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Existence and analyticity of solutions of the Kuramoto–Sivashinsky equation with singular data

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We prove the existence of solutions to the Kuramoto–Sivashinsky equation with low regularity data in function spaces based on the Wiener algebra and in pseudomeasure spaces. In any spatial dimension, we allow the data to have its antiderivative in the Wiener algebra. In one spatial dimension, we also allow data that are in a pseudomeasure space of negative order. In two spatial dimensions, we also allow data that are in a pseudomeasure space one derivative more regular than in the one-dimensional case. In the course of carrying out the existence arguments, we show a parabolic gain of regularity of the solutions as compared to the data. Subsequently, we show that the solutions are in fact analytic at any positive time in the interval of existence.

Keywords: Analyticity; existence; Kuramoto–Sivashinsky; pseudomeasure; singular data

2010 Mathematics subject classification: 35K46; 35R05; 35B65; 35A01; 35A20

1. Introduction

The scalar form of the Kuramoto–Sivashinsky equation is

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + \Delta^2\phi + \Delta\phi = 0. \tag{1.1}$$

This is taken with initial data

$$\phi(\cdot, 0) = \phi_0. \tag{1.2}$$

The spatial domain we consider is the n-dimensional torus $\mathbb{T}^n$, which is given by

$$\mathbb{T}^n = \prod_{i=1}^{n} [0, L_i]$$

for some given lengths $L_i > 0$, $i \in \{1, \ldots, n\}$, with periodic boundary conditions. Equation (1.1) was introduced separately by Kuramoto and Tsuzuki when studying pattern formation in reaction–diffusion equations [24] and by Sivashinsky in modelling the evolution of flame fronts [34]. As a model of flame fronts, the surface $(\vec{x}, \phi(\vec{x}, t))$ gives the location of the interface between the burnt and unburnt phases of a gas undergoing combustion. As such, the physical cases are $n = 1$ (representing the interface between 2 two-dimensional gases) and $n = 2$ (representing the interface between 2 three-dimensional gases).

Demonstrating local well-posedness of the initial value problems (1.1) and (1.2) is straightforward for relatively smooth data. For global well-posedness, the situation is only clear in dimension $n = 1$. In this case, the nonlinearity has a simpler structure, and there are many results, especially those of Tadmor [35], as well as those of Bronski and Gambill [10], Goodman [19], and Nicolaenko, Scheurer, and Temam [30]. All of these articles assume the same regularity on the initial data, which is that $\phi_0 \in H^1$.

In two space dimensions, there are two types of global existence results, both of which have limitations. The earliest global existence result in two dimensions was the thin domain result of Sell and Taboada [33]; this was then followed by the other thin-domain results [8, 23, 29]. Other than these, D.M.A. and Mazzucato have demonstrated global existence of small solutions for the two-dimensional Kuramoto–Sivashinsky equation for certain domain sizes (i.e., placing certain conditions on L_1 and L_2), but without the anisotropy inherent in the thin-domain results [2, 3].

Other global results rely upon modifying either the linear or nonlinear parts of (1.1). For instance, by no longer considering fourth-order linear terms, a maximum principle may be introduced, leading to global existence of solutions [25, 28]. Changing the power in the nonlinear term leads to global existence or singularity formation, depending on the power, as demonstrated in [7]. Global existence also follows from the introduction of appropriate transport terms, as shown in [14, 16].

Grujić and Kukavica demonstrated existence of solutions for the Kuramoto–Sivashinsky equation in one dimension, with $\partial_x \phi_0 \in L^\infty$, and also demonstrated analyticity of the solutions at positive times [20]. Biswas and Swanson considered the Kuramoto–Sivashinsky equation in general dimension. Their results include improving the assumption made by Grujić and Kukavica, in dimension one, on the regularity of the data (by one); Biswas and Swanson also studied higher regularity through estimates of Gevrey norms [9].

This research is naturally related to work on the Navier–Stokes equations, for which there have been many studies of existence of solutions starting from low-regularity data. The optimal result in critical spaces is due to Koch and Tataru for

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data in BMO^{-1} [22]. The present work draws more from other studies, such as by Cannone and Karch for data in PM^2 and by Lei and Lin for data in X^{-1} [13, 26] (see § 2 and § 4 in the present work for the definition of these spaces). In [5], Bae proved a version of the Lei–Lin result using a two-norm approach, which also gives analyticity of the solution at positive times and draws upon earlier work [6]. The authors of the present article adapted the work of [5] to the spatially periodic case, finding an improved estimate for the radius of analyticity [1].

The primary contribution of the present work is to weaken the assumed regularity of the initial data as compared to prior works on existence of solutions for the Kuramoto–Sivashinsky equation (1.1). Our primary motivation is to examine how the two-norm approach may be used to improve regularity requirements and analyticity estimates beyond the Navier–Stokes system.

D.M.A. and Mazzucato proved the existence of solutions for the two-dimensional Kuramoto–Sivashinsky equations in the case of small domain sizes with data that have one derivative in the Wiener algebra or one derivative in L^2 [2, 3]. Subsequently, Coti Zelati, Dolce, Feng, and Mazzucato treated situations (for an equation with added advection) with data in L^2 [14]; Feng and Mazzucato also treated a different class of advective equations, again with L^2 data, in [16]. Biswas and Swanson treated the whole-space case rather than the spatially periodic case and took data such that the Fourier transform is in an L^p space [9], with $p \neq 1$ and $p \neq \infty$ (we treat the complementary cases of periodic data with Fourier coefficients in ℓ^1 or ℓ^∞). Existence of solutions for the Kuramoto–Sivashinsky equation with pseudomeasure data was treated by Miao and Yuan, but only in non-physical spatial dimensions, specifically $n=4$, $n=5$, and $n=6$ [27]. In the present work, we deal with the physically relevant spatial dimensions $n=1$ and $n=2$. It is notable that our one-dimensional existence theorem allows initial data with Fourier coefficients, which grow as the Fourier variable, k, goes to infinity. The Kuramoto–Sivashinsky equation in dimension $n=3$ with pseudomeasure-type data remains unaddressed; while this is not a physical case when regarding the equation as a model for the motion of flame fronts, it may be of physical interest in other areas of application (such as pattern formation in reaction-diffusion equations). We discuss in remark 4.3 how our result unfortunately does not readily extend to the case $n=3$.

We prove that our solutions are global in time in the case that the linearized problem has no growing Fourier modes. This amounts to an assumption of smallness of the periodic cell that comprises the spatial domain. In the general case of larger period cells, our results are valid up until a finite time. This is consistent with the lack of general global existence theory for the Kuramoto–Sivashinsky equation in dimension two and higher. In addition to proving that solutions exist, we also prove that they are analytic at positive times, following the approach of Bae [5], which the authors also used previously for the Navier–Stokes equations [1].

The plan of the article is as follows: we establish some preliminaries in § 2, which includes introducing a number of function spaces and giving an abstract fixed point result; in § 3, we establish existence of solutions with data in a space related to the Wiener algebra; in § 4, we treat existence of solutions with data in pseudomeasure spaces; we establish the associated linear estimates in § 4.1, the nonlinear estimates in one spatial dimension in § 4.2, and the nonlinear estimates in two spatial dimensions in § 4.3; analyticity of all of these solutions at positive times is demonstrated in § 5; the main theorems are the existence theorems, i.e., theorem 3.1 at the beginning of § 3, theorem 4.4 at the beginning of § 4.2, and theorem 4.5 at the beginning of § 4.3, and the analyticity theorems, i.e., theorem 5.3 and theorem 5.4, at the end of § 5; and we close with some concluding remarks in § 6.

