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ON NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract

In this paper we obtain some normality criteria of families of meromorphic functions, which improve and generalize the related results of Gu and Bergweiler, respectively. Some examples are given to show the sharpness of our results.

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1. Introduction

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j}, such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

In 1979, Gu [5] proved the following well-known normality criterion, which was a conjecture of Hayman [8].

THEOREM G. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, and let k be a positive integer. If, for every function $f \in \mathcal{F}$, $f \neq 0, f^{(k)} \neq 1$, then $\mathcal{F}$ is normal.*

Recently, Bergweiler [2] improved the above result for the case $k = 1$, by allowing f to have zeros, but restricting the values f' can take at the zeros of f.

THEOREM B. *Let K and ε be positive numbers, and let $\mathcal{F}$ be the family of all functions meromorphic in D which satisfy the following conditions*:

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(i) *If* $z \in D$, *then* $f'(z) \neq 1$.

(ii) *If* $z \in D$, *and* $f(z) = 0$, *then* $0 < |f'(z)| \leq K$.

(iii) *If* Δ *is a disk in D and if f has* $m \geq 2$ *zeros* $z_1, z_2, \ldots, z_m \in \Delta$, *then there exists* $k \in \{-1\} \bigcup \{1, \ldots, m-2\}$ *such that* $\left| \sum_{i=1}^{m} f'(z_i)^k - m^{k+1} \right| \geq \varepsilon$.

Then $\mathscr{F}$ *is normal in D.*

A natural problem arises: *what can we say if* f' *is replaced by k-th derivative* $f^{(k)}$ *in Theorem* B? In this paper, we obtain the following results, which improve and generalize Theorem G and Theorem B.

For the case $k \geq 3$, we have

THEOREM 1. *Let k be a positive integer such that* $k \geq 3$ *and K be a positive number. Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain D and* $a(z)$ *be a non-vanishing analytic function in D. Suppose that, for every function* $f \in \mathscr{F}$, f *has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) *If* $z \in D$, *then* $f^{(k)}(z) \neq a(z)$.

(b) *If* $z \in D$ *and* $f(z) = 0$, *then* $0 < |f^{(k)}(z)| \leq K$.

Then $\mathscr{F}$ *is normal in D.*

REMARK 1. Theorem 1 shows that for $k \geq 3$ the conclusion of Theorem B is still valid without the condition such as (iii).

The following example shows that condition (b) cannot be omitted in Theorem 1.

EXAMPLE 1 (see [11]). Let $n, k \in \mathbb{N}$, $D = \{z : |z| < 1\}$, and $a_n (n = 1, 2, \ldots)$ satisfy $(k! a_n^{k+1})/n = 1$. Set

$$\mathscr{F} = \left\{ \frac{(a_n z + 1)^{k+1}}{nz}, n = 1, 2, \ldots, z \in D \right\}.$$

Then for each $f_n(z) \in \mathscr{F}$, $f_n(z) = (a_n z + 1)^{k+1}/(nz)$, we have

(1) the zeros of $f_n(z)$ are of multiplicity at least $k+1$;

(2) $f_n^{(k)}(z) \neq 1$.

But $\mathscr{F}$ is not normal in D. In fact, for each $f_n(z) \in \mathscr{F}$, by a simple computation, we deduce that $f_n^{\#}(0) = n \to \infty$, as $n \to \infty$. By Marty's criterion, $\mathscr{F}$ is not normal in D.

For $k = 2$, Theorem 1 is not valid. But we have the following two results.

THEOREM 2. *Let K be a positive number. Let* $\mathscr{F}$ *be a family of meromorphic functions in a domain D and* $a(z)$ *be a non-vanishing analytic function in D. Suppose* *that, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 2 and satisfies the following conditions*:

(a) *If $z \in D$, then $f''(z) \neq a(z)$.*
(b) *If $z \in D$ and $f(z) = 0$, then $|f''(z)| \leq K$.*
(c) *All poles of f are of multiplicity at least 3.*

Then $\mathcal{F}$ is normal in D.

The following example shows that condition (c) in Theorem 2 is necessary and the number 3 is sharp.

EXAMPLE 2. Let $D = \{z : |z| < 1\}$ and

$$\mathcal{F} = \left\{ \frac{(nz+1)^2(z-1/n)^2}{2n^2z^2}, n = 2, 3, \ldots, z \in D \right\}.$$

Then for each $f_n(z) \in \mathcal{F}$, $f_n(z) = (nz+1)^2(z-1/n)^2/(2n^2z^2)$, we have

(1) $f_n''(z) = 1 + 3/(n^4z^4)$, then $f_n''(z) \neq 1$.
(2) $z_1 = 1/n$, $z_2 = -1/n$ are the zeros of $f_n(z)$ of multiplicity 2 in D, and $|f_n''(z_i)| = 4$ $(i = 1, 2)$.

