from [5, Appendix A] we find that

$$\tfrac12\check\zeta = \overline{\check\Gamma}{}^1 + (\partial_r + \check\kappa + \tfrac12\tau)\overline{g}^{11}. \tag{7.29}$$

Comparing with (7.24), in view of (7.6) and (7.28), we conclude that

$$\overline{\mathring W}{}^1 = \tfrac12(\zeta-\check\zeta) + \overline{\check\Gamma}{}^1 + (\check\kappa-\kappa)\overline{g}^{11} - \overline{\hat W}{}^1 = \overline{\check H}{}^1 + O_\infty(r^\infty). \tag{7.30}$$

Summarising, given the fields $\overline{g}_{\mu\nu}$ on C_O, we have found a vector field $\overline{\mathring W}$ on C_O satisfying

$$\overline{\mathring W}{}^\mu = \overline{\check H}{}^\mu + O_\infty(r^\infty).$$

The field $\overline{\check H}{}^\mu$ extends trivially to the smooth vector field $\check H^\mu$, while a vector field with components which are $O_\infty(r^\infty)$ extends to a smooth vector field in space–time by Lemma A.1. We conclude that there exists a smooth vector field, which we call $\mathring W$, defined in a neighbourhood $\mathscr{O}$ of O, which coincides with $\overline{\mathring W}$ on $C_O\cap\mathscr{O}$.

We apply the existence and uniqueness theorem of [13] to the reduced Einstein equations $R^{(H)}_{\alpha\beta} = 0$, with initial data $\overline{g}$, where

$$R^{(H)}_{\alpha\beta} := R_{\alpha\beta} - \tfrac12(g_{\alpha\lambda}\hat D_\beta H^\lambda + g_{\beta\lambda}\hat D_\alpha H^\lambda), \tag{7.31}$$

with H^μ defined by (7.4), where $\hat D$ is the Levi-Civita covariant derivative in the metric $\hat g$. Indeed, it follows from [3, p. 163] that $R^{(H)}_{\alpha\beta}$ is a quasi-linear, quasi-diagonal operator on g, tensor-valued, depending on $\hat g$, of the form

$$R^{(H)}_{\alpha\beta} \equiv -\tfrac12 g^{\lambda\mu}\hat D_\lambda\hat D_\mu g_{\alpha\beta} + \hat f[g,\hat Dg]_{\alpha\beta}, \tag{7.32}$$

where $\hat f[g,\hat Dg]_{\alpha\beta}$ is a tensor quadratic in $\hat Dg$ with coefficients depending upon g, $\hat g$, $\mathring W$, $\hat D\hat W$, and $\hat D\mathring W$, which is of the right form for [13].

Now, the metric g so obtained will solve the vacuum Einstein equations if and only if H^μ vanishes on C_O. It should be clear that κ then equals $\overline{\Gamma}{}^1_{11}$ and ξ_A equals $-\frac{1}{2}\overline{\Gamma}{}^1_{1A}$, but, to avoid ambiguities, we will justify these inequalities explicitly in what follows.

Note that at this stage a smooth metric g and smooth vector fields $\mathring{W}^\mu$ and H^μ are known in a neighbourhood $\mathscr{U}$ of O, with g satisfying the reduced Einstein equations in $\mathscr{U} \cap J^+(O)$.

The proof of the vanishing of $\overline{H}$ is essentially the same as the one in [5]; we outline it here for completeness.

In order to prove that $\overline{H}^0 = 0$ holds, we note the identity (see (7.31))

$$\overline{R}_{11} \equiv \underbrace{\overline{R}{}^{(H)}_{11}}_{=0} + \nu_0 \hat{D}_1 \overline{H}^0. \tag{7.33}$$

The reader will note that this equation, as well as (7.36) and (7.37) below, are identical with the corresponding equations in [5], even though *our H is not the same as the corresponding vector field in* [5]. This is due to the fact that our operator $\overline{R}{}^{(H)}_{11}$ in (7.31) is constructed using our vector field $\Gamma^\mu - \hat{W}^\mu - \mathring{W}^\mu$, while in [5] the vector field $\Gamma^\mu - \hat{W}^\mu$ is used for H^μ.