2. Preliminaries

We observe that the mean of ϕ does not influence the evolution of ϕ. We thus introduce the projection $\mathbb{P}$, which removes the mean of a periodic function, as follows:

$$\mathbb{P}f = f - \frac{1}{L_1 \cdots L_n} \int_{\mathbb{T}^n} f(x)\, dx.$$

We let $\psi = \mathbb{P}\phi$, and we note that $\nabla\psi = \nabla\phi$; we then see that ψ satisfies the equation

$$\psi_t + \frac{1}{2}\mathbb{P}|\nabla\psi|^2 + \Delta^2\psi + \Delta\psi = 0. \tag{2.1}$$

Recall the Fourier series of a periodic function, given in terms of its Fourier coefficients:

$$f(x) \sim \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{2\pi i k_1 x_1/L_1} \cdots e^{2\pi i k_n x_n/L_n}.$$

From this we see directly that the symbol of the partial differential operator ∂_{x_i} is

$$\sigma(\partial_{x_j}) = \frac{2\pi i}{L_j} k_j.$$

Therefore, the symbols of the Laplacian and bi-Laplacian are

$$\sigma(\Delta) = -\sum_{j=1}^{n} \left(\frac{2\pi}{L_j}\right)^2 k_j^2,$$

$$\sigma(\Delta^2) = \left(\sum_{j=1}^{n} \left(\frac{2\pi}{L_j}\right)^2 k_j^2\right)^2.$$

We next introduce some spaces based on the Wiener algebra, which we denote as Y^m for $m \in \mathbb{R}$. We define $\mathbb{Z}^n_* = \mathbb{Z}^n \setminus \{0\}$. A periodic function, f, is in Y^m if the norm given by

$$\|f\|_{Y^m} = |\hat{f}(0)| + \sum_{k\in\mathbb{Z}^n_*} |k|^m |\hat{f}(k)|$$

is finite. If $m = 0$, then this space is exactly the Wiener algebra. We let $T > 0$ be given, with T possibly being infinite. On the space–time domain $[0, T] \times \mathbb{T}^n$, we also have a related function space, $\mathcal{Y}^m$. The norm for this space is

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$$\|f\|_{\mathcal{Y}^m} = \sup_{t\in[0,T]} |\hat{f}(t,0)| + \sum_{k\in\mathbb{Z}^n} \sup_{t\in[0,T]} |k|^m |\hat{f}(t,k)|.$$

In practice, we will be dealing with functions with zero mean, so it will be equivalent for us to treat the norms as

$$\|f\|_{Y^m} = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^m |\hat{f}(k)|,$$

$$\|f\|_{\mathcal{Y}^m} = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \sup_{t\in[0,T]} |k|^m |\hat{f}(t,k)|.$$

We note that the space X^{-1} as used in [1, 5, 26] is equal to our space Y^{-1}.

Given $m \in \mathbb{R}$, we also have a related function space on space–time, $\mathcal{X}^m$. We define the space $\mathcal{X}^m$ according to the norm

$$\|f\|_{\mathcal{X}^m} = \int_0^T |\hat{f}(t,0)|\ dt + \sum_{k\in\mathbb{Z}^n_*} \int_0^T |k|^m |\hat{f}(t,k)|\ dt. \tag{2.2}$$

If f has zero mean for all times, then this becomes simply

$$\|f\|_{\mathcal{X}^m} = \sum_{k\in\mathbb{Z}^n_*} \int_0^T |k|^m |\hat{f}(t,k)|\ dt. \tag{2.3}$$

In the results to follow, we will typically take $m=2$ or $m=4$.

We will consider two cases in what follows. We first describe Case A. In Case A, we assume that all $L_i < 2\pi$, and we take $T=\infty$. Because of the size of the L_i, we have $\sigma(\Delta^2+\Delta)(k) > 0$ for all $k \in \mathbb{Z}^n_*$. Then, we have

$$\sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n_*} e^{-t\sigma(\Delta^2+\Delta)(k)} = \sup_{t\in[0,\infty)} \sup_{k\in\mathbb{Z}^n_*} e^{-t\sigma(\Delta^2+\Delta)(k)} = 1.$$

In Case B, we let $T \in (0,\infty)$ be given, and we assume there exists at least one $i \in \{1,\ldots,n\}$ such that $L_i \geq 2\pi$. Then, there exists $M_1 > 0$ such that

$$\sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n_*} e^{-t\sigma(\Delta^2+\Delta)(k)} \leq M_1. \tag{2.4}$$

In Case B, we make the decomposition $\mathbb{Z}^n_* = \Omega_F \cup \Omega_I$, where for all $k \in \Omega_F$, the symbol is non-positive, i.e., $\sigma(\Delta^2+\Delta)(k) \leq 0$. Then, on the complement, of course, we have for all $k \in \Omega_I$, $\sigma(\Delta^2+\Delta)(k) > 0$. Of course, the set Ω_F is finite and Ω_I is infinite.

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We may, of course, also consider the decomposition $\mathbb{Z}^n_* = \Omega_F \cup \Omega_I$ in Case A as well, and then we simply have $\Omega_F = \emptyset$. In either case, we have $M_2 > 0$ such that

$$\sigma(\Delta^2 + \Delta)(k) > M_2|k|^4 \qquad \forall k \in \Omega_I. \tag{2.5}$$

We also introduce M_3 to be the maximum value of $|k|$ for $k \in \Omega_F$,

$$|k| \leq M_3 \qquad \forall k \in \Omega_F. \tag{2.6}$$

We will rely on the following classical abstract result:

LEMMA 2.1. *Let* $(X, |||\cdot|||_X)$ *be a Banach space. Assume that* $\mathcal{B} : X \times X \to X$ *is a continuous bilinear operator and let* $\eta > 0$ *satisfy* $\eta \geq \|\mathcal{B}\|_{X\times X\to X}$. *Then, for any* $x_0 \in X$ *such that*

$$4\eta|||x_0|||_X < 1,$$

there exists one and only one solution to the equation

$$x = x_0 + \mathcal{B}(x, x) \qquad \textit{with } |||x|||_X < \frac{1}{2\eta}.$$

Moreover, $|||x|||_X \leq 2|||x_0|||_X$.

See [11, p. 37, lemma 1.2.6] and [4, 12].

We may write the mild formulation of the Kuramoto–Sivashinsky equation (2.1) as

$$\psi = S\psi_0 - \frac{1}{2}B(\psi, \psi). \tag{2.7}$$

Here, the semigroup operator is

$$S\psi_0 = e^{-t(\Delta^2+\Delta)}\psi_0, \tag{2.8}$$

and the bilinear term is

$$B(F, G) = \int_0^t e^{-(t-s)(\Delta^2+\Delta)}\mathbb{P}(\nabla F \cdot \nabla G)\, ds. \tag{2.9}$$

The Fourier coefficients of $B(F, G)$ are

$$\widehat{B(F,G)}(t,k) = \int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\mathcal{F}[\mathbb{P}(\nabla F \cdot \nabla G)](s,k)\, ds \tag{2.10}$$

$$= \int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}^n_*, j\neq k} \sum_{\ell=1}^n \frac{2\pi i}{L_\ell}(k_\ell - j_\ell)\hat{F}(s, k-j)\frac{2\pi i}{L_\ell} j_\ell \hat{G}(s,j)\, ds$$

$$= -\int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}^n_*, j\neq k} \sum_{i=1}^n \frac{4\pi^2}{L_i^2}(k_i - j_i)\hat{F}(s, k-j) j_i \hat{G}(s,j)\, ds.$$

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In all of the estimates we will perform, we will use only bounds from above, with respect to the frequency variable k, of $|\widehat{B(F,G)}(t,k)|$. We thus ignore, hereafter, the constants $\frac{4\pi^2}{L_i^2}$, absorbing them into a positive constant C, which is then normalized to 1.

3. Existence of solutions with data in Y^{-1}

In this section, we will prove the following theorem, giving existence of solutions with initial data taken from the space Y^{-1}.

THEOREM 3.1 *Let $T>0$ be given. (If the conditions of Case A hold, then T may be taken to be $T=\infty$.) Let $n\geq 1$. There exists $\varepsilon>0$ such that for any ϕ_0 with $\mathbb{P}\phi_0 \in Y^{-1}$, if $\|\mathbb{P}\phi_0\|_{Y^{-1}} < \varepsilon$, then there exists a unique ϕ with $\mathbb{P}\phi \in \mathcal{Y}^{-1}\cap\mathcal{X}^3$ such that ϕ is a mild solution to the initial value problem (1.1) and (1.2).*

Proof. To use lemma 2.1, we need to establish the bilinear estimate, and also that $x_0 = S\mathbb{P}\phi_0 \in \mathcal{Y}^{-1}\cap\mathcal{X}^3$. Note that the lemma gives both existence and uniqueness of the solution.

For the semigroup, we let $\psi_0 \in Y^{-1}$ be given, and we must show $S\psi_0 \in \mathcal{Y}^{-1}\cap\mathcal{X}^3$. We begin by computing the norm in $\mathcal{Y}^{-1}$:

$$\|S\psi_0\|_{\mathcal{Y}^{-1}} = \sum_{k\in\mathbb{Z}^n_*} \sup_{t\in[0,T]} \frac{e^{-t\sigma(\Delta^2+\Delta)(k)}}{|k|}|\hat{\psi}_0(k)| \leq M_1\|\psi_0\|_{Y^{-1}}.$$

We next compute the norm in $\mathcal{X}^3$:

$$\begin{aligned}\|S\psi_0\|_{\mathcal{X}^3} &= \sum_{k\in\mathbb{Z}^n_*}\int_0^T |k|^3 e^{-t\sigma(\Delta^2+\Delta)(k)}|\hat{\psi}_0(k)|\ dt\\ &\leq \left(\sum_{k\in\mathbb{Z}^n_*}\frac{|\hat{\psi}_0(k)|}{|k|}\right)\sup_{k\in\mathbb{Z}^n_*}\left(\frac{|k|^4(1-e^{-T\sigma(\Delta^2+\Delta)(k)})}{\sigma(\Delta^2+\Delta)(k)}\right).\end{aligned}$$

In Case A, the supremum is finite because we may neglect the exponential and $\sigma(\Delta^2+\Delta)(k) > M_2|k|^4$. In Case B, we may take the supremum separately over the sets Ω_F and Ω_I, and the reasoning from Case A applies to the supremum over Ω_I. For the supremum over Ω_F, we find that it is finite because k is in a bounded set and T is finite. In either case, we have concluded that there exists $C>0$ such that $S\psi_0 \in \mathcal{X}^3$ and

$$\|S\psi_0\|_{\mathcal{X}^3} \leq C\|\psi_0\|_{Y^{-1}}. \tag{3.1}$$

This completes the proof of the needed semigroup properties.