However, $\mathcal{F}$ is not normal in D. In fact, for each $f_n(z) \in \mathcal{F}$, we have

$$f_n^{\#}\left(\frac{2}{n}\right) = \frac{384}{145} n^3 \to \infty,$$

as $n \to \infty$. Then by Marty's criterion, $\mathcal{F}$ is not normal in D.

THEOREM 3. *Let K be a positive number. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and $a(z)$ be a non-vanishing analytic function in D. Suppose that, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 2 and satisfies the following conditions*:

(1) *If $z \in D$, then $f''(z) \neq a(z)$.*
(2) *If $z \in D$ and $f(z) = 0$, then $|f''(z)| \leq K$.*
(3) *If Δ is a disk in D and if f has $m \geq 3$ zeros $z_1, z_2, \ldots, z_m \in \Delta$, then there exists $h \in \{1, 2, \ldots, m-2\}$ such that $\left|\sum_{i=1}^{m} f''(z_i)^h - m^{h+1}\right| \geq \varepsilon$.*

Then $\mathcal{F}$ is normal in D.

REMARK 2. If f has only zeros of multiplicity at least 3 in Theorem 2 and Theorem 3, it is obvious that condition (b) can be omitted. In fact, Wang and Fang [11] proved that: Let $\mathcal{F}$ be a family of meromorphic functions defined in D. If for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 3 and only poles of multiplicity at least 2 and $f'' \neq 1$, then $\mathcal{F}$ is normal.

For $k = 1$, we obtain the following result, which is a generalization of Theorem B.

THEOREM 4. *Let K, ε be positive numbers, $a(z)$ be a non-vanishing analytic function in D, and let $\mathscr{F}$ be the family of all functions meromorphic in D which satisfy the following conditions*:

(i) *If $z \in D$, then $f'(z) \neq a(z)$.*
(ii) *If $z \in D$, and $f(z) = 0$, then $0 < |f'(z)| \leq K$.*
(iii) *If Δ is a disk in D and if f has $m \geq 2$ zeros $z_1, z_2, \dots, z_m \in \Delta$, then there exists $k \in \{-1\} \bigcup \{1, \dots, m-2\}$ such that $\left|\sum_{i=1}^{m} f'(z_i)^k - m^{k+1}\right| \geq \varepsilon$.*

Then $\mathscr{F}$ is normal in D.

2. Some lemmas

To prove our results, we need some lemmas.

LEMMA 1 ([3]). *Let f be meromorphic in $\mathbb{C}$ and of finite order. If f has only finitely many critical values, then f has only finitely many asymptotic values.*

The following lemma is due to Rippon and Stallard ([10]; see also [1]).

LEMMA 2. *Let f be meromorphic in $\mathbb{C}$ and suppose that the set of all finite critical and asymptotic values of f is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|f(z)| > R$, then*

$$|f'(z)| \geq \frac{|f(z)| \log |f(z)|}{16\pi |z|}.$$

LEMMA 3 ([11]). *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$, where $a_0, a_1, \dots, a_n$ are constants, $p(z)$ and $q(z)$ are two coprime polynomials with $\deg q(z) < \deg p(z)$, and let k be a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{1}{k!} z^k + \cdots + a_0 + \frac{b}{(z-c)^m},$$

where b ($\neq 0$), c are two constants and $m \in \mathbb{N}$.

We denote the residue of a meromorphic function f at a point z by $\operatorname{res}(f, z)$. By an elementary computation, we have

LEMMA 4 (see also [2]). *Let $f(z) = z + a + b/(z-c)^l$ with $a, b, c \in \mathbb{C}$, $b \neq 0$, $l \in \mathbb{N}$, and let $p \in \{0, 1, \dots, l\}$. Then*

$$\operatorname{res}\left(\frac{(f')^p}{f}, -c\right) = 1 - (l+1)^p.$$

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LEMMA 5. *Let f be meromorphic in $\mathbb{C}$ and of finite order, and let $k \geq 2$ be a positive integer and K be a positive number. Suppose that f has only zeros of multiplicity at least k, $|f^{(k)}(z)| < K$ wherever $f(z) = 0$, and $f^{(k)}(z) \neq 1$. Then one of the following two cases must occur:*

(i)

$$f(z) = \alpha(z - \beta)^k, \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$, and $\alpha k! \neq 1$.