Equations (7.9) and (7.33) imply that $\overline{H}^0$ satisfies a linear homogeneous differential equation on C_O, namely,

$$\hat{D}_1 \overline{H}^0 + \tfrac{1}{2}\tau \overline{H}^0 = 0. \tag{7.34}$$

As explained in [5, Section 7.6], the only bounded solution of this equation is $\overline{H}^0 \equiv 0$. The equality $\Gamma^1_{11}|_{C_O} = \kappa$ follows trivially now from (7.8),

$$H^0|_{C_O} \equiv 2\nu^0(\kappa - \Gamma^1_{11}). \tag{7.35}$$

To establish the vanishing of $\overline{H}_A$ we invoke the following identity [5, Equation (9.8)]:

$$\overline{R}_{1A} \equiv \underbrace{\overline{R}{}^{(H)}_{1A}}_{=0} + \frac{1}{2}(\nu_0 \overline{\hat{D}_A H^0} + \nu_A \overline{\hat{D}_1 H^0} + \overline{g}_{AB} \overline{\hat{D}_1 H^B}). \tag{7.36}$$

Combined with (7.15), and taking into account that $\overline{H}^0 = 0$ has already been established, this gives a radial homogeneous ODE for $\overline{H}^A$, with $\overline{H}^A \equiv 0$ being the only solution with the relevant asymptotic behaviour at O. We can now conclude that $\xi_A = -2\overline{\Gamma}{}^1_{1A}$ from (7.16).

Finally, we have the identity [5, Equation (11.18)] (recall that $S_{\mu\nu}$ denotes the Einstein tensor)

$$\overline{S}_{01} \equiv \underbrace{\overline{S}{}^{(H)}_{01}}_{=0} + \frac{1}{2}(\overline{g}_{00} \overline{\hat{D}_1 H^0} + \nu_A \overline{\hat{D}_1 H^A} - \nu_0 \overline{\hat{D}_A H^A}). \tag{7.37}$$

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Combining this with (7.26), one similarly concludes that $\overline{H}^1 = 0$; see also [5, Section 11.3]. The vanishing of $\overline{H}^0$ and $\overline{H}^1$, together with the identity (7.25), implies that on C_O the field ζ coincides with $2\overline{g}^{AB}\overline{\Gamma}^1_{AB} + \tau\nu^0$.

Thus H^μ vanishes on C_0, and by the usual arguments (see, for example, [5, Theorem 3.3]) we have $H^\mu \equiv 0$.

This completes the proof of Theorem 1.3. □

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Appendix A. On Taylor expansions

To proceed, some terminology will be needed. We say that a function g defined on a space–time neighbourhood of the origin is $o_m(|y|^k)$ if g is C^m and if for $0 \leqslant \ell \leqslant m$ we have

$$\lim_{|y|\to 0} |y|^{\ell-k}\partial_{\mu_1}\dots\partial_{\mu_\ell} g = 0,$$

where $|y| := \sqrt{\sum_{\mu=0}^n (y^\mu)^2}$.

A similar definition will be used for functions defined in a neighbourhood of O on the light-cone C_O: we parameterise C_O by coordinates $y^i \in \mathbb{R}^n$, and we say that a function g defined on a neighbourhood of O within C_O is $o_m(r^k)$ if g is a C^m function of the coordinates y^i and if for $0 \leqslant \ell \leqslant m$ we have $\lim_{r\to 0} r^{\ell-k}\partial_{\mu_1}\dots\partial_{\mu_\ell} g = 0$, where $r := \sqrt{\sum_{i=1}^n (y^i)^2}$.

We consider a light-cone C_O which is smooth away from its tip. The following lemma will be used repeatedly (recall that $\Theta^i = y^i/r$).