We next need to compute $\|B(F,G)\|_{\mathcal{Y}^{-1}}$ and $\|B(F,G)\|_{\mathcal{X}^3}$. We begin to compute the norm in $\mathcal{Y}^{-1}$:

$$\|B(F,G)\|_{\mathcal{Y}^{-1}} = \sum_{k\in\mathbb{Z}^n_*} \sup_{t\in[0,T]} \frac{|\widehat{B(F,G)}(t,k)|}{|k|}.$$

We substitute from (2.10) and make some elementary bounds, arriving at

$$\|B(F,G)\|_{\mathcal{Y}^{-1}} \leq nM_1 \sum_{k\in\mathbb{Z}^n_*} \int_0^T \sum_{j\in\mathbb{Z}^n_*} \frac{1}{|k|}|k-j||\hat{F}(s,k-j)|j||\hat{G}(s,j)|\ ds.$$

Since we have

$$|k-j||j| \leq \frac{1}{2}\left(|k-j|^2+|j|^2\right), \tag{3.2}$$

we find

$$\begin{aligned}\|B(F,G)\|_{\mathcal{Y}^{-1}} \leq \frac{nM_1}{2} \sum_{k\in\mathbb{Z}^n_*} \int_0^T \sum_{j\in\mathbb{Z}^n_*} \frac{|k-j|^2}{|k|}|\hat{F}(s,k-j)||\hat{G}(s,j)|\ ds \\ +\frac{nM_1}{2} \sum_{k\in\mathbb{Z}^n_*} \int_0^T \sum_{j\in\mathbb{Z}^n_*} |\hat{F}(s,k-j)|\frac{|j|^2}{|k|}|\hat{G}(s,j)|\ ds.\end{aligned}$$

We then multiply and divide both terms by $|k-j||j|$, arriving at

$$\begin{aligned}\|B(F,G)\|_{\mathcal{Y}^{-1}} \leq \frac{nM_1}{2} \sum_{k\in\mathbb{Z}^n_*} \int_0^T \sum_{j\in\mathbb{Z}^n_*} |k-j|^3|\hat{F}(s,k-j)|\frac{|\hat{G}(s,j)|}{|j|}\left(\frac{|j|}{|k||k-j|}\right)\ ds \\ +\frac{nM_1}{2} \sum_{k\in\mathbb{Z}^n_*} \int_0^T \sum_{j\in\mathbb{Z}^n_*} \frac{|\hat{F}(s,k-j)|}{|k-j|}|j|^3|\hat{G}(s,j)|\left(\frac{|k-j|}{|k||j|}\right)\ ds.\end{aligned}$$

We note the elementary bounds

$$\frac{|j|}{|k||k-j|} \leq \frac{|k|+|k-j|}{|k||k-j|} \leq 2, \qquad \frac{|k-j|}{|k||j|} \leq \frac{|k|+|j|}{|k||j|} \leq 2. \tag{3.3}$$

This, then, immediately yields the bound

$$\begin{aligned}\|B(F,G)\|_{\mathcal{Y}^{-1}} &\leq nM_1\|F\|_{\mathcal{X}^3}\|G\|_{\mathcal{Y}^{-1}} + nM_1\|F\|_{\mathcal{Y}^{-1}}\|G\|_{\mathcal{X}^3} \\ &\leq 2nM_1(\|F\|_{\mathcal{Y}^{-1}}+\|F\|_{\mathcal{X}^3})(\|G\|_{\mathcal{Y}^{-1}}+\|G\|_{\mathcal{X}^3}).\end{aligned} \tag{3.4}$$

We next will consider the higher norm of $B(F,G)$, attempting to bound $\|B(F,G)\|_{\mathcal{X}^3}$. To begin, we have

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$$\|B(F,G)\|_{\mathcal{X}^3} = \sum_{k\in\mathbb{Z}^n_*}\int_0^T |k|^3|\widehat{B(F,G)}(k)|\ dt$$

$$= \sum_{k\in\mathbb{Z}^n_*}\int_0^T \left|\int_0^t |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}^n_*}\sum_{i=1}^n (k_i-j_i)\hat{F}(s,k-j)j_i\hat{G}(s,j)\ ds\right| dt.$$

We use the triangle inequality and (3.2), finding

$$\begin{aligned}
&\|B(F,G)\|_{\mathcal{X}^3}\\
&\quad\le \frac{1}{2}\sum_{k\in\mathbb{Z}^n_*}\int_0^T\int_0^t |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)||\hat{G}(s,j)|\ dsdt\\
&\qquad+\frac{1}{2}\sum_{k\in\mathbb{Z}^n_*}\int_0^T\int_0^t |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|\hat{F}(s,k-j)||j|^2|\hat{G}(s,j)|\ dsdt\\
&\qquad= A_1 + A_2.
\end{aligned}$$

We will only include the details for the estimate of A_1, as the estimate of A_2 is exactly the same. We decompose A_1 further, using the decomposition $\mathbb{Z}^n_* = \Omega_F\cup\Omega_I$. We have

$$\begin{aligned}
A_1 &= \frac{1}{2}\sum_{k\in\Omega_F}\int_0^T\int_0^t |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)||\hat{G}(s,j)|\ dsdt\\
&\quad+\frac{1}{2}\sum_{k\in\Omega_I}\int_0^T\int_0^t |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)||\hat{G}(s,j)|\ dsdt\\
&= A_3 + A_4.
\end{aligned}$$

We use the definitions of M_1, (2.4), and M_3, (2.6), to immediately bound A_3 as

$$A_3 \le \frac{1}{2}M_1M_3^3T\int_0^T\sum_{k\in\Omega_F}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)|\left[\sup_{\tau\in[0,T]}|\hat{G}(\tau,j)|\right]\ ds.$$

We then multiply and divide by $|k-j||j|$ and rearrange, finding

$$\begin{aligned}
A_3 &\le \frac{1}{2}M_1M_3^3(M_3+1)T\int_0^T\sum_{k\in\mathbb{Z}^n_*}\sum_{j\in\mathbb{Z}^n_*}|k-j|^3|\hat{F}(s,k-j)|\left[\sup_{\tau\in[0,T]}\frac{|\hat{G}(\tau,j)|}{|j|}\right]\ ds.\\
&\le \frac{1}{2}M_1M_3^4(M_3+1)T\|F\|_{\mathcal{X}^3}\|G\|_{\mathcal{Y}^{-1}}.
\end{aligned}$$

Here we have used the definition of M_3 to make the elementary bound

$$\sup_{k\in\Omega_F}\ \sup_{j\in\mathbb{Z}^n,\ j\neq k}\frac{|j|}{|k-j|} = \sup_{k\in\Omega_F}\ \sup_{\ell\in\mathbb{Z}^n,\ell\neq 0}\frac{|\ell+k|}{|\ell|}\le M_3+1.$$

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We next bound A_4. We exchange the order of integration and compute the integral with respect to t, finding

$$A_4 = \frac{1}{2}\sum_{k\in\Omega_I}\int_0^T\int_s^T |k|^3 e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)||\hat{G}(s,j)|\,dtds$$
$$= \frac{1}{2}\sum_{k\in\Omega_I}\int_0^T \frac{|k|^4}{|k|}\frac{1-e^{-(T-s)\sigma(\Delta^2+\Delta)(k)}}{\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}^n_*}|k-j|^2|\hat{F}(s,k-j)||\hat{G}(s,j)|\,ds.$$

Since $\sigma(\Delta^2+\Delta)(k)$ is positive for $k\in\Omega_I$, we may neglect the exponential in the numerator, and use the definition of M_2, finding

$$A_4 \le \frac{1}{2M_2}\int_0^T\sum_{k\in\mathbb{Z}^n_*}\sum_{j\in\mathbb{Z}^n_*}\frac{|k-j|^2}{|k|}|\hat{F}(s,k-j)||\hat{G}(s,j)|\,ds.$$

