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Algebraic integrability of Macdonald operators and representations of quantum groups

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Abstract. In this paper we construct examples of commutative rings of difference operators with matrix coefficients from representation theory of quantum groups, generalizing the results of our previous paper [ES] to the q-deformed case. A generalized Baker–Akhiezer function Ψ is realized as a matrix character of a Verma module and is a common eigenfunction for a commutative ring of difference operators.

In particular, we obtain the following result in Macdonald theory: at integer values of the Macdonald parameter k, there exist difference operators commuting with Macdonald operators which are not polynomials of Macdonald operators. This result generalizes an analogous result of Chalyh and Veselov for the case $q = 1$, to arbitrary q. As a by-product, we prove a generalized Weyl character formula for Macdonald polynomials (= Conjecture 8.2 from [FV]), the duality for the Ψ-function, and the existence of shift operators.

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1. Introduction

Let N be a positive integer. Let $\mathfrak{D}_q^N$ be the algebra over the field $\mathbb{C}(q)$ generated by the field of rational functions $\mathbb{C}(q, X_1, \ldots, X_N)$ and commuting operators $T_1^{\pm 1}, \ldots, T_N^{\pm 1}$, with commutation relations

$$T_i \circ f(q, X_1, \ldots, X_i, \ldots, X_N) = f(q, X_1, \ldots, qX_i, \ldots, X_N) \circ T_i.$$

This algebra is called the algebra of q-difference operators in N variables with rational coefficients. Elements of this algebra are called difference operators.

Let V be a finite-dimensional vector space over $\mathbb{C}$. Introduce the algebra $\mathfrak{D}_q^N(V)$ of difference operators with matrix coefficients

$$\mathfrak{D}_q^N(V) = \mathfrak{D}_q^N \otimes \mathrm{End}(V).$$

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ of rank r, and let $\mathcal{U}_q\mathfrak{g}$ be the corresponding quantum group. In [EK], to any finite dimensional representation

U of $\mathcal{U}_q\mathfrak{g}$ was assigned a family of commuting difference operators D_c parametrized by Weyl group invariant trigonometric polynomials c on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. These operators are constructed as follows.

Let M_λ be the Verma module over $\mathcal{U}_q\mathfrak{g}$ with highest weight λ and highest weight vector v_λ. Let $U[0]$ be the zero weight subspace of U. For any $u \in U[0]$, define the intertwining operator $\Phi_\lambda^u\colon M_\lambda \to M_\lambda \otimes U$ by the condition $\Phi_\lambda^u v_\lambda = v_\lambda \otimes u + \sum w_i \otimes u_i$, where w_i are homogeneous vectors of weights $\mu_i < \lambda$. This operator is defined for generic λ. For any weight ν, let $\mathrm{Proj}|_{M_\lambda[\nu]}\colon M_\lambda \to M_\lambda$ be the homogeneous projector to the subspace $M_\lambda[\nu]$ of weight ν. Let $\tilde{\psi}_\lambda(X_1,\ldots,X_r)$ be the function with values in $\mathrm{End}(U[0])$ such that for any $u \in U[0]$

$$\tilde{\psi}_\lambda(X_1,\ldots,X_r)u = \sum_\mu X_1^{\mu_1},\ldots,X_r^{\mu_r}\, Tr|_{M_\lambda}(\mathrm{Proj}|_{M_\lambda[\mu]} \circ \Phi_\lambda^u \circ \mathrm{Proj}|_{M_\lambda[\mu]}).$$

Let $\mathbf{P}$ be the weight lattice of φ.

PROPOSITION 1.1. [EK] *For any Weyl group invariant function $c(\lambda)$ on $\mathfrak{h}^*$ of the form*

$$c(\lambda) = \sum_{\mu\in\mathbf{P}} c_\mu q^{2\langle\lambda,\mu\rangle}, \quad c_\mu \in \mathbb{C}(q), \tag{1.1}$$

there exists a unique difference operator $D_c \in \mathfrak{D}_q^r(U[0])$ such that

$$D_c\tilde{\psi}_\lambda = c(\lambda+\rho)\tilde{\psi}_\lambda.$$

For any root α of $\mathfrak{g}$, let $k_\alpha = \max\{n|U[n\alpha] \neq 0\}$, where $U[\mu]$ is the subspace of weight μ in U. Let $R(U)$ be the ring of functions on $\mathfrak{h}^*$ of the form (1.1) such that for any positive root α of $\mathfrak{g}$

$$c\left(\lambda - \frac{n\alpha}{2}\right) = c\left(\lambda + \frac{n\alpha}{2}\right), \quad n = 1,\ldots,k_\alpha,$$

whenever $\langle\lambda,\alpha\rangle = 0$. The main result of this paper is the following theorem, proved in Chapter 4 of this paper.

THEOREM 1.2. *There exists an injective homomorphism $\xi\colon R(U) \to \mathfrak{D}_q^r(U[0])$ such that for any Weyl group invariant element $c \in R$ one has $\xi(c) = D_c$. For any $c \in R(U)$, the operator $\xi(c)$ is defined by the equation*

$$\xi(c)\tilde{\psi}_\lambda = c(\lambda+\rho)\tilde{\psi}_\lambda.$$

We will denote $\xi(c)$ by D_c for any $\xi \in R$.