Lemma A.1 [7, Lemma A.1]. *A function φ defined on a light-cone C_O is the trace $\overline{f}$ on C_O of a C^k space–time function f if and only if φ admits an expansion of the form*

$$\varphi = \sum_{p=0}^{k} f_p r^p + o_k(r^k), \tag{A.1}$$

with

$$f_p \equiv f_{i_1\dots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + f'_{i_1\dots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}}, \tag{A.2}$$

where $f_{i_1\dots i_p}$ and $f'_{i_1\dots i_{p-1}}$ are numbers.

The claim remains true with $k = \infty$ if (A.1) holds for all k. □

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We will also need the following.

LEMMA A.2 (Borel summation; see for example [7, Lemma D.1]). *For any given sequence*

$$\{c_{i_1\dots\, i_k}\}_{k\in\mathbf{N}} = \{c, c_i, c_{ij}, \dots\}$$

there exists a smooth function f *such that, for all* $k \in \mathbf{N}$,

$$f - \sum_{p=0}^{k} c_{i_1\dots\, i_p} y^{i_1} \cdots y^{i_p} = o_k(r^k).$$

Appendix B. ODE lemmas

For $k \in \mathbb{N} \cup \{\infty, \omega\}$ we will say that a function $\varphi\colon C_O \to \mathbb{R}$ is C^k-cone differentiable if there exists a C^k function on space–time ϕ such that φ is the restriction to C_O of ϕ. We shall say 'cone-smooth' for C^∞-cone differentiable.

We start with the following elementary result.

LEMMA B.1. *Let* $k \in \mathbb{N} \cup \{\infty, \omega\}$, *and let* φ *be a* C^k*-cone differentiable function on* C_O. *Then the integrals*

$$\psi(r, x^A) = \int_0^r \frac{\varphi(s, x^A)}{s}\, ds \quad \text{and} \quad \chi(r, x^A) = \frac{1}{r}\int_0^r \varphi(s, x^A)\, ds \tag{B.1}$$

are C^k*-cone differentiable, assuming moreover that* $\varphi(0) = 0$ *in the case of the integral defining* ψ.

Proof. Let, first, $k \in \mathbb{N}$. By Lemma A.1 we have

$$\varphi = \sum_{p=0}^{k} f_p r^p + o_k(r^k), \tag{B.2}$$

where the coefficients f_p are of the form (A.2). Inserting (B.2) into (B.1), we find that

$$\psi(r, x^A) = \sum_{p=1}^{k} \frac{f_p r^p}{p} + o_k(r^k), \qquad \chi(r, x^A) = \sum_{p=0}^{k} \frac{f_p r^p}{p+1} + o_k(r^k), \tag{B.3}$$

and the result follows from Lemma A.1.

The case $k = \infty$ is established in a similar way using Borel summation.

The case $k = \omega$ is the contents of [6, Lemma 6.5]. □

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We will need the following result about systems of Fuchsian ODEs.

LEMMA B.2. *Let $r_0 > 0$, $k \in \mathbb{N} \cup \{\infty\}$, $N \in \mathbb{N}$, $0 > a \in \mathbf{R}$, $\psi \in C^k([0, r_0], \mathbb{R}^N)$, and $\alpha \in C^k([0, r_0], \mathrm{End}(\mathbb{R}^N))$ with*

$$\alpha(0) = a\mathrm{Id},$$

where Id *is the identity matrix in* $\mathrm{End}(\mathbb{R}^N)$. *If $\phi \in C^1((0, r_0], \mathbb{R}^N)$ is a solution of*

$$\phi' = \frac{\alpha}{r}\phi + \psi, \tag{B.4}$$

then:

(1) *the limit*

$$\lim_{r\to 0} r^{-a}\phi \tag{B.5}$$

exists;

(2) *there exists a solution such that the last limit is zero. For such solutions, ϕ extends by continuity to a function in $C^{k+1}([0, r_0])$. If moreover $\psi = O(r^m)$, respectively $o(r^k)$, then $\phi = O(r^{m+1})$, respectively $o(r^{k+1})$. Here, by $o(r^\infty)$ we mean a function which is $o(r^k)$ for all k.*

REMARK B.3. The fact that $\phi \in C^k((0, r_0)$ is standard, so the only issue is at $r = 0$. Similarly the case $a = 0$ is standard. It is easy to analyse the equation with $a > 0$ using similar methods, but the results are more complicated to describe, and will not be needed in this work.