We then multiply and divide by $|k-j||j|$ and again use (3.3), finding

$$A_4 \le \frac{1}{2M_2}\int_0^T\sum_{k\in\mathbb{Z}^n_*}\sum_{j\in\mathbb{Z}^n_*}|k-j|^3|\hat{F}(s,k-j)|\frac{|\hat{G}(s,j)|}{|j|}\left(\frac{|j|}{|k||k-j|}\right)\,ds$$
$$\le \frac{1}{M_2}\|F\|_{\mathcal{X}^3}\|G\|_{\mathcal{Y}^{-1}}.$$

We have concluded the bound

$$A_1 \le \left(M_1M_3^3(M_3+1)T+\frac{1}{M_2}\right)\|F\|_{\mathcal{X}^3}\|G\|_{\mathcal{Y}^{-1}},$$

and we have by symmetry the corresponding estimate for A_2, namely

$$A_2 \le \left(M_1M_3^3(M_3+1)T+\frac{1}{M_2}\right)\|F\|_{\mathcal{Y}^{-1}}\|G\|_{\mathcal{X}^3}.$$

These bounds immediately imply the desired conclusion, which is

$$\|B(F,G)\|_{\mathcal{X}^3} \le \left(M_1M_3^3(M_3+1)T+\frac{1}{M_2}\right) \times(\|F\|_{\mathcal{Y}^{-1}}+\|F\|_{\mathcal{X}^3})(\|G\|_{\mathcal{Y}^{-1}}+\|G\|_{\mathcal{X}^3}). \tag{3.5}$$

(It is understood that if the set Ω_F is empty, then we may take $T=\infty$, and in this case, the combination M_1T is understood as $M_1T=0$.) □

4. Existence of solutions with data in pseudomeasure spaces

We prove existence theorems for the Kuramoto–Sivashinsky equation with pseudomeasure data in dimensions $n=1$ and $n=2$. We first define the pseudomeasure spaces.

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For any $m \in \mathbb{R}$, we define the sets PM^m and $\mathcal{PM}^m$ by their norms,

$$\|f\|_{PM^m} = |\hat{f}(0)| + \sup_{k\in\mathbb{Z}^n} |k|^m |\hat{f}(k)|,$$
$$\|f\|_{\mathcal{PM}^m} = \sup_{t\in[0,T]} |\hat{f}(t,0)| + \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n} |k|^m |\hat{f}(t,k)|.$$

As before, we will hereafter assume that all functions considered have zero mean and will therefore only need to use these norms on spaces of functions with zero mean. We then get the simpler expressions

$$\|f\|_{PM^m} = \sup_{k\in\mathbb{Z}^n_*} |k|^m |\hat{f}(k)|,$$
$$\|f\|_{\mathcal{PM}^m} = \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n_*} |k|^m |\hat{f}(t,k)|.$$

In § 4.1, we give the linear estimates, which are relevant for both dimensions $n=1$ and $n=2$. In § 4.2, we state the existence theorem in dimension $n=1$ and demonstrate the needed bilinear estimates. Then, in § 4.3, we state the existence theorem for dimension $n=2$ and again give the needed bilinear estimates.

4.1. Linear estimates

We give the linear estimates for pseudomeasure data in the following two lemmas.

LEMMA 4.1. *For any $m \in \mathbb{R}$, the semigroup operator S satisfies $S : PM^m \to \mathcal{PM}^m$, with the estimate*

$$\|S\psi_0\|_{\mathcal{PM}^m} \le M_1 \|\psi_0\|_{PM^m}.$$

Proof. We begin to estimate $\|S\psi_0\|_{\mathcal{PM}^m}$. We have

$$\begin{aligned}\|S\psi_0\|_{\mathcal{PM}^m} &= \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n_*} |k|^m e^{-t\sigma(\Delta^2+\Delta)(k)} |\hat{\psi}_0(k)| \\ &\le \|\psi_0\|_{PM^m} \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}^n_*} e^{-t\sigma(\Delta^2+\Delta)(k)}.\end{aligned}$$

Using the definition of M_1, then, this estimate becomes the following:

$$\|S\psi_0\|_{\mathcal{PM}^m} \le M_1 \|\psi_0\|_{PM^m}.$$

□

We next estimate $\|S\psi_0\|_{\mathcal{X}^{m_2}}$, with $\psi_0 \in PM^{m_1}$.

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LEMMA 4.2. *Let m_1 and m_2 be real numbers satisfying $m_2 - m_1 - 4 < -n$. There exists $K > 0$ such that for any $\psi_0 \in PM^{m-1}$, we have $S\psi_0 \in \mathcal{X}^{m_2}$ with the estimate*

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} \leq K\|\psi_0\|_{PM^{m_1}}.$$

Proof. We begin by writing out the norm of $S\psi_0$ in the space $\mathcal{X}^{m_2}$:

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} = \int_0^T \sum_{k\in\mathbb{Z}_*^n} |k|^{m_2} e^{-t\sigma(\Delta^2+\Delta)(k)} |\hat{\psi}_0(k)| \; dt.$$

We multiply and divide the integrand by $|k|^{m_1}$, and we take out the norm of ψ_0, finding

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} \leq \|\psi_0\|_{PM^{m_1}} \int_0^T \sum_{k\in\mathbb{Z}_*^n} |k|^{m_2-m_1} e^{-t(\sigma(\Delta^2+\Delta)(k))} \; dt.$$

We then decompose $\mathbb{Z}_*^n$ into $\Omega_F \cup \Omega_I$, and we estimate the portion over the finite set Ω_F. This yields

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} \leq \|\psi_0\|_{PM^{m_1}} \left(M_1 C(\Omega_F) T + \int_0^T \sum_{k\in\Omega_I} |k|^{m_2-m_1} e^{-t\sigma(\Delta^2+\Delta)(k)} \; dt \right).$$

(If we are in Case A, then since $\Omega_F = \emptyset$, then in this case with $T = \infty$, we may take $C(\Omega_F) = 0$, with the understanding that this would mean that $M_1 C(\Omega_F) T = 0$.) Then, we evaluate the remaining integral, finding

$$\|S\psi_0\|_{\mathcal{X}_{m_2}} \leq \|\psi_0\|_{PM^{m_1}} \left(M_1 C(\Omega_F) T + \sum_{k\in\Omega_I} \frac{|k|^{m_2-m_1}(1 - e^{-T(\sigma(\Delta^2+\Delta)(k))})}{\sigma(\Delta^2+\Delta)(k)} \right).$$

We may then neglect the exponential and use (2.5) to find

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} \leq \|\psi_0\|_{PM^{m_1}} \left(M_1 C(\Omega_F) T + \frac{1}{M_2} \sum_{k\in\Omega_I} |k|^{m_2-m_1-4} \right).$$

The series on the right-hand side converges, so we have concluded that there exists $K > 0$ such that

$$\|S\psi_0\|_{\mathcal{X}^{m_2}} \leq K\|\psi_0\|_{PM^{m_1}}.$$

□

REMARK 4.3. The quantity $m_2 - m_1$ describes how many derivatives the solution gains at positive times compared to the data. Of course, one may expect, in L^1-based spaces, to gain four derivatives from a fourth-order parabolic evolution; at the same time, because our nonlinearity only contains first derivatives, we do not

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require this full gain of four derivatives. Our requirement $m_2 - m_1 - 4 < -n$ implies that in one space dimension, we may take $m_2 - m_1 < 3$, and in two space dimensions, we may take $m_2 - m_1 < 2$. In these cases, this is sufficient gain of regularity to establish the bilinear estimates. Unfortunately, in three space dimensions, we find no result, as when $n = 3$, we have $m_2 - m_1 < 1$, and this is less than the gain of one full derivative which we need to estimate the nonlinearity by the present method.

4.2. Existence of solutions with pseudomeasure data with $n = 1$

In one space dimension, we can find the existence of solutions with PM^{-p} data, for any $p \in (0, 1/2)$. With the parabolic gain of regularity, we will also have that the solutions are in $\mathcal{X}^{2+p}$; this gain of $2 + 2p$ derivatives is less than the four full derivatives, which might be possible, but this is sufficient gain to deal with the nonlinearity. Note that these choices satisfy the constraints as discussed in remark 4.3. Specifically, with $m_1 = -p$ and $m_2 = 2 + p$, and with $p < 1/2$, we have $m_2 - m_1 = 2 + 2p < 3$, as desired.

THEOREM 4.4 *Let $p \in (0, 1/2)$ and $T > 0$ be given. (If the conditions of Case A hold, then T may be taken to be $T = \infty$.) Let $n = 1$. There exists $\varepsilon > 0$ such that for any ϕ_0 and $\mathbb{P}\phi_0 \in PM^{-p}$, if $\|\mathbb{P}\phi_0\|_{PM^{-p}} < \varepsilon$, then there exists a unique ϕ with $\mathbb{P}\phi \in \mathcal{PM}^{-p} \cap \mathcal{X}^{2+p}$ such that ϕ is a mild solution to the initial value problem (1.1) and (1.2).*

Proof. We will again use lemma 2.1; recall that this implies both existence and uniqueness of solutions. To apply lemma 2.1, we need to conclude that $x_0 = S\mathbb{P}\phi_0$ is in the space $\mathcal{PM}^{-p} \cap \mathcal{X}^{2+p}$. This follows from lemmas 4.1 and 4.2 with $m_1 = -p$ and $m_2 = 2 + p$, since these parameters satisfy the condition $m_2 - m_1 - 4 < -n$.

