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Extrapolation of L^p Data from a Modular Inequality

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Abstract. If an operator T satisfies a modular inequality on a rearrangement invariant space $L^\rho(\Omega,\mu)$, and if p is strictly between the indices of the space, then the Lebesgue inequality $\int|Tf|^p \le C\int|f|^p$ holds. This extrapolation result is a partial converse to the usual interpolation results. A modular inequality for Orlicz spaces takes the form $\int\Phi(|Tf|)\le\int\Phi(C|f|)$, and here, one can extrapolate to the (finite) indices $i(\Phi)$ and $I(\Phi)$ as well.

1 Introduction

Let (Ω,μ) be a complete σ-finite, nonatomic measure space. Denote by $M(\Omega,\mu)$, the class of real-valued measurable functions on Ω and by $S(\Omega,\mu)$ the simple, integrable functions in $M(\Omega,\mu)$. Suppose T is a sublinear operator mapping $S(\Omega,\mu)$ into $M(\Omega,\mu)$. The classical (Lebesgue) norm inequality for such T,

$$\left[\int_\Omega |Tf(x)|^p\,d\mu(x)\right]^{1/p} \le C\left[\int_\Omega |f(x)|^p\,d\mu(x)\right]^{1/p} \tag{1}$$

is equivalent to the modular inequality

$$\int_\Omega |Tf(x)|^p\,d\mu(x) \le \int_\Omega |Cf(x)|^p\,d\mu(x). \tag{2}$$

In both, p is a fixed index, $1\le p<\infty$, and $C>0$ is independent of f.

One generalization of (2) replaces the modular t^p by any Young's function $\Phi(x)=\int_0^x\phi(t)\,dt$, where ϕ is increasing on $R^+=(0,\infty)$, $\phi(0+)=0$, and $\lim_{x\to\infty}\phi(x)=\infty$. This results in

$$\int_\Omega \Phi\big(|Tf(x)|\big)\,d\mu(x) \le \int_\Omega \Phi\big(C|f(x)|\big)\,d\mu(x), \tag{3}$$

the general modular inequality. The corresponding generalization of (1) is the Orlicz norm inequality

$$\|Tf\|_{\Phi,\mu} \le C\|f\|_{\Phi,\mu}, \tag{4}$$

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in which, for g in the Orlicz space $L_\Phi = L_\Phi(\Omega,\mu)$,

$$L_\Phi(\Omega,\mu) = \left\{ g \in M(\Omega,\mu) : \int_\Omega \Phi\left(\frac{|g(x)|}{C}\,d\mu(x)\right) < \infty \text{ for some } C > 0 \right\},$$

we have

$$\|g\|_{\Phi,\mu} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|g(x)|}{\lambda}\,d\mu(x)\right) \le 1 \right\}.$$

A recent book on modular inequalities is Kokilashvili and Krbec [4]. See also the earlier one by Musielak [9]. For a detailed account of Orlicz spaces see Krasnosel'skii and Rutickii [5], Lindenstrass and Tzafiri [6], or Rao and Ren [10].

Writing

$$\Phi(t) = t^{\log \frac{\Phi(t)}{\log t}}, \tag{5}$$

one can think of (3) as an inequality like (2) in which the index p varies with t. In Section 2, we recall the definition of the Orlicz-Matuszewska-Maligranda indices $i(\Phi)$ and $I(\Phi)$, for which $1 \le i(\Phi) \le I(\Phi) \le \infty$. In some sense, the lower index $i(\Phi)$, and the upper index, $I(\Phi)$, are, respectively, the least possible and the greatest possible exponent in (5). Our main result, Theorem 3, asserts, among other things, that, if $i(\Phi) < I(\Phi)$, then (3) implies (2) for all p with $i(\Phi) < p < I(\Phi)$.