(ii) *If $k = 2$, then*

$$f(z) = \frac{(z - c_1)^2 (z - c_2)^2}{2(z - c)^2}, \tag{2}$$

or

$$f(z) = \frac{(z - c_1)^3}{2(z - c)}. \tag{3}$$

If $k \geq 3$, then

$$f(z) = \frac{1}{k!} \frac{(z - c_1)^{k+1}}{(z - c)}. \tag{4}$$

Here c_1, c_2 and c are distinct complex numbers.

PROOF. If $g(z) = z - f^{(k-1)}(z)$, then $g'(z) = 1 - f^{(k)}(z) \neq 0$ for all $z \in \mathbb{C}$. First, we prove that f is not transcendental. Suppose that f is transcendental, then g is also transcendental. By Hayman's inequality ([6], see also [7]), f has infinitely many zeros $z_n (n = 1, 2, \ldots)$. Since f has only zeros of multiplicity at least k, we have $g(z_n) = z_n$. Since $g'(z) \neq 0$, by Lemma 1, g has only finitely many asymptotic values, and then satisfies the hypotheses of Lemma 2 for some $R > 0$. We get

$$|g'(z_n)| \geq \frac{\log |z_n|}{16\pi}$$

for large n. Thus $g'(z_n) \to \infty$, as $n \to \infty$. On the other hand, we know $|f^{(k)}(z_n)| < K$ and thus $|g'(z_n)| \leq 1 + K$ for all n, a contradiction.

Thus f is rational. If f is a polynomial, then since $f^{(k)}(z) \neq 1$ and f has only zeros of multiplicity at least k, f has the form (1). If f is not a polynomial, we can write $f = R + P/Q$ with polynomials P, Q, R satisfying $\deg P < \deg Q$. Since $f^{(k)}(z) \neq 1$, from Lemma 3, we have

$$f(z) = \frac{1}{k!} z^k + \cdots + a_0 + \frac{b}{(z - c)^m},$$

where b ($\neq 0$), c are two constants and $m \in \mathbb{N}$. Set

$$p_k(z) = \frac{1}{k!}z^k + \cdots + a_0,$$

so that

$$f(z) = \frac{p_k(z)(z-c)^m + b}{(z-c)^m}.$$

Obviously, $f(z)$ and $p_k(z)(z-c)^m + b$ have the same zeros. If $c_1, c_2, \ldots, c_q$ are the zeros of $p_k(z)(z-c)^m + b$, with multiplicity $n_1, n_2, \ldots, n_q$, then $n_i \geq k$ ($i = 1, 2, \ldots, q$). Hence c_1 is a zero of $[p_k(z)(z-c)^m + b]'$ with multiplicity $n_1 - 1$ ($\geq k-1$). Since

$$[p_k(z)(z-c)^m + b]' = (z-c)^{m-1}[p_k'(z)(z-c) + mp_k(z)],$$

and it is easy to see that $c_1 \neq c$, then c_1 is a zero of $p_k'(z)(z-c) + mp_k(z)$ with multiplicity $n_1 - 1$ ($\geq k-1$). Note that $\deg[p_k'(z)(z-c) + mp_k(z)] = k$. If $k = 2$, we deduce that $p_2(z)(z-c)^m + b$ has two zeros c_1, c_2 with multiplicity 2 or only one zero c_1 with multiplicity 3, where c_1, c_2 and c are three distinct constants. Thus $p_2(z)(z-c)^m + b = \frac{1}{2}(z-c_1)^2(z-c_2)^2$ or $p_2(z)(z-c)^m + b = \frac{1}{2}(z-c_1)^3$. It follows that $m = 2$ and f has the form (2) or $m = 1$ and f has the form (3). If $k \geq 3$, then c_1 is the only zero of $p_k(z)(z-c)^m + b$, with multiplicity $k+1$. Thus $p_k(z)(z-c)^m + b = (z-c_1)^{k+1}/k!$, and hence f has the form (4). This completes the proof of the lemma. □

The following result is a generalization of the well-known Zalcman's lemma, which is due to Pang and Zalcman [9].

LEMMA 6. *Let k be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain D, such that each function $f \in \mathcal{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then, for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. Moreover, g has finite order.

REMARK 3. The above result improves the result of Chen and Gu [4].