REMARK B.4. We will be using Lemma B.2 in the following equivalent form. Suppose that there exist matrices α_i such that α has an expansion

$$\alpha = a\mathrm{Id} + \alpha_1 r + \cdots + \alpha_k r^k + o_k(r^k), \tag{B.6}$$

and suppose that there exist vectors $\psi_i \in \mathbb{R}^N$ such that ψ has an expansion

$$\psi = \psi_0 + \psi_1 r + \cdots + \psi_k r^k + o_k(r^k). \tag{B.7}$$

Here we write $f = o_k(r^m)$ if for $0 \leqslant i \leqslant k$ we have $\partial_r^i f = o(r^{m-i})$. Then the limit (B.5) exists. If this limit vanishes, then there exist vectors $\phi_i \in \mathbb{R}^N$ such that ϕ has an expansion

$$\phi = \phi_1 r + \cdots + \phi_k r^k + o_k(r^k). \tag{B.8}$$

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Proof. Let us denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^N$, with $|\phi|^2 = \langle\phi, \phi\rangle$. Set $f := r^{-2a}|\phi|^2$. From (B.4) we have, for some constant C,

$$\begin{aligned} r\partial_r(\underbrace{r^{-2a}|\phi|^2}_{f}) &= 2r^{-2a}\langle\phi, \underbrace{(\alpha - a\mathrm{Id})}_{\geqslant -Cr}\phi + r\psi\rangle \\ &\geqslant r(-2Cr^{-2a}|\phi|^2 + r^{-2a}\underbrace{2\langle\phi,\psi\rangle}_{\geqslant -|\phi|^2-|\psi|^2}) \\ &\geqslant -r(2(C+1)f + r^{-2a}|\psi|^2); \end{aligned}$$

equivalently,

$$\partial_r\underbrace{\left(e^{2(C+1)r}f + \int_{r_0}^r e^{2(C+1)s}s^{-2a}|\psi(s)|^2\,ds\right)}_{=:h} \geqslant 0.$$

So the function h defined in the last equation is monotonic and nondecreasing. Monotonicity and positivity of h imply that of existence of the nonnegative limit $\lim_{r\to 0} h(r)$, and we conclude that $r^{-a}|\phi|$ has a finite limit as $r \to 0$. In particular, $|\phi| \leqslant Cr^a$ for some constant C.

We rewrite (B.4) as

$$\partial_r(r^{-a}\phi) = r^{-a}(\psi + (\alpha - a\mathrm{Id})\phi).$$

Integrating, for $0 < r_1 < r \leqslant r_0$, one finds that

$$\frac{\phi(r)}{r^a} = \frac{\phi(r_1)}{r_1^a} + \int_{r_1}^r s^{-a}\psi(s)\,ds + \int_{r_1}^r \underbrace{(\alpha - a\mathrm{Id})s^{-a}\phi(s)}_{\leqslant Cs^1}\,dx. \qquad \text{(B.9)}$$

Passing with r_1 to zero, using convergence of the integrals above in the limit, we find that the limit

$$\tilde\phi := \lim_{r_1\to 0}\frac{\phi(r_1)}{r_1^a}$$

exists. Hence point (1) holds, and moreover

$$\phi(r) = r^a\tilde\phi + r^a\int_0^r s^{-a}\psi(s)\,ds + r^a\int_0^r (a\mathrm{Id} - \alpha)s^{-a}\phi(s)\,dx. \qquad \text{(B.10)}$$

(2) It is standard that solutions of the homogeneous equation can be uniquely parameterised by $\tilde\phi$. So, given any solution of the nonhomogeneous equation (B.4), we can subtract from it a solution of the homogeneous equation with the

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same value of $\tilde{\phi}$, obtaining a solution with $\tilde{\phi} = 0$. It follows from (B.10) that we then have

$$\begin{aligned}\phi(r) &= r^a \int_0^r s^{-a}\psi(s)\,ds + r^a \int_0^r (a\mathrm{Id} - \alpha)s^{-a}\phi(s)\,dx \\ &= \frac{r}{1-a}\psi_0 + o(r).\end{aligned} \tag{B.11}$$