As a simple illustration of this extrapolation result, consider the Young function

$$\Phi(t) = \begin{cases} t^q & \text{if } t \le 1 \\ t^r & \text{if } t > 1, \end{cases}$$

$1 < r < q < \infty$. Here, $i(\Phi) = r$, $I(\Phi) = q$, and so, by Theorem 3, (3) yields (2) whenever $r < p < q$ (in fact, one has (2) for the endpoints r and q also; see Theorem 7). We observe that, for this particular Φ, the Orlicz norm inequality (4) is

$$\|Tf\|_{L^q+L^r} \le C\|f\|_{L^q+L^r}, \tag{6}$$

where

$$\|h\|_{L^q+L^r} \simeq \left[\int_{[|h|<1]} |h|^q\,d\mu\right]^{1/q} + \left[\int_{[|h|\ge 1]} |h|^r\,d\mu\right]^{1/r}.$$

Taking (Ω,μ) to be the real line R with Lebesgue measure and

$$Tf(x) = g(x)\int_0^1 f(y)\,dy, \quad x \in R,$$

for $g \in (L^q+L^r)(R)$, we have, in strong contrast to the modular case, (6) not implying (2) for *any* p, $r < p < q$.

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The precise connection between modular and Orlicz norm inequalities will be given at the end of this section. In Section 2, we use this theorem to define modular inequalities in the general setting of rearrangement-invariant spaces. The extrapolation result, Theorem 3 is then proven in Section 3. An example is given to show that Theorem 3 is best possible in a certain sense. The last section has a proof, due to Nigel Kalton, that one can extrapolate to the (finite) indices when dealing with modular inequalities of the form (3).

We remark that the theory presented here for nonatomic measures can be carried out along similar lines for purely atomic measures with all atoms having equal measure.

Theorem 1 *Suppose that (Ω, μ) is a complete, σ-finite, nonatomic measure space and that T is a sublinear operator mapping $S(\Omega, \mu)$ into $M(\Omega, \mu)$. Let Φ be a Young function. For each $\epsilon > 0$, define the measure $\epsilon\mu$ of the measurable set $E \subset \Omega$ by $(\epsilon\mu)(E) = \epsilon \cdot \mu(E)$. Then the modular inequality (3) holds if and only if*

$$\|Tf\|_{\Phi,\epsilon\mu} \le C\|f\|_{\Phi,\epsilon\mu} \quad f \in S(\Omega, \mu), \tag{7}$$

with $C > 0$ independent of f and ϵ, $0 < \epsilon < \mu(\Omega)$.

Proof The proof is similar to the one for the case $\mu(\Omega) = \infty$ in Proposition 2.5 of [2]. Suppose (7) holds. For $f \in S(\Omega, \mu)$, with $\int_\Omega \Phi(C|f|)\, d\mu > \frac{1}{\mu(\Omega)}$, put $\epsilon = \left(\int_\Omega \Phi(C|f|)\, d\mu\right)^{-1}$. Since $\|f\|_{\Phi,\epsilon\mu} \le \frac{1}{C}$, $\|Tf\|_{\Phi,\epsilon\mu} \le 1$. Thus,

$$\int_\Omega \Phi(|Tf|)\, d\mu \le \frac{1}{\epsilon} = \int_\Omega \Phi(C|f|)\, d\mu.$$

Conversely, given (3), fix an $f \in S(\Omega)$ and an ϵ, $0 < \epsilon < \mu(\Omega)$, with $0 < \alpha = \|f\|_{\Phi,\epsilon\mu} < \infty$. Then, $\int_\Omega \Phi(\frac{|f|}{\alpha})\epsilon\, d\mu = 1$, and so

$$\int_\Omega \Phi\left(\frac{|Tf|}{C\alpha}\right) \epsilon\, d\mu \le \epsilon \int_\Omega \Phi\left(\frac{|f|}{\alpha}\right) d\mu \le 1$$

which shows that

$$\|Tf\|_{\Phi,\epsilon\mu} \le C\alpha = C\|f\|_{\Phi,\epsilon\mu}.$$

■

2 Rearrangement Invariant Spaces

Let (Ω, μ) be a complete, σ-finite measure space, and let $M^+(\Omega, \mu)$ denote the class of all nonnegative, measurable functions on Ω. A functional $\rho\colon M^+(\Omega, \mu) \to [0, \infty]$ is called a function norm if it satisfies