We begin with the estimate of $B(F, G)$ in $\mathcal{PM}^{-p}$. Using the definition of $\mathcal{PM}^{-p}$ and the triangle inequality, we have

$$\begin{aligned} &\|B(F,G)\|_{\mathcal{PM}^{-p}} \\ &\quad = \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}_*} \left| \int_0^t \frac{1}{|k|^p} e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}_*} j\hat{F}(s,j)(k-j)\hat{G}(s,k-j)\, ds \right| \\ &\quad \le \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}_*} \int_0^t \frac{e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}}{|k|^p} \sum_{j\in\mathbb{Z}_*} |j||\hat{F}(s,j)||k-j||\hat{G}(s,k-j)|\, ds. \end{aligned}$$

We then use Young's inequality on $j(k - j)$ and bound the exponentials by M_1 (recall the definition of M_1 in (2.4)), finding

$$\begin{aligned} &\|B(F,G)\|_{\mathcal{PM}^{-p}} \\ &\quad \le \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}_*} \frac{M_1}{2} \int_0^t \frac{1}{|k|^p} \sum_{j\in\mathbb{Z}_*} |j|^2|\hat{F}(s,j)||\hat{G}(s,k-j)|\, ds \\ &\quad + \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}_*} \frac{M_1}{2} \int_0^t \frac{1}{|k|^p} \sum_{j\in\mathbb{Z}_*} |\hat{F}(s,j)||k-j|^2|\hat{G}(s,k-j)|\, ds. \end{aligned} \tag{4.1}$$

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We multiply and divide by the appropriate powers of $|j|$ and $|k-j|$:

$$\|B(F,G)\|_{\mathcal{PM}^{-p}} \tag{4.2}$$
$$\leq \sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}_*}\frac{M_1}{2}\int_0^t\sum_{j\in\mathbb{Z}_*}\left(\frac{|k-j|^p}{|k|^p|j|^p}\right)|j|^{2+p}|\hat{F}(s,j)|\left(\frac{|\hat{G}(s,k-j)|}{|k-j|^p}\right)\,ds$$
$$+\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}_*}\frac{M_1}{2}\int_0^t\sum_{j\in\mathbb{Z}_*}\left(\frac{|j|^p}{|k|^p|k-j|^p}\right)\left(\frac{|\hat{F}(s,j)|}{|j|^p}\right)|k-j|^{2+p}|\hat{G}(s,k-j)|\,ds.$$

For the first term on the right-hand side of (4.2), we take the supremum with respect to s and k for the factor $\frac{|\hat{G}(s,k-j)|}{|k-j|^p}$ in the integrand. For the second term on the right-hand side, we take the supremum with respect to s and j for the factor $\frac{|\hat{F}(s,j)|}{|j|^p}$ in the integrand. Also, we again use (3.3). These considerations lead to the following bound:

$$\|B(F,G)\|_{\mathcal{PM}^{-p}}\leq 2^{p-1}M_1\left(\|G\|_{\mathcal{PM}^{-p}}\|F\|_{\mathcal{X}^{2+p}}+\|F\|_{\mathcal{PM}^{-p}}\|G\|_{\mathcal{X}^{2+p}}\right).$$

We may further bound this as

$$\|B(F,G)\|_{\mathcal{PM}^{-p}}\leq 2^{p-1}M_1(\|F\|_{\mathcal{PM}^{-p}}+\|F\|_{\mathcal{X}^{2+p}})(\|G\|_{\mathcal{PM}^{-p}}+\|G\|_{\mathcal{X}^{2+p}}). \tag{4.3}$$

We next must estimate $B(F,G)$ in the space $\mathcal{X}^{2+p}$. We begin with the definition and use the triangle inequality:

$$\|B(F,G)\|_{\mathcal{X}^{2+p}}$$
$$\leq\sum_{k\in\mathbb{Z}_*}\int_0^T|k|^{2+p}\int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}_*}|j||\hat{F}(s,j)||k-j||\hat{G}(s,k-j)|\,dsdt.$$

Then, as before, we use Young's inequality on $j(k-j)$,

$$\|B(F,G)\|_{\mathcal{X}^{2+p}} \tag{4.4}$$
$$\leq\frac{1}{2}\sum_{k\in\mathbb{Z}_*}\int_0^T|k|^{2+p}\int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}_*}|j|^2|\hat{F}(s,j)||\hat{G}(s,k-j)|\,dsdt$$
$$+\sum_{k\in\mathbb{Z}_*}\frac{1}{2}\int_0^T|k|^{2+p}\int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}_*}|k-j|^2|\hat{F}(s,j)||\hat{G}(s,k-j)|\,dsdt.$$

For the first term on the right-hand side, we find the $\mathcal{PM}^{-p}$-norm of G by multiplying and dividing by $|k-j|^p$ and taking a supremum, and we also multiply and divide by $|j|^p$. We treat the second term on the right-hand side similarly, and we arrive at

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$$
\begin{aligned}
&\|B(F,G)\|_{\mathcal{X}^{2+p}} \\
&\le \frac{\|G\|_{\mathcal{PM}^{-p}}}{2}\sum_{k\in\mathbb{Z}_*}\int_0^T\int_0^t |k|^{2+p}e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}_*}\frac{|k-j|^p|j|^{2+p}}{|j|^p}|\hat{F}(s,j)|\ dsdt \\
&\qquad + \frac{\|F\|_{\mathcal{PM}^{-p}}}{2}\sum_{k\in\mathbb{Z}_*}\int_0^T\int_0^t |k|^{2+p}e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \\
&\qquad\quad \times\sum_{j\in\mathbb{Z}_*}\frac{|k-j|^{2+p}|j|^p}{|k-j|^p}|\hat{G}(s,k-j)|\ dsdt.
\end{aligned} \tag{4.5}
$$

Using (3.3) with (4.5), we have

$$
\begin{aligned}
&\|B(F,G)\|_{\mathcal{X}^{2+p}} \\
&\le 2^{p-1}\|G\|_{\mathcal{PM}^{-p}}\sum_{k\in\mathbb{Z}_*}\int_0^T\int_0^t |k|^{2+2p}e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\sum_{j\in\mathbb{Z}_*}|j|^{2+p}|\hat{F}(s,j)|\ dsdt \\
&\qquad + 2^{p-1}\|F\|_{\mathcal{PM}^{-p}}\sum_{k\in\mathbb{Z}_*}\int_0^T\int_0^t |k|^{2+2p}e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \\
&\qquad\quad \times\sum_{j\in\mathbb{Z}_*}|k-j|^{2+p}|\hat{G}(s,k-j)|\ dsdt.
\end{aligned}
$$

In the second term on the right-hand side, we change the variable in the final summation, and we also change the order of integration in both terms on the right-hand side, finding

$$
\begin{aligned}
&\|B(F,G)\|_{\mathcal{X}^{2+p}} \\
&\le 2^{p-1}\|G\|_{\mathcal{PM}^{-p}}\int_0^T\left(\sum_{j\in\mathbb{Z}_*}|j|^{2+p}|\hat{F}(s,j)|\right) \\
&\qquad \times\left(\sum_{k\in\mathbb{Z}_*}|k|^{2+2p}\int_s^T e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\ dt\right)\ ds \\
&\qquad + 2^{p-1}\|F\|_{\mathcal{PM}^{-p}}\int_0^T\left(\sum_{j\in\mathbb{Z}_*}|j|^{2+p}|\hat{G}(s,j)|\right) \\
&\qquad \times\left(\sum_{k\in\mathbb{Z}_*}|k|^{2+2p}\int_s^T e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\ dt\right)\ ds.
\end{aligned} \tag{4.6}
$$

We will work now with the sum with respect to k, which is the same in both of the terms on the right-hand side. We split it into the sum over Ω_F and the sum over Ω_I. Considering $k\in\Omega_F$, we have

$$
\sum_{k\in\Omega_F}k^{2+2p}\int_s^T e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\ dt \le |\Omega_F|M_1M_3^{2+2p}T.
$$

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(We have said that if $\Omega_F = \emptyset$, then we may take $T = \infty$, and then this product is to be understood as $|\Omega_F|T = 0$.) Considering $k \in \Omega_I$, we evaluate the integral, finding

$$\sum_{k\in\Omega_I} k^{2+2p} \int_s^T e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\, dt = \sum_{k\in\Omega_I} k^{2+2p} \left(\frac{1 - e^{-(T-s)\sigma(\Delta^2+\Delta)(k)}}{\sigma(\Delta^2+\Delta)(k)} \right).$$

Since the denominator is positive for $k \in \Omega_I$, we may neglect the exponential in the numerator. Then, we use the definition of M_2, finding

$$\sum_{k\in\Omega_I} k^{2+2p} \int_s^T e^{-(t-s)\sigma(\Delta^2+\Delta)(k)}\, dt \leq \frac{1}{M_2} \sum_{k\in\mathbb{Z}_*} \frac{1}{k^{2-2p}} = \frac{c(p)}{M_2} < \infty.$$

We, of course, have used here that $p < 1/2$.