Suppose, now, that

$$\phi = \phi_1 r + \cdots + \phi_j r^j + o(r^j) \tag{B.12}$$

holds for some $1 \leqslant j < k+1$; we have just shown that this holds with $j = 1$. Inserting (B.6)–(B.7) and (B.12) into (B.11), one then finds by elementary manipulations that (B.12) holds with j replaced by $j+1$. Lemma B.2 follows now by induction, using Remark B.4. □

Let M be any smooth compact manifold; in our applications M will be a sphere S^{n-1}. By commuting (B.4) with differential operators tangential to M one immediately obtains the following corollary to Remark B.4.

PROPOSITION B.5. *Let $r_0 > 0$, $k, N \in \mathbb{N}$, $0 > a \in \mathbf{R}$. Suppose that there exist matrices $\alpha_i \in C^\infty(M, \mathrm{End}(\mathbb{R}^N))$ such that α has an expansion*

$$\alpha = a\mathrm{Id} + \alpha_1 r + \cdots + \alpha_k r^k + o_k(r^k), \tag{B.13}$$

and suppose that there exist vectors $\psi_i \in C^\infty(M, \mathbb{R}^N)$ so that ψ has an expansion

$$\psi = \psi_0 + \psi_1 r + \cdots + \psi_k r^k + o_k(r^k). \tag{B.14}$$

We assume moreover that, for any $\ell \in \mathbb{N}$ and for any smooth differential operator X on M of order ℓ, the error terms in (B.13) and in (B.14) satisfy, for $0 \leqslant i \leqslant k$ and $0 \leqslant i + \ell \leqslant k+1$,

$$\partial_r^i X(o_k(r^k)) = o(r^{k-i}). \tag{B.15}$$

Let $\phi \in C^0(M \times (0, r_0], \mathbb{R}^N)$ be differentiable in r and satisfy

$$\phi' = \frac{\alpha}{r}\phi + \psi. \tag{B.16}$$

Then:

(1) *the limit*

$$\lim_{r\to 0} r^{-a}\phi \tag{B.17}$$

exists;

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(2) *there exists a solution* $\phi \in C^{k+1}(M \times (0, r_0], \mathbb{R}^N)$ *of* (B.4) *such that the last limit is zero. Fur such solutions* ϕ *has an expansion*

$$\phi = \phi_1 r + \cdots + \phi_{k+1} r^{k+1} + o_{k+1}(r^{k+1}),$$

where $\phi_i \in C^\infty(M \times [0, r_0])$, *and where the error term satisfies* (B.15), *with* k *in the exponent replaced by* $k+1$. *If moreover* $\psi = O(r^m)$, *respectively* $o(r^k)$, *then* $\phi = O(r^{m+1})$, *respectively* $o(r^{k+1})$.

Appendix C. Prescribing ν_0

Let ν_0 be the restriction of a smooth space–time function to the future light-cone of a smooth metric C. In this appendix we show how to deform C to achieve $\overline{C}_{01} = \nu_0$ without changing $\overline{C}_{AB}\, dx^A\, dx^B$.

Let y be a coordinate system in which the future light-cone of C takes the Minkowskian form $y^0 = |\vec{y}|$, and let x be coordinates as in (3.3). Using the notation

$$\underline{C_{\mu\nu}} = C\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\mu}\right), \quad C_{\mu\nu} = C\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\mu}\right),$$

we have the transformation formulae

$$C_{00} \equiv \underline{C_{00}}, \quad C_{01} \equiv -\underline{C_{00}} - \underline{C_{0i}}\Theta^i, \quad C_{0A} \equiv -\underline{C_{0i}} r \frac{\partial \Theta^i}{\partial x^A}, \tag{C.1}$$

$$C_{11} \equiv \underline{C_{00}} + 2\underline{C_{0i}}\Theta^i + \underline{C_{ij}}\Theta^i\Theta^j, \quad C_{1A} \equiv \underline{C_{0i}} r \frac{\partial \Theta^i}{\partial x^A} + \underline{C_{ji}} r \Theta^j \frac{\partial \Theta^i}{\partial x^A}, \tag{C.2}$$

$$C_{AB} \equiv \underline{C_{ij}} r^2 \frac{\partial \Theta^i}{\partial x^A}\frac{\partial \Theta^j}{\partial x^B}. \tag{C.3}$$