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1.

$$\rho(f+g) = \rho(f) + \rho(g), \quad \text{for } f, g \in M^+(\Omega, \mu)$$
$$\rho(cf) = c\rho(f), \quad \text{for } c \geq 0, f \in M^+(\Omega, \mu)$$
$$\rho(f) = 0 \quad \text{if and only if } f = 0 \text{ a.e.;}$$

2. $0 \leq f_n \uparrow f$ μ-a.e. implies $\rho(f_n) \uparrow \rho(f)$;
3. E measurable with $\mu(E) < \infty$ implies $\rho(\chi_E) < \infty$ and $\int_\Omega f\chi_E \, d\mu < \infty$ whenever $\rho(f) < \infty$.

The Banach function space $L^\rho(\Omega, \mu)$, determined by ρ, is defined by

$$L^\rho(\Omega, \mu) = \{f \in M(\Omega, \mu) : \rho(|f|) < \infty\},$$

where the norm $\|\cdot\|_{\rho,\mu}$ at f is

$$\|f\|_{\rho,\mu} = \rho(|f|).$$

The space $L^\rho(\Omega, \mu)$ is said to be a rearrangement-invariant (r.i.) space if $f \in L^\rho(\Omega, \mu)$ implies $g \in L^\rho(\Omega, \mu)$ and $\|g\|_{\rho,\mu} = \|f\|_{\rho,\mu}$, whenever g is equimeasurable with f; that is

$$\mu_f(\lambda) := \mu\Big([x \in \Omega : |f(x)| > \lambda]\Big) = \mu\Big([x \in \Omega : |g(x)| > \lambda]\Big)$$

for all $\lambda > 0$. In this case, it is shown in [1, pp. 62–64] that there is a function norm σ on $M^+\Big(\big(0, \mu(\Omega)\big), m\Big)$ such that $L^\sigma\Big(\big(0, \mu(\Omega)\big), m\Big)$ is an r.i. space and

$$\|f\|_{\rho,\mu} = \sigma(f^*) \quad \text{for } f \in L^\rho(\Omega, \mu). \tag{8}$$

Here, f^* is the usual nonincreasing rearrangement of f,

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad 0 < t < \mu(\Omega),$$

so that

$$\mu\Big([x \in \Omega : |f(x)| > \lambda]\Big) = m\Big(\big[t \in \big(0, \mu(\Omega)\big) : f^*(t) > \lambda\big]\Big). \tag{9}$$

Orlicz spaces are important examples of r.i. spaces, as are the Lorentz spaces $L^{pq} = L^{pq}(\Omega, \mu)$, $1 < p < \infty$, $1 \leq q \leq \infty$, whose norms are given by

$$\|f\|_{pq} = \begin{cases} \Big[\int_0^\infty \big(s\mu_f(s)^{\frac{1}{p}}\big)^q \frac{ds}{s}\Big]^{\frac{1}{q}} & \text{when } q < \infty \\ \sup_{s>0} s\mu_f(s)^{\frac{1}{p}} & \text{when } q = \infty. \end{cases}$$

The associate norm ρ' of the function norm ρ is defined for $g \in M^+(\Omega, \mu)$ by

$$\rho'(g) = \sup\Big\{ \int_\Omega |fg| \, d\mu : f \in M^+(\Omega, \mu), \rho(f) \leq 1 \Big\}. \tag{10}$$