Returning to (4.6), we conclude with the bound

$$\|B(F,G)\|_{\mathcal{X}^{2+p}} \leq 2^{p-1} \left(|\Omega_F| M_1 M_3^{2+2p} T + \frac{c(p)}{M_2} \right) \cdot (\|F\|_{\mathcal{PM}^{-p}} + \|F\|_{\mathcal{X}^{2+p}})(\|G\|_{\mathcal{PM}^{-p}} + \|G\|_{\mathcal{X}^{2+p}}). \tag{4.7}$$

□

4.3. Existence of solutions with pseudomeasure data with $n = 2$

We again let $p \in (0, 1/2)$ be given. In the case of two space dimensions, we will be taking data in PM^{1-p} and finding solutions in $\mathcal{PM}^{1-p} \cap \mathcal{X}^{2+p}$. As regards remark 4.3, this means that we have $m_1 = 1 - p$ and $m_2 = 2 + p$, so that $m_2 - m_1 = 1 + 2p < 2$, as required.

THEOREM 4.5 *Let $p \in (0, 1/2)$ and $T > 0$ be given. (If the conditions of Case A hold, then T may be taken to be $T = \infty$.) Let $n = 2$. There exists $\varepsilon > 0$ such that for any ϕ_0 with $\mathbb{P}\phi_0 \in PM^{1-p}$, if $\|\mathbb{P}\phi_0\|_{PM^{1-p}} < \varepsilon$, then there exists a unique ϕ with $\mathbb{P}\phi \in \mathcal{PM}^{1-p} \cap \mathcal{X}^{2+p}$ such that ϕ is a mild solution to the initial value problem (1.1) and (1.2).*

Proof. To apply lemma 2.1, we need to conclude that $x_0 = S\mathbb{P}\phi_0$ is in the space $\mathcal{PM}^{1-p} \cap \mathcal{X}^{2+p}$. This follows from lemmas 4.1 and 4.2 with $m_1 = 1 - p$ and $m_2 = 2 + p$, since these parameters satisfy the condition $m_2 - m_1 - 4 < -n$.

We estimate $\|B(F,G)\|_{\mathcal{PM}^{1-p}}$. From the definition of the norm and $B(F,G)$, we have

$$\|B(F,G)\|_{\mathcal{PM}^{1-p}}$$
$$= \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}_*^2} |k|^{1-p} \left| \int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}_*^2} \sum_{i=1}^2 (k_i - j_i)\hat{F}(s, k-j) j_i \hat{G}(s,j)\, ds \right|.$$

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We use the triangle inequality and the definition of the constant M_1 to find

$$\|B(F,G)\|_{\mathcal{PM}^{1-p}} \leq 2M_1 \sup_{k\in\mathbb{Z}^2_*} |k|^{1-p} \int_0^T \sum_{j\in\mathbb{Z}^2_*} |k-j||\hat{F}(s,k-j)||j||\hat{G}(s,j)|\ ds.$$

Bounding $|k|$ as $|k| \leq |k-j| + |j|$, this becomes

$$\begin{aligned}\|B(F,G)\|_{\mathcal{PM}^{1-p}} \leq{}& 2M_1 \sup_{k\in\mathbb{Z}^2_*} \int_0^T \sum_{j\in\mathbb{Z}^2_*} \frac{|k-j|^2|j|}{|k|^p}|\hat{F}(s,k-j)||\hat{G}(s,j)|\ ds \\ &+ 2M_1 \sup_{k\in\mathbb{Z}^2_*} \int_0^T \sum_{j\in\mathbb{Z}^2_*} \frac{|k-j||j|^2}{|k|^p}|\hat{F}(s,k-j)||\hat{G}(s,j)|\ ds.\end{aligned}$$

We then proceed as in the proof of theorem 4.4, i.e., we adjust the factors of $|k-j|$ and $|j|$ and use (3.3) as appropriate, until we are able to conclude that

$$\|B(F,G)\|_{\mathcal{PM}^{1-p}} \leq 2^{p+1}M_1(\|F\|_{\mathcal{PM}^{1-p}} + \|F\|_{\mathcal{X}^{2+p}})(\|G\|_{\mathcal{PM}^{1-p}} + \|G\|_{\mathcal{X}^{2+p}}). \tag{4.8}$$

Next we bound $B(F,G)$ in $\mathcal{X}^{2+p}$. From the definition of the norm and of $B(F,G)$, and again using $|k| \leq |k-j| + |j|$ for just one factor of $|k|$, we have

$$\begin{aligned}&\|B(F,G)\|_{\mathcal{X}^{2+p}} \\ &\leq 2\int_0^T \sum_{k\in\mathbb{Z}^2_*} |k|^{1+p} \int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}^2_*} |k-j|^2|\hat{F}(s,k-j)||j||\hat{G}(s,j)|\ dsdt \\ &+2\int_0^T \sum_{k\in\mathbb{Z}^2_*} |k|^{1+p} \int_0^t e^{-(t-s)\sigma(\Delta^2+\Delta)(k)} \sum_{j\in\mathbb{Z}^2_*} |k-j||\hat{F}(s,k-j)||j|^2|\hat{G}(s,j)|\ dsdt.\end{aligned}$$

We then follow the corresponding steps of the proof of theorem 4.4, including adjusting factors of $|k-j|$ and $|j|$ and using (3.3) as appropriate, until we reach the conclusion

$$\begin{aligned}\|B(F,G)\|_{\mathcal{X}^{2+p}} \leq 2^{p+1}&\left(|\Omega_F|M_1M_3^{1+2p}T + \frac{1}{M_2}\sum_{k\in\mathbb{Z}^2_*}\frac{1}{|k|^{3-2p}}\right) \\ &\cdot(\|F\|_{\mathcal{PM}^{1-p}} + \|F\|_{\mathcal{X}^{2+p}})(\|G\|_{\mathcal{PM}^{1-p}} + \|G\|_{\mathcal{X}^{2+p}}).\end{aligned} \tag{4.9}$$

Note that the sum on the right-hand side converges because $p < 1/2$. □

REMARK 4.6. We note that the proofs of theorems 4.4 and 4.5 overall look quite similar, but there is a subtle difference. In the proof of theorem 4.4, in both (4.1) and (4.4), we have bounded $|k-j||j|$ by $\frac{1}{2}(|k-j|^2 + |j|^2)$, while in the proof of theorem 4.5, we have not done so in the corresponding places. In the proof of theorem 4.5, we then take a factor of $|k|$ and bound it as $|k| \leq |k-j| + |j|$.

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This first set of manipulations produces a different result from the second set of manipulations. The first set of manipulations requires a greater gain of regularity and allows lower initial regularity than the second set of manipulations. That is, if we were to proceed as in the second way for the $n=1$ theorem, then the proof would not work for PM^{-p} data. If we were to proceed in the first way for the $n=2$ theorem, then the proof would not work because, for $n=2$, we only gain $1+2p<2$ derivatives as compared to the data rather than the $2+2p$ derivatives we gain in the $n=1$ case. Recall that this limit on the gain of regularity is determined from our linear theory and is therefore a constraint on how we may conduct the nonlinear estimates.

5. Analyticity

In this section we will show that the solutions produced earlier are analytic within their time of existence, if needed by further restricting the size of the initial data.

Given initial data ψ_0, we recall the mild formulation of the Kuramoto–Sivashinsky equation (2.7):

$$\psi = S\psi_0 - \frac{1}{2}B(\psi,\psi),$$

where the semigroup S was introduced in (2.8) and the bilinear term $B = B(F,G)$ was given in (2.9).

Our approach to establish analyticity follows the one used by Bae in [5], in which one revisits the existence proofs but for an exponentially weighted modification of ψ. More precisely, let $g = g(t)$ be a given function and consider

$$V \equiv e^{g(t)|D|}\psi, \tag{5.1}$$

where $|D| = \sqrt{-\Delta}$. Then, V should satisfy the equation

$$V = e^{[g(t)|D|-t(\Delta^2+\Delta)]}V_0 - \frac{1}{2}\int_0^t e^{[g(t)|D|-(t-s)(\Delta^2+\Delta)]}[\mathbb{P}(|\nabla e^{-g(s)|D|}V|^2)]\,ds, \tag{5.2}$$

with $V_0 = \psi_0$. Existence of a solution to this equation for suitable g and sufficiently small V_0 in certain function spaces then implies analyticity of ψ, as will be made precise at the end of this section. The radius of analyticity is bounded from below by $g(t)$.