3. Proof of theorems

PROOF OF THEOREM 1. Suppose that $\mathscr{F}$ is not normal at a point $z_0 \in D$. Then by Lemma 6, for $\alpha = k$, there exist a sequence of functions $f_n \in \mathscr{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = \rho_n^{-k} f_n(z_n + \rho_n \zeta)$$

converges locally uniformly to a non-constant function $g(\zeta)$, which is meromorphic in $\mathbb{C}$ and of finite order. Moreover, $g^{\#}(\zeta) \le g^{\#}(0) = k(K+1)+1$ for all $\zeta \in \mathbb{C}$. Since $g_n(\zeta)$ has only zeros of multiplicity at least k, by Hurwitz's theorem, the zeros of $g(\zeta)$ are of multiplicity at least k.

Let ζ_1 be a zero of $g(\zeta)$. Then there exist $\zeta_n, \zeta_n \to \zeta_1$, such that $g_n(\zeta_n) = \rho_n^{-k} f_n(z_n + \rho_n \zeta_n) = 0$ for n sufficiently large. Thus $f_n(z_n + \rho_n \zeta_n) = 0$ for sufficiently large n. Since

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) \to g^{(k)}(\zeta_1),$$

we deduce from condition (b) that $|g^{(k)}(\zeta_1)| \le K$.

Obviously, $a(z_0) \ne 0, \infty$. Now we distinguish two cases.

Case 1. There exists ζ_0 such that $g^{(k)}(\zeta_0) = a(z_0)$.

Then there exists $\delta > 0$, such that $g(\zeta)$ is analytic on $D_{2\delta} = \{\zeta : |\zeta - \zeta_0| < 2\delta\}$. Hence $g_n^{(k)}(\zeta)$ are analytic on $D_\delta = \{\zeta : |\zeta - \zeta_0| < \delta\}$ for sufficiently large n. Since

$$g_n^{(k)}(\zeta) - a(z_n + \rho_n \zeta) = f_n^{(k)}(z_n + \rho_n \zeta) - a(z_n + \rho_n \zeta) \ne 0,$$

and $g_n^{(k)}(\zeta) - a(z_n + \rho_n \zeta)$ converges uniformly to $g^{(k)}(\zeta) - a(z_0)$ on $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$, we conclude that $g^{(k)}(\zeta) - a(z_0) \equiv 0$ on $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$, and then

$$g^{(k)}(\zeta) - a(z_0) \equiv 0$$

for all $\zeta \in \mathbb{C}$. Note that $g(\zeta)$ has only zeros of multiplicity at least k, so we have

$$g(\zeta) = \frac{a(z_0)}{k!}(z - \alpha)^k, \quad (\alpha \in \mathbb{C}),$$

and $|g^{(k)}(\zeta)| = |a(z_0)| \le K$ (if $|a(z_0)| > K$, we have already obtain a contradiction). A simple calculation shows that

$$g^{\#}(0) \le \begin{cases} k/2 & \text{if } |\alpha| \ge 1; \\ |a(z_0)| & \text{if } |\alpha| < 1. \end{cases}$$

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This contradicts $g^{\#}(0) = k(K+1)+1$.

Case 2. $g^{(k)}(\zeta) \neq a(z_0)$.

Without loss of generality, we may assume $a(z_0) = 1$. Then by Lemma 5, we know that g has the the form (1) or (4) in Lemma 5. Similarly as in Case 1, we exclude the case that g has the form (1). Then

$$g(\zeta) = \frac{1}{k!}\frac{(\zeta - c_1)^{k+1}}{(\zeta - c)},$$

where c_1 and c are two distinct constants. Thus $g(\zeta)$ has only one zero c_1 with multiplicity $k+1$. On the other hand, by the assumption of Theorem 1 and Hurwitz's theorem, $g_n(\zeta)$ has only zeros of multiplicity k. We arrive at a contradiction. This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. Suppose that $\mathscr{F}$ is not normal at a point $z_0 \in D$. Then by Lemma 6, for $\alpha = 2$, there exist a sequence of functions $f_n \in \mathscr{F}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = \rho_n^{-2} f_n(z_n + \rho_n\zeta)$$

converges locally uniformly to a non-constant meromorphic function $g(\zeta)$. Moreover, $g(\zeta)$ is of finite order, and $g^{\#}(\zeta) \leq g^{\#}(0) = k(K+1)+1$ for all $\zeta \in \mathbb{C}$. By Hurwitz's theorem, $g(\zeta)$ has only zeros of multiplicity at least 2. Similarly as in the proof of Theorem 1, we know that $|g''(\zeta)| \leq K$ wherever $g(\zeta) = 0$.