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As proved in [1, pp. 8–13], ρ' is a function norm. The space $L^{\rho'}(\Omega,\mu)$ is called the associate space of $L^{\rho}(\Omega,\mu)$. The generalized Hölder inequality, which is an immediate consequence of (10), says that $f \in L^{\rho}(\Omega,\mu)$, $g \in L^{\rho'}(\Omega,\mu)$ implies $\int_\Omega |fg|\,d\mu < \infty$ and

$$\left|\int_\Omega fg\,d\mu\right| \le \|f\|_{L^{\rho}(\Omega,\mu)}\|g\|_{L^{\rho'}(\Omega,\mu)}. \tag{11}$$

Further, $\rho'' = \rho$. If

$$\rho(f) = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}, \tag{12}$$

where $\Phi(t) = \int_0^t \phi(u)\,du$ is a Young's function, then ρ' is equivalent to the function norm

$$\rho^*(g) = \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\},$$

where $\Psi(t) = \int_0^t \phi^{-1}(u)\,du$ is the Young's function complementary to Φ; in fact,

$$\rho^*(g) \le \rho'(g) \le 2\rho^*(g).$$

Let ρ be a function norm on $M^+(\Omega,\mu)$ and suppose σ is a function norm on $M^+(\Omega,\mu)$ for which (8) holds. The fundamental function, τ_ρ, of $L^\rho(\Omega,\mu)$ is

$$\tau_\rho(t) = \|\chi_{(0,t)}\|_\sigma \quad 0 < t < \mu(\Omega).$$

For $t > 0$, set

$$h_\rho(t) = \sup_{s>0} \frac{\tau_\rho(\frac{s}{t})}{\tau_\rho(s)} \quad \text{when } \mu(\Omega) = \infty$$

and

$$h_\rho(t) = \limsup_{s\to 0^+} \frac{\tau_\rho(\frac{s}{t})}{\tau_\rho(s)} \quad \text{when } \mu(\Omega) < \infty.$$

Then $h(st) \le h(s)h(t)$ for all $s, t > 0$. The fundamental indices of $L^\rho(\Omega,\mu)$ are

$$i(\rho) = i(\rho,\mu) = \lim_{t\to 0^+} \frac{-\log t}{\log h_\rho(t)} = \inf_{0<t<1} \frac{-\log t}{\log h_\rho(t)}$$

and

$$I(\rho) = I(\rho,\mu) = \lim_{t\to\infty} \frac{-\log t}{\log h_\rho(t)} = \sup_{t>1} \frac{-\log t}{\log h_\rho(t)};$$

they satisfy $1 \le i(\rho) \le I(\rho) \le \infty$ and $i(\rho') = \frac{I(\rho)}{I(\rho)-1}$, $I(\rho') = \frac{i(\rho)}{i(\rho)-1}$. Also,

$$h_\rho(t) \ge t^{-\frac{1}{p}} \quad \text{if } t > 0,\ p < \infty,\ i(\rho) \le p \le I(\rho). \tag{13}$$

Using the methods of [3, Lemmas 1 and 2], it can be shown that these indices are the reciprocals of the fundamental indices (as defined in [1, p. 127]). Finally, in the case in which ρ is defined by (12), and so gives rise to an Orlicz space, the fundamental indices coincide with those of Orlicz-Matuszewska-Maligranda; see [7] and [8].

3 The Extrapolation Theorem

Let $\epsilon > 0$. We begin by observing that $S(\Omega, \epsilon\mu) = S(\Omega, \mu)$ and $M(\Omega, \epsilon\mu) = M(\Omega, \mu)$. Again, if ρ is a function norm on $M^+(\Omega, \mu)$, then it is also one on $M^+(\Omega, \epsilon\mu)$; moreover, as sets, $L^\rho(\Omega, \epsilon\mu) = L^\rho(\Omega, \mu)$. Suppose T is a sublinear operator mapping $S(\Omega, \mu)$ into $M(\Omega, \mu)$. With Theorem 1 in mind, we say T satisfies a modular inequality with respect to the r.i. space $L^\rho(\Omega, \mu)$ if there exists a $C > 0$ such that

$$\|Tf\|_{\rho,\epsilon\mu} \leq C\|f\|_{\rho,\epsilon\mu}, \tag{14}$$

for all $f \in S(\Omega, \mu)$ and $0 < \epsilon < E(\Omega)$, where

$$E(\Omega) = \begin{cases} 1 & \text{if } \mu(\Omega) < \infty \\ \infty & \text{if } \mu(\Omega) = \infty. \end{cases}$$

The following simple result concerning $\|\cdot\|_{\rho,\epsilon\mu}$ will be important.