We rewrite (5.2), separating the linear from the nonlinear term, as

$$V = \mathcal{L}V_0 - \frac{1}{2}\mathcal{B}(V,V),$$

with

$$\mathcal{L}V_0 = e^{[g(t)|D|-t(\Delta^2+\Delta)]}V_0 \tag{5.3}$$

and

$$\mathcal{B}(U,W) = \int_0^t e^{[g(t)|D|-(t-s)(\Delta^2+\Delta)]}[\mathbb{P}(\nabla e^{-g(s)|D|}U\cdot\nabla e^{-g(s)|D|}W)]\,ds.$$

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We will prove existence of a solution V to (5.2) for initial data $V_0 = \psi_0$ in Y^{-1}, in any dimension, and in $PM^{(n-1)-p}$, in dimensions $n=1$ and $n=2$, with $0 < p < 1/2$. As before, we use lemma 2.1, so we require the following bounds:

$$\|\mathcal{L}V_0\|_{\mathcal{Y}^{-1}} \leq C\|V_0\|_{Y^{-1}} \tag{5.4}$$

$$\|\mathcal{L}V_0\|_{\mathcal{X}^3} \leq C\|V_0\|_{Y^{-1}} \tag{5.5}$$

$$\|\mathcal{L}V_0\|_{\mathcal{PM}^{(n-1)-p}} \leq C\|V_0\|_{PM^{(n-1)-p}} \tag{5.6}$$

$$\|\mathcal{L}V_0\|_{\mathcal{X}^{2+p}} \leq C\|V_0\|_{PM^{(n-1)-p}}, \tag{5.7}$$

as well as

$$\|\mathcal{B}(U,W)\|_{\mathcal{Y}^{-1}} \leq C(\|U\|_{\mathcal{Y}^{-1}} + \|U\|_{\mathcal{X}^3})(\|W\|_{\mathcal{Y}^{-1}} + \|W\|_{\mathcal{X}^3}) \tag{5.8}$$

$$\|\mathcal{B}(U,W)\|_{\mathcal{X}^3} \leq C(\|U\|_{\mathcal{Y}^{-1}} + \|U\|_{\mathcal{X}^3})(\|W\|_{\mathcal{Y}^{-1}} + \|W\|_{\mathcal{X}^3}) \tag{5.9}$$

$$\begin{aligned}\|\mathcal{B}(U,W)\|_{\mathcal{PM}^{(n-1)-p}} \leq C(&\|U\|_{\mathcal{PM}^{(n-1)-p}} + \|U\|_{\mathcal{X}^{2+p}})\\ &\times (\|W\|_{\mathcal{PM}^{(n-1)-p}} + \|W\|_{\mathcal{X}^{2+p}})\end{aligned} \tag{5.10}$$

$$\|\mathcal{B}(U,W)\|_{\mathcal{X}^{2+p}} \leq C(\|U\|_{\mathcal{PM}^{(n-1)-p}} + \|U\|_{\mathcal{X}^{2+p}})(\|W\|_{\mathcal{PM}^{(n-1)-p}} + \|W\|_{\mathcal{X}^{2+p}}). \tag{5.11}$$

The Fourier coefficients of $(\mathcal{L}V_0)(t,\cdot)$ are given by

$$\mathcal{F}(\mathcal{L}V_0)(t,k) = e^{g(t)|k| - t\sigma(\Delta^2+\Delta)(k)}\hat{V}_0(k). \tag{5.12}$$

The Fourier coefficients of the nonlinear term are

$$\begin{aligned}\mathcal{F}(\mathcal{B}(U,W))(t,k) = -\int_0^t e^{g(t)|k| - (t-s)\sigma(\Delta^2+\Delta)(k)}\Big[\\ \sum_{j\in\mathbb{Z}^n_*, j\neq k} ((k-j)\cdot j)\, e^{-g(s)|k-j|}\hat{U}(s,k-j)e^{-g(s)|j|}\hat{W}(s,j)\Big]\, ds.\end{aligned} \tag{5.13}$$

In what follows, we will consider two kinds of temporal weights:

$$g(t) = a\sqrt[4]{t}, \text{for some constant } a > 0; \tag{5.14}$$

$$g(t) = bt, \text{for some constant } b > 0. \tag{5.15}$$

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In order to estimate the linear term $\mathcal{L}$, we will make use of the following technical lemma:

Lemma 5.1. *Let $k \in \Omega_I$. Then, if M_2 is as in (2.5), it holds that:*

(i) if $g(t) = bt$, with $b < \frac{M_2}{2}$, then

$$g(t)|k| - t\sigma(\Delta^2 + \Delta)(k) \le -\frac{M_2 t}{2}|k|^4;$$

(ii) if $g(t) = a\sqrt[4]{t}$, then there exists $C = C(a) > 0$ such that

$$g(t)|k| - t\sigma(\Delta^2 + \Delta)(k) \le C - \frac{M_2 t}{2}|k|^4$$

for all $t \ge 0$.

Proof. Recall the definition of M_2, from (2.5), such that

$$\sigma(\Delta^2 + \Delta)(k) \ge M_2|k|^4.$$

Let us first consider the case $g(t) = bt$ with $b < \frac{M_2}{2}$. Then, clearly, if k is such that $|k| \ge 1$, it follows that $bt|k| - \frac{M_2}{2}t|k|^4 \le 0$. It then follows easily that, if $|k| \ge 1$,

$$g(t)|k| - t\sigma(\Delta^2 + \Delta)(k) \le g(t)|k| - tM_2|k|^4 \le -\frac{M_2 t}{2}|k|^4,$$

as desired. This establishes item (i).

Next, consider the case $g(t) = a\sqrt[4]{t}$. Then, of course, we have

$$g(t)|k| - t\sigma(\Delta^2 + \Delta)(k) \le a\sqrt[4]{t}|k| - M_2 t|k|^4. \tag{5.16}$$

Consider the function $f = f(z) = az - \frac{M_2}{2}z^4$. This function is globally bounded from above. Let

$$C = C(a) = \max\{\sup f(z), 1\}.$$

Noting that $a\sqrt[4]{t}|k| - \frac{M_2}{2}t|k|^4 = f(\sqrt[4]{t}|k|)$ we obtain item (ii). □

In view of lemma 5.1, all the estimates for $\mathcal{L}V_0 \equiv \mathcal{L}\psi_0$, (5.4), (5.5), (5.6), and (5.7), can be reduced to the corresponding estimates already obtained for $S\psi_0$, namely (3.1) and those obtained in lemmas 4.1 and 4.2.

Next, we prove another technical lemma, which will be used for the nonlinear term $\mathcal{B}(U, W)$.

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LEMMA 5.2. *Let $k \in \Omega_I$. Then, if M_2 is as in (2.5), it holds that:*

(i) if $g(t) = bt$, with $b < \frac{M_2}{2}$, then it follows that

$$(g(t) - g(s))|k| - (t-s)\sigma(\Delta^2 + \Delta)(k) \leq -\frac{M_2(t-s)}{2}|k|^4$$

for all t s such that $0 \leq s \leq t$ and all $k \in \Omega_I$;

(ii) if $g(t) = a\sqrt[4]{t}$, then there exists $C = C(a) > 0$ such that

$$(g(t) - g(s))|k| - (t-s)\sigma(\Delta^2 + \Delta)(k) \leq C - \frac{M_2(t-s)}{2}|k|^4$$

for all t s such that $0 \leq s \leq t$ and all $k \in \Omega_I$.

Proof. Let us begin with item (i), the case $g(t) = bt$, $b < M_2/2$. In view of lemma 5.1 item (i), this is trivial since

$$(g(t) - g(s))|k| - (t-s)\sigma(\Delta^2 + \Delta)(k) = (t-s)b|k| - (t-s)\sigma(\Delta^2 + \Delta)(k),$$

and $t - s \geq 0$.