We consider two cases.

Case 1. There exists ζ_0 such that such that $g''(\zeta_0) = a(z_0)$.

Using the same argument as in the proof of Theorem 1, we arrive at a contradiction.

Case 2. $g''(\zeta) \neq a(z_0)$.

Without loss of generality, we may assume $a(z_0) = 1$. Then by Lemma 5, we know that g has the the form (1) (here $k = 2$), (2) or (3) in Lemma 5. As in the proof of Theorem 1 (Case 1), we exclude the case that g has the form (1). Then

$$g(\zeta) = \frac{(\zeta - c_1)^2(\zeta - c_2)^2}{2(\zeta - c)^2},$$

or

$$g(\zeta) = \frac{(\zeta - c_1)^3}{2(\zeta - c)},$$

where c_1, c_2, and c are distinct constants. Thus $g(\zeta)$ has only one pole c with multiplicity 1 or 2. However, since all poles of $g_n(\zeta)$ are of multiplicity at least 3, Hurwitz's theorem guarantees that $g(\zeta)$ has only poles with multiplicity at least 3, a contradiction. This completes the proof of Theorem 2. □

PROOF OF THEOREM 3. Suppose that $\mathscr{F}$ is not normal at a point $z_0 \in D$. The first part of the proof is almost the same as the proof of Theorem 2. Here we only need to consider Case 2. Suppose that $g''(\zeta) \neq a(z_0)$. Without loss of generality, we may assume $a(z_0) = 1$. Then by Lemma 5,

$$g(\zeta) = \frac{1}{2}\zeta^2 + a_1\zeta + a_0 + \frac{b}{(\zeta - c)^l},$$

where a_1, a_0, b $(\neq 0)$, c are constants and $l = 1$ or 2. (The form (1) can be excluded as in the proof of Theorem 1.) Thus

$$g'(\zeta) = \zeta + a_0 + \frac{b_1}{(\zeta - c)^{l+1}}, \quad (b_1 = -bl).$$

Let $m = l+2$. Then $g'(\zeta)$ has m zeros $\zeta_1, \zeta_2, \ldots, \zeta_m$, counted according to multiplicity. Choose R such that $\max_{1\leq i\leq m} |\zeta_i| < R$. By Hurwitz's theorem, for large n, there exist m distinct zeros $\zeta_{n,i} \to \zeta_i$ as $n \to \infty$ for $1 \leq i \leq m$. Thus $f_n'(z_n + \rho_n\zeta_{n,i}) = 0$ for $1 \leq i \leq m$. Set $\Delta_n := D(z_n, \rho_n R)$, then $z_n + \rho_n\zeta_{n,i} \in \Delta_n$ $(1 \leq i \leq m)$, $\Delta_n \subset D$ (for sufficiently large n), and f_n' has no further zeros in Δ_n.

For $h \in \{1, 2, \ldots, m-2\}$, we have

$$\sum_{i=1}^{m} \left(f_n''(z_n + \rho_n\zeta_{n,i})\right)^h = \sum_{i=1}^{m} \left(g_n''(\zeta_{n,i})\right)^h = \sum_{i=1}^{m} \operatorname{res}\left(\frac{(g_n'')^{h+1}}{g_n'}, \zeta_{n,i}\right)$$
$$\to \sum_{\zeta\in(g')^{-1}(0)} \operatorname{res}\left(\frac{(g'')^{h+1}}{g'}, \zeta\right)$$

as $n \to \infty$, where in the last sum multiple zeros ζ of g' occur only once. Obviously,

$$\frac{(g''(\zeta))^{h+1}}{g'(\zeta)} = \frac{1}{\zeta} + O\left(\frac{1}{\zeta^2}\right)$$

as $\zeta \to \infty$, so $\operatorname{res}\left((g'')^{h+1}/g', \infty\right) = -1$. By the residue theorem and Lemma 4, we have

$$\sum_{\zeta\in(g')^{-1}(0)} \operatorname{res}\left(\frac{(g'')^{h+1}}{g'}, \zeta\right) = 1 - \operatorname{res}\left(\frac{(g'')^{h+1}}{g'}, c\right) = (l+1)^{h+1} = m^{h+1}.$$

Thus

$$\sum_{i=1}^{m} \left(f_n''(z_n + \rho_n\zeta_{n,i})\right)^h \to m^{h+1}.$$

This contradicts condition (c) and completes the proof of Theorem 3. □

PROOF OF THEOREM 4. Using the same argument as in this paper and [2], we can prove Theorem 4. We omit the details. □

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