Lemma 2 *Fix ϵ, $0 < \epsilon < E(\Omega)$. Suppose ρ is a function norm on $M^+(\Omega, \mu)$, and hence on $M^+(\Omega, \epsilon\mu)$. Let σ be a function norm on $M^+\Big(\big(0, \mu(\Omega)\big), m\Big)$ satisfying (8). Given $g \in M^+\Big(\big(0, \mu(\Omega)\big), m\Big)$ and $0 < t < \mu(\Omega)$, define*

$$D_\epsilon g(t) = \begin{cases} g(\frac{t}{\epsilon}) & \text{if } 0 < t < \epsilon\mu(\Omega) \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $f \in L^\rho(\Omega, \mu)$,

$$\|f\|_{\rho,\epsilon\mu} = \sigma(D_\epsilon f^*). \tag{15}$$

The proof is straightforward, and follows easily from the observation that

$$f^{*\epsilon}(t) = D_\epsilon f^*(t) \quad 0 < t < \mu(\Omega).$$

We leave the details to the reader.

We are now in a position to state and prove our main result.

Theorem 3 *Let (Ω, μ) and T be as in Theorem 1. Suppose T satisfies the modular inequality (14) with respect to the r.i. space $L^\rho(\Omega, \mu)$. Then there also holds, for each $p < \infty$, $i(\rho) \leq p \leq I(\rho)$, the restricted weak-type inequality*

$$\lambda^p \mu(E_\lambda) \leq C^p \mu(E), \tag{16}$$

in which $E_\lambda = [x \in \Omega : |T\chi_E(x)| > \lambda]$ and the positive constant C, the same as in (14), is independent of the measurable set $E \subset \Omega$ and $\lambda > 0$. In particular, we then have the Lebesgue inequality (1) if $i(\rho) < p < I(\rho)$.

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Proof Since (14) holds, we have, for $0 < \epsilon < E(\Omega)$, and for $\mu(E) < \infty$,

$$\begin{aligned}\lambda\|\chi_{E_\lambda}\|_{\rho,\epsilon\mu} &\le \|T\chi_E\|_{\rho,\epsilon\mu} \\ &\le C\|\chi_E\|_{\rho,\epsilon\mu}.\end{aligned} \tag{17}$$

In view of (15), if $0 < \epsilon < E(\Omega)$ and if $\mu(F) < \infty$,

$$\begin{aligned}\|\chi_F\|_{\rho,\epsilon\mu} &= \Big\| \chi_{(0,\mu(F))}\Big(\frac{t}{\epsilon}\Big) \Big\|_\sigma \\ &= \|\chi_{(0,\epsilon\mu(F))}(t)\|_\sigma \\ &= \tau_\rho\big(\epsilon\mu(F)\big),\end{aligned}$$

and so (17) says that $\mu(E_\lambda) < \infty$ with

$$\lambda\frac{\tau_\rho\big(\epsilon\mu(E_\lambda)\big)}{\tau_\rho\big(\epsilon\mu(E)\big)} \le C.$$