Next consider $g(t) = a\sqrt[4]{t}$. We first note that

$$(g(t) - g(s))|k| - (t-s)\sigma(\Delta^2 + \Delta)(k) \leq (g(t) - g(s))|k| - (t-s)M_2|k|^4,$$

using, again, (2.5) and the fact that $t - s \geq 0$. Next, we observe that

$$(g(t) - g(s))|k| - (t-s)M_2|k|^4 = f(a\sqrt[4]{t}) - f(a\sqrt[4]{s}) - \frac{M_2(t-s)}{2}|k|^4,$$

where f was introduced in the proof of lemma 5.1. There are two possibilities: either $f(a\sqrt[4]{s}) \geq 0$, in which case we may ignore this term and use the boundedness from above of f to obtain (ii), or $f(a\sqrt[4]{s}) < 0$. Let us assume the latter and note that f has only two real roots, namely 0 and $\sqrt[3]{\frac{2a}{M_2}} > 0$. In addition, f restricted to the positive real axis is only negative on the interval $\left(\sqrt[3]{\frac{2a}{M_2}}, +\infty\right)$, on which it is also decreasing. Therefore, since $s \leq t$, we have $f(a\sqrt[4]{t}) \leq f(a\sqrt[4]{s})$, from which (ii) follows immediately. □

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Now we rewrite (5.13) in a more convenient form and estimate:

$$|\mathcal{F}(\mathcal{B}(U,W))(t,k)| = \Bigg| - \int_0^t e^{(g(t)-g(s))|k|-(t-s)\sigma(\Delta^2+\Delta)(k)} \Bigg[\sum_{j\in\mathbb{Z}^n_*, j\neq k} ((k-j)\cdot j)\, e^{g(s)(|k|-|k-j|-|j|)} \hat{U}(s,k-j)\hat{W}(s,j)\Bigg] ds \Bigg| \tag{5.18}$$

$$\leq \int_0^t e^{(g(t)-g(s))|k|-(t-s)\sigma(\Delta^2+\Delta)(k)} \Bigg[\sum_{\substack{j\in\mathbb{Z}^n_* \\ j\neq k}} |k-j||j||\hat{U}(s,k-j)||\hat{W}(s,j)|\Bigg] ds, \tag{5.18}$$

where we used the triangle inequality to estimate $|k|-|k-j|-|j|\leq 0$.

In view of lemma 5.2, it is easy to see that the estimates on the term (5.18), namely (5.8), (5.9), (5.10) and (5.11), can be reduced to the corresponding ones for $B(F,G)$, (3.4), (3.5), (4.3), (4.7), (4.8) and (4.9), established in the previous sections.

We now comment on how these results imply analyticity of solutions. By the periodic analogue of theorem IX.13 of [32], a function is analytic with radius of analyticity at least ρ if its Fourier series decays like $e^{-\tilde{\rho}|k|}$ for all $\tilde{\rho} < \rho$. With a solution $V \in \mathcal{Y}^{-1}$, then at each time t, we have, for any $\varepsilon > 0$, the existence of $c > 0$ such that

$$c \sum_{k\in\mathbb{Z}^n_*} e^{(g(t)-\varepsilon)|k|}|\hat{\psi}(t,k)| \leq \sum_{k\in\mathbb{Z}^n_*} e^{(g(t)-\varepsilon)|k|} \frac{e^{\varepsilon|k|}}{|k|}|\hat{\psi}(t,k)| \leq \|V\|_{\mathcal{Y}^{-1}}.$$

We see from this that $\hat{\psi}(t,\cdot)$ decays like $e^{-\tilde{\rho}|k|}$ for any $\tilde{\rho} < g(t)$, and thus ψ is analytic with radius of analyticity at least $g(t)$. Similarly, for solutions with $V \in \mathcal{PM}^{(n-1)-p}$, the solution of Kuramoto–Sivashinsky, ψ, is again analytic with radius of analyticity at least $g(t)$.

By virtue of these considerations, we have established the following results.

THEOREM 5.3 *Let $T > 0$ be given. (If the conditions of Case A hold, then T may be taken to be $T = \infty$.) Let $n \geq 1$. There exists $\varepsilon > 0$ such that for any ϕ_0 with $\mathbb{P}\phi_0 \in Y^{-1}$, if $\|\mathbb{P}\phi_0\|_{Y^{-1}} < \varepsilon$, then there exists a unique ϕ with $\mathbb{P}\phi \in \mathcal{Y}^{-1} \cap \mathcal{X}^3$ such that ϕ is an analytic mild solution to the initial value problem (1.1) and (1.2) with radius of analyticity at least $R(t) = \max\{a\sqrt[4]{t}, bt\}$, with $b < M_2$ and $a > 0$.*

THEOREM 5.4 *Let $n \in \{1, 2\}$. Let $p \in (0, 1/2)$ and $T > 0$ be given. (If the conditions of Case A hold, then T may be taken to be $T = \infty$.) There exists $\varepsilon > 0$ such that for any ϕ_0 with $\mathbb{P}\phi_0 \in PM^{(n-1)-p}$, if $\|\mathbb{P}\phi_0\|_{PM^{(n-1)-p}} < \varepsilon$, then there exists a unique ϕ with $\mathbb{P}\phi \in \mathcal{PM}^{(n-1)-p} \cap \mathcal{X}^{2+p}$ such that ϕ is an analytic mild solution to the initial value problem (1.1) and (1.2) with radius of analyticity at least $R(t) = \max\{a\sqrt[4]{t}, bt\}$, with $b < M_2$ and $a > 0$.*

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6. Concluding remarks

We close now with a few remarks.

First, we comment on our bound for the radius of analyticity. We have shown that our solutions have radius of analyticity that grows at least like $t^{1/4}$ and also at least like t. Of course, the rate $t^{1/4}$ is faster for times near zero, and the rate t is faster for large times. A fractional-power rate has previously been observed for the Navier–Stokes equations (where the rate is $t^{1/2}$) and for the Kuramoto–Sivashinsky equation (where the rate is $t^{1/4}$) for solutions on $\mathbb{R}^n$ [5, 18, 20]. For spatially periodic problems, rates like t have been observed previously for the Navier–Stokes equations [18] or for more general parabolic equations [17]. We have previously observed for the Navier–Stokes equations that in the periodic case, one gets the improvement that both of these rates hold [1]. The present work shows this improvement in the periodic case of the Kuramoto–Sivashinsky equation. The radius of analyticity of solutions is relevant for the convergence rate of numerical simulations [15].

Let $n \in \{1, 2\}$. Then, we note that the two function spaces for the initial data we have considered in this work, Y^{-1} and $PM^{(n-1)-p}$, with $p < 1/2$, are not comparable. Consider, for instance, f such that

$$|\hat{f}(k)| = \frac{1}{|k|^{n-1}} \text{ for all } k.$$

Then, $f \in PM^{(n-1)-p}$, but $f \notin Y^{-1}$. On the other hand, let f be such that

$$|\hat{f}(j)| = \begin{cases} |j|^{3/4} & \text{if } |j| = 2^{4\ell},\ \ell = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f \in Y^{-1}$, but $f \notin PM^{-1/2}$, and thus $f \notin PM^{(n-1)-p}$.

In the introduction, we mentioned the Navier–Stokes results of Koch and Tataru [22], Cannone and Karch [13], and Lei and Lin [26] as works proving existence of solutions for the Navier–Stokes equations with low-regularity data. It should be noted that the function spaces considered in the aforementioned works, BMO^{-1}, PM^2, and X^{-1}, respectively, are all critical spaces for the Navier–Stokes equations.

If we discard the unstable Laplacian and consider (1.1) in full space $\mathbb{R}^n$, then it is easy to see that this modified Kuramoto–Sivashinsky equation is invariant under the scaling

$$\lambda \mapsto \lambda^2 \psi(\lambda x, \lambda^4 t).$$

Thus, among the hierarchy of spaces considered in this work, the spaces Y^{-2} and PM^{n-2} are *critical spaces*, i.e., whose norms are invariant under this scaling. In the present work, we lower the regularity requirements for existence theory for the Kuramoto–Sivashinsky equation as compared to the prior literature and have proved existence of solutions in spaces of negative index, namely Y^{-1} and $PM^{(n-1)-p}$, for any $0 < p < 1/2$, but these spaces are not critical. Therefore, there remains work to be done to continue lowering the regularity threshold for the initial data.

We also mentioned in the introduction that solutions of the Kuramoto–Sivashinsky equation have been proved to be global in one spatial dimension, when starting from H^1 data. In the present work, we have shown existence of solutions with rough data, but only until a short time (unless the spatial domain $[0, L]$ satisfies $L < 2\pi$). But we have shown that the solutions are analytic at positive times, and thus, the solutions instantaneously become H^1 solutions, which could then be continued for all time. So, our one-dimensional solutions are in fact global. However, the present method would not extend on its own to demonstrate this. The radius of analyticity that we prove grows in time, like both $t^{1/4}$ and t. This growth of the radius for all time is possible in the small-domain case (again, $L < 2\pi$), but in the presence of linearly growing modes ($L > 2\pi$), one would not expect this. Instead, the solution, in some cases, tends towards coherent structures such as travelling waves or time-periodic waves, and these attracting solutions tend to have finite radius of analyticity. The long-time behaviour of the radius of analyticity for the initial value problem, then, is to tend towards this value of the radius of analyticity rather than to tend towards infinity. This can be seen from computational work such as [21, 31]. Understanding in more detail the time evolution of the radius of analyticity of solutions of the Kuramoto-Sivashinsky problem will be a subject of future work.

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Statements and Declarations

Data availability.

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests.

The authors declare that they have no competing interests.

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