Conversely, $\underline{C_{\lambda\mu}} = (\partial x^\alpha/\partial y^\lambda)(\partial x^\beta/\partial y^\mu) C_{\alpha\beta}$ gives

$$\underline{C_{00}} \equiv C_{00}, \quad \underline{C_{0i}} \equiv -(C_{00} + C_{01})\Theta^i - C_{0A}\frac{\partial x^A}{\partial y^i}, \tag{C.4}$$

$$\begin{aligned}\underline{C_{ij}} = {} & (C_{00} + 2C_{01} + C_{11})\Theta^i\Theta^j + (C_{0A} + C_{1A})\left(\Theta^i \frac{\partial x^A}{\partial y^j} + \Theta^j \frac{\partial x^A}{\partial y^i}\right) \\ & + C_{AB}\frac{\partial x^A}{\partial y^i}\frac{\partial x^B}{\partial y^j}.\end{aligned} \tag{C.5}$$

As the first step of our argument, we need to write ν_0 as

$$\nu_0 = 1 + r^2 \bar{f}_0 + \sum_{i=1}^{n} \bar{f}_i y^i, \tag{C.6}$$

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where $\bar{f}_0$, respectively $\bar{f}_i$, are restrictions to the light-cone of functions f_0, respectively f_i, which are smooth on space–time. (It follows from (4.10) that $\bar{f}_0 = O(r^2)$ when $\kappa = 0$, a harmonic gauge, and the vacuum Raychaudhuri equation are assumed, but this will not be needed in what follows.) To prove (C.6), let f be any smooth function on space–time; Taylor expanding f with respect to y^1, we can write

$$f(y^0, y^1, \dots, y^n) = f(y^0, 0, y^2, \dots, y^n) + f_1(y^0, y^1, \dots, y^n) y^1,$$

where

$$f_1 = \int_0^1 \frac{\partial f}{\partial y^1}(y^0, s y^1, y^2, \dots, y^n)\, ds \in C^\infty.$$

Similarly,

$$f(y^0, 0, y^2 \dots, y^n) = f(y^0, 0, 0, y^3, \dots, y^n) + f_2(y^0, y^2, \dots, y^n) y^2,$$

with

$$f_2 = \int_0^1 \frac{\partial f}{\partial y^2}(y^0, 0, s y^2, y^3, \dots, y^n)\, ds \in C^\infty.$$

Continuing in this way, after n steps the function

$$f - \sum_{i=1}^{n} f_i y^i = f(y^0, 0, \dots, 0) \tag{C.7}$$

depends only upon y^0. A final Taylor expansion allows us to rewrite the right-hand side as $f(0, 0, \dots, 0) + (y^0)^m f_0$, where f_0 is a smooth function of y^0 and m is the order of the zero of $f(y^0, 0, \dots, 0) - f(0, 0, \dots, 0)$. Keeping in mind that $y^0|_{C_O} = r$, (C.6) for $\nu_0 = \overline{f}$ follows.

Let, now, $C_{\mu\nu}$ be given, and consider

$$\tilde{C}_{\mu\nu} := \Omega^2 C_{\mu\nu} + \delta C_{\mu\nu},$$

where $\Omega = 1$ if one wishes to keep $\overline{C}_{AB}\, dx^A\, dx^B$ fixed, or Ω is a smooth space–time function with prescribed $\overline{\Omega}$ (for example the conformal factor determined in Section 3.3), if one only wishes to prescribe $\overline{C}_{AB}\, dx^A\, dx^B$ up to a conformal factor.