Now, given $0 < u < \mu(E)\mu(\Omega)$, take $\epsilon = \frac{u}{\mu(E)}$ to get

$$\lambda\frac{\tau_\rho\big(u \cdot \mu(E_\lambda)/\mu(E)\big)}{\tau_\rho(u)} \le C,$$

and, therefore,

$$\lambda h_\rho\left(\frac{\mu(E_\lambda)}{\mu(E)}\right) \le C.$$

Hence, if $p < \infty$, $i(\rho) \le p \le I(\rho)$,

$$\lambda\left[\frac{\mu(F_\lambda)}{\mu(E)}\right]^{\frac{1}{p}} \le C,$$

or

$$\lambda^p\mu(E_\lambda) \le C^p\mu(E).$$

Finally, having (16), the interpolation theorem of Stein-Weiss [12] ensures (1), if $i(\rho) < p < I(\rho)$. ∎

An r.i. norm inequality—that is (14) for a fixed ϵ—is typically obtained by interpolation from weak-type conditions such as (16); [1, Chapters 5 and 8]. It is interesting that the stronger inequality (14) (and so, in the Orlicz case (3)), also holds, given the same hypotheses. Theorem 3 is a partial converse to this modular interpolation theorem.

Now, it is, in fact, not the fundamental indices, but the Boyd indices $p(\rho)$ and $q(\rho)$, that figure in interpolation theory [6, Vol. II, Chapter 26]. The latter satisfy only $1 \le p(\rho) \le i(\rho) \le I(\rho) \le q(\rho) \le \infty$, though $p(\rho) = i(\rho)$ and $q(\rho) = I(\rho)$ when $L^\rho(\Omega,\mu)$ is an Orlicz space or a Lorentz space. A natural question is whether Theorem 3 can be strengthened by replacing $i(\rho)$ and $I(\rho)$ by $p(\rho)$ and $q(\rho)$, respectively. The following example answers this question in the negative.

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Example 4 In [11], Shimogaki constructed an r.i. space $L^\rho\big((0,1),m\big)$ for which $\tau_\rho(t) = \tau_{\rho'}(t) = \sqrt{t}$, $0 < t < 1$ (so that $i(\rho) = I(\rho) = i(\rho') = I(\rho') = 2$), yet $p(\rho) = 1$ and $q(\rho) = \infty$. Since the Lorentz space $L^{21} = L^{21}\big((0,1),m\big)$ is the smallest r.i. space on $(0,1)$ with fundamental function $\sqrt{t}$ [1, p. 79], we have the continuous embedding

$$\max[\|f\|_{\rho,m}, \|f\|_{\rho',m}] \le C\|f\|_{21}. \tag{18}$$

Now suppose g is nonincreasing on $(0,1)$ with $g \in L^{21} - \bigcup_{p>2} L^p$. Define the linear operator T by

$$Tf(x) = g(x)\int_0^1 f(y)g(y)\,dy, \quad 0 < x < 1.$$

We claim the modular inequality (14) holds for T, though clearly T is unbounded on L^p for all $p \neq 2$. In view of Lemma 2, with $\sigma = \rho$, the claim amounts to showing

$$\|D_\epsilon(Tf)^*\|_{\rho,m} \le C\|D_\epsilon f^*\|_{\rho,m},$$

where C is independent of $f \in L^\rho\big((0,1),m\big)$ and ϵ, $0 < \epsilon < 1$. A simple change of variables gives

$$\begin{aligned} D_\epsilon(Tf)^*(x) &= \epsilon^{-\frac12} g\Big(\frac{x}{\epsilon}\Big)\left[\epsilon^{-\frac12}\int_0^\epsilon f\Big(\frac{y}{\epsilon}\Big) g\Big(\frac{y}{\epsilon}\Big)\,dy\right] \\ &\le \epsilon^{-\frac12} g\Big(\frac{x}{\epsilon}\Big)\left[\epsilon^{-\frac12}\int_0^1 D_\epsilon f^*(s) D_\epsilon g(s)\,ds\right] \end{aligned}$$

for $0 < x < \epsilon$. Finally, the generalized Hölder inequality (11), together with (18), yields

$$\begin{aligned} \|D_\epsilon(Tf)^*\|_{\rho,m} &\le [\epsilon^{-\frac12}\|D_\epsilon g\|_{\rho,m}][\epsilon^{-\frac12}\|D_\epsilon g\|_{\rho',m}]\|D_\epsilon f^*\|_{\rho,m} \\ &\le C\epsilon^{-1}\|D_\epsilon g\|_{21}^2\,\|D_\epsilon f^*\|_{\rho,m} \\ &\le C\|g\|_{21}^2\,\|D_\epsilon f^*\|_{\rho,m}. \end{aligned}$$

4 Lebesgue Inequalities at the Indices

We begin by observing that the general modular inequality (14) with respect to an r.i. space $L^\rho(\Omega,\mu)$ need not imply (1) for $p = i(\rho), I(\rho)$. Thus, consider

Example 5 Take $\Omega = R^+$, $\mu = m$, and fix p and q, $1 < p < q < \infty$. Given $\Lambda(t) = \min\{t^{\frac1p}, t^{\frac1q}\}$, define the function norm ρ by

$$\rho(f) = \int_0^\infty f^*(t)\,d\Lambda(t), \quad f \in M^+(R^+, m).$$

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The r.i. space $L^\rho(R^+, m)$ is a generalized Lorentz space (see [1, p. 72]) having $i(\rho) = p$, $I(\rho) = q$. We claim the operator T, given at $f \in S(R^+, m)$ by

$$Tf(t) = \left(\int_0^\infty f(s) \frac{\Lambda(s)}{s} \, ds \right) \chi_{(0,1)}(t),$$

satisfies (14), but not (1) for p or q. Since $\frac{\Lambda(s)}{s} \notin L^{\frac{p}{p-1}} + L^{\frac{q}{q-1}}$, T cannot be bounded on L^p or L^q. Yet, T satisfies (14). Indeed, we have, for $f \in S(R^+, m)$ and all $\epsilon > 0$,

$$\begin{aligned}
\|Tf\|_{\rho,\epsilon m} &= \int_0^\infty (Tf)^* \left(\frac{t}{\epsilon}\right) d\Lambda(t) \\
&= \left| \int_0^\infty f(s) \frac{\Lambda(s)}{s} \, ds \right| \int_0^\infty \chi_{(0,1)} \left(\frac{t}{\epsilon}\right) d\Lambda(t) \\
&= \Lambda(\epsilon) \left| \int_0^\infty f(s) \frac{\Lambda(s)}{s} \, ds \right| \\
&\leq \Lambda(\epsilon) \int_0^\infty f^*(s) \frac{\Lambda(s)}{s} \, ds \\
&= \Lambda(\epsilon) \int_0^\infty f^* \left(\frac{t}{\epsilon}\right) \Lambda\left(\frac{t}{\epsilon}\right) \frac{dt}{t}.
\end{aligned}$$

But, $\Lambda(\epsilon)\Lambda(\frac{t}{\epsilon}) \leq \Lambda(t)$, so

$$\|Tf\|_{\rho,\epsilon m} \leq \int_0^\infty f^* \left(\frac{t}{\epsilon}\right) \Lambda(t) \frac{dt}{t} \leq q \|f\|_{\rho,\epsilon m}.$$

In spite of this example, extrapolation always extends to the (finite) indices for Orlicz spaces. The proof is due to Nigel Kalton, who has kindly allowed us to give it here.

We require an alternate definition of the indices for ρ, given by (12), which is equivalent to the one in Section 2. Let

$$k(t) = \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \quad \text{when } \mu(\Omega) = \infty$$

and

$$k(t) = \limsup_{u\to\infty} \frac{\Phi(tu)}{\Phi(u)} \quad \text{when } \mu(\Omega) < \infty.$$

Then, in view of the remark on p. 322 of [3], the lower and upper fundamental indices of the Orlicz space L_Φ, denoted by $i(\Phi)$ and $I(\Phi)$, respectively, can be defined as

$$i(\Phi) = i(\Phi, \mu) = \sup_{0<t<1} \frac{\log k(t)}{\log t} = \lim_{t\to 0^+} \frac{\log k(t)}{\log t}$$

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and

$$I(\Phi) = I(\Phi, \mu) = \inf_{t>1} \frac{\log k(t)}{\log t} = \lim_{t\to\infty} \frac{\log k(t)}{\log t}.$$

The following deep result is implicit in [6, Vol. I, Theorems 4.a.8 and 4.a.9].