Suppose, momentarily, that the components

$$\underline{\delta C_{0\mu}}$$

are prescribed smooth functions on space–time, and suppose that δC satisfies

$$\delta C_{AB} = 0 = \overline{\delta C_{11}} = \overline{\delta C_{1A}}. \tag{C.8}$$

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The first equality guarantees that the initial data $\tilde{C}_{AB}$ defined by $\tilde{C}$ coincide with the metric g_{AB} solving the first constraint equation, while the last two guarantee that the cone $\{y^0 = |\vec{y}|\}$ remains characteristic for $\tilde{C}$.

Then, by (C.1),

$$\delta C_{00} = \underline{\delta C_{00}}, \quad \delta C_{01} = -\underline{\delta C_{00}} - \underline{\delta C_{0k}}\Theta^k, \quad \delta C_{0A} = -\underline{\delta C_{0k}}\frac{\partial y^k}{\partial x^A}, \tag{C.9}$$

and so all components $\overline{\delta C_{\mu\nu}}$ are known. We can now find the restrictions to the light-cone of the missing components $\underline{\delta C_{ij}}$ of δC using (C.5):

$$\begin{aligned}
\underline{\overline{\delta C_{ij}}} &= \overline{(\delta C_{00} + 2\delta C_{01})\Theta^i\Theta^j + \delta C_{0A}\left(\Theta^i\frac{\partial x^A}{\partial y^j} + \Theta^j\frac{\partial x^A}{\partial y^i}\right)} \\
&= \overline{-(\underline{\delta C_{00}} + 2\underline{\delta C_{0k}}\Theta^k)\Theta^i\Theta^j - \underline{\delta C_{0k}}\frac{\partial y^k}{\partial x^A}\left(\Theta^i\frac{\partial x^A}{\partial y^j} + \Theta^j\frac{\partial x^A}{\partial y^i}\right)} \\
&= \overline{-(\underline{\delta C_{00}} + 2\underline{\delta C_{0k}}\Theta^k)\Theta^i\Theta^j - \underline{\delta C_{0k}}((\delta^k_j - \Theta^k\Theta^j)\Theta^i + (\delta^k_i - \Theta^k\Theta^i)\Theta^j)} \\
&= \overline{-\underline{\delta C_{00}}\Theta^i\Theta^j - \underline{\delta C_{0i}}\Theta^j - \underline{\delta C_{0j}}\Theta^i}.
\end{aligned} \tag{C.10}$$

Keeping in mind that $\delta C_{\mu\nu}$ is required to satisfy (C.8), we chose the tensor field $\delta C_{\mu\nu}$ now so that in addition to this last equation it holds that

$$\overline{\tilde{C}}_{01} = \nu_0, \tag{C.11}$$

where ν_0 is the restriction to the light-cone of a smooth function f. Equivalently,

$$\delta C_{01} = r^2 f_0 + \sum_{i=1}^{n} f_i y^i + 1 - \Omega^2, \tag{C.12}$$

where f_0 and f_i are given by (C.6). As in that last equation, we can also write

$$\Omega^2 = 1 + r^2 h_0 + \sum_{i=1}^{n} h_i y^i,$$

which allows us to rewrite (C.12) as

$$\delta C_{01} = r^2(f_0 - h_0) + \sum_{i=1}^{n}(f_i - h_i)y^i. \tag{C.13}$$

Comparing with (C.9), we see that (C.12) will hold if we choose

$$\underline{\delta C_{00}} = r^2(h_0 - f_0), \quad \underline{\delta C_{0i}} = t(h_i - f_i),$$

while, in view of (C.10), (C.8) will be satisfied if $\underline{\delta C_{ij}}$ is further chosen to be

$$\underline{\delta C_{ij}} = (f_0 - h_0)y^i y^j + (f_i - h_i)y^j + (f_j - y_j)y^i. \qquad \text{(C.14)}$$

The reader might wish to verify by a direct calculation that, with these choices, (C.8) and (C.11) hold.

The metric $\tilde{C}$ will clearly be Lorentzian in a sufficiently small neighbourhood of the vertex of the cone.

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