Lemma 6 *Let (Ω, μ) be as in Theorem 1. Suppose Φ is a Young's function with associated lower and upper indices $i(\Phi)$ and $I(\Phi)$, respectively. Then, if $p < \infty$, $i(\Phi) \le p \le I(\Phi)$, there exists a sequence of probability measures $\{\mu_n\}$ on the Borel subsets of R such that*

$$\lim_{n\to\infty} \int_R \frac{\Phi(t|x|)}{\Phi(|x|)}\, d\mu_n(x) = t^p,$$

uniformly in t on compact subsets of R^+.

Theorem 7 *Let (Ω, μ), Φ and T be as in Theorem 1. If $p < \infty$, $i(\Phi) \le p \le I(\Phi)$, then the modular inequality (3) implies the Lebesgue inequality (1).*

Proof Fix $p < \infty$ with $i(\Phi) \le p \le I(\Phi)$. Let $f \in S(\Omega, \mu)$. From (3) we have, for each $t \in R$, $t \ne 0$,

$$\int_\Omega \frac{\Phi\big(|t|\,|Tf(x)|\big)}{\Phi(|t|)}\, d\mu(x) \le \int_\Omega \frac{\Phi\big(|t|\,|Cf(x)|\big)}{\Phi(|t|)}\, d\mu(x). \tag{19}$$

Integrating both sides of (19) over R, with respect to μ_n, yields

$$\int_R d\mu_n(t) \int_\Omega \frac{\Phi\big(|t|\,|Tf(x)|\big)}{\Phi(|t|)}\, d\mu(x) \le \int_R d\mu_n(t) \int_\Omega \frac{\Phi\big(|t|\,|Cf(x)|\big)}{\Phi(|t|)}\, d\mu(x),$$

and so, applying Fubini's theorem,

$$\int_\Omega \int_R \frac{\Phi\big(|t|\,|Tf(x)|\big)}{\Phi(|t|)}\, d\mu_n(t)\, d\mu(x) \le \int_\Omega \int_R \frac{\Phi\big(|t|\,|Cf(x)|\big)}{\Phi(|t|)}\, d\mu_n(t)\, d\mu(x).$$

By Lemma 6,

$$\int_R \frac{\Phi\big(|t|\,|Cf(x)|\big)}{\Phi(|t|)}\, d\mu_n(t) \to |Cf(x)|^p$$

uniformly on Ω, while, for $m = 1, 2, \dots,$

$$\int_R \frac{\Phi\big(|t|\,|Tf(x)|\big)}{\Phi(|t|)}\, d\mu_n(t) \to |Tf(x)|^p$$

uniformly on

$$\Omega_m = \Big[x \in \Omega : \frac{1}{m} \le |Tf(x)| \le m\Big].$$

Thus,

$$\int_{\Omega_m} |Tf(x)|^p \, d\mu(x) \le \int_{\Omega} |Cf(x)|^p \, d\mu(x), \quad m = 1, 2, \ldots,$$

whence (1) follows on letting $m \to \infty$. ■

We conclude by remarking that if the operator T in Theorem 7 is linear and $I(\Phi) = \infty$, then one obtains the Lebesgue inequality

$$\operatorname*{ess\,sup}_{x\in\Omega} |Tf(x)| \le \operatorname*{ess\,sup}_{x\in\Omega} |Cf(x)|$$

for $f \in S(\Omega, \mu)$, by standard duality arguments.

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