In the case when $\mathfrak{g} = \mathfrak{sl}_N$ (type A_{N-1}), we can choose representation U to be $S^{kN}V$, where V is the fundamental representation, in which case the space $U[0]$

is 1-dimensional. Then the operators D_c for symmetric functions c are conjugate to Macdonald operators, corresponding to $t = q^{k+1}$. Namely, if c_l are elementary symmetric functions, then $\{D_{c_l}\}$ are simultaneously conjugate to

$$\mathcal{M}_l = \sum_{I\subset 1,\ldots,N,|I|=l} \prod_{i\in I, j\notin I} \frac{q^{k+1}X_i - q^{-k-1}X_j}{X_i - X_j} \prod_{i\in I} T_i^2,$$

(in suitable coordinates). In this case, the numbers k_α are all equal to k; so we will denote the algebra $R(U)$ by R_k. From Theorem 1.2 we get (See Chapter 5):

THEOREM 1.3. *For any positive integer k, there exists an injective homomorphism $\xi\colon R_k \to \mathfrak{D}_q^N$ such that $\xi(c_l) = \mathcal{M}_l$, $l = 1, \ldots, N$. The function $\tilde{\psi}_\lambda$ is a common eigenfunction of the operators $\xi(c), c \in R_k$ with eigenvalue $c(\lambda + \rho)$.*

Note that Theorem 1.3 is a special property of Macdonald's operators at integer values of k. If k is not an integer, one can show that the centralizer of $\mathcal{M}_1, \ldots, \mathcal{M}_N$ in $\mathfrak{D}_q^N$ reduces to the polynomial algebra of $\mathcal{M}_1, \ldots, \mathcal{M}_N$. We call this special property at integer values of k 'algebraic integrability of Macdonald operators', by analogy with the case differential operators which was treated in [CV1, CV2, VSC, ES]. In this sense, the results of this paper are precisely a q-deformation of the results of [ES].

As a by-product, we obtain several results in Macdonald's theory. Namely, we prove the partial Weyl group symmetry of the $\tilde{\psi}$-function, a generalized Weyl character formula for Macdonald's polynomials (which coincides with Conjecture 8.2 in [FV]), an explicit formula for the $\tilde{\psi}$-function in terms of shift operators, and symmetry of the $\tilde{\psi}$-function with respect to the interchange $\lambda \leftrightarrow x$.

The paper is organized as follows. In Section 2 we recall basic facts about representations of quantum groups and intertwining operators. In Section 3 we introduce the Ψ-function as matrix trace of an intertwining operator, and prove its properties. In Section 4 we explain how to construct a commutative ring of difference operators from the Ψ-function. In Section 5 we review some facts from Macdonald theory for root system A_n and explain how to obtain them from our construction. In Appendix we show how our technique works in the simplest example.

2. Quantum groups and their representations

Notation. Let $\mathfrak{g}$ be a simple (finite-dimensional) complex Lie algebra of rank r with fixed diagonalizable Cartan matrix $A = (a_{ij}), i, j = 1, \ldots, r$, and let $d_1, \ldots, d_r$ be positive relatively prime integers such that the matrix $B = (b_{ij}) = (d_i a_{ij})$ is symmetric. We denote its Cartan subalgebra by $\mathfrak{h}$. Let $\alpha_i \in \mathfrak{h}^*, i = 1, \ldots, r$ denote simple roots, R be the corresponding root system, R^+ and R^- be the sets of positive and negative roots, respectively.

The invariant form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*$ is defined by $\langle\alpha_i,\alpha_j\rangle = d_i a_{ij}$. Let $\Lambda_1,\dots,\Lambda_r \in \mathfrak{h}^*$ be fundamental weights, i.e. $\langle\Lambda_i,\alpha_j\rangle = \delta_{ij}, i,j = 1,\dots,r$. Put $\rho = \sum_{i=1}^r \Lambda_i$. Denote

$$\mathbf{Q} = \sum \mathbb{Z}\alpha_i, \mathbf{Q}_+ = \sum \mathbb{Z}_+\alpha_i, \qquad \mathbf{P} = \sum \mathbb{Z}\Lambda_i, \qquad \mathbf{P}_+ = \sum \mathbb{Z}_+\Lambda_i.$$

For $\mu,\nu \in \mathbf{P}$ we write $\mu \geq \nu$ if $\mu - \nu \in \mathbf{Q}_+$.

Let W be the Weyl group of $\mathfrak{g}$. The Weyl group generators s_i act on $\mathfrak{h}^*$ by simple root reflections

$$s_i \cdot \mu = \mu - 2\frac{\langle\alpha_i,\mu\rangle}{\langle\alpha_i,\alpha_i\rangle}\alpha_i.$$

We also introduce a shifted action of Weyl group by

$$w^\rho \cdot \mu = w(\mu+\rho) - \rho.$$

For $w \in W$ let $l(w)$ denote the length of w, i.e. the number of generators in a reduced decomposition $w = s_{i_1}\cdot,\dots,\cdot s_{i_l}$.

Quantum groups. The quantum group $\mathcal{U}_q\mathfrak{g}$, associated to a simple Lie algebra $\mathfrak{g}$, is a Hopf algebra over $\mathbb{C}(q)$ with generators $E_i, F_i, K_i, i = 1,\dots,r$ and relations:

$$K_iK_j = K_jK_i, \qquad K_iE_j = q_i^{a_{ij}}E_jK_i, \qquad K_iF_j = q_i^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k}E_jE_i^k = 0, \quad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k}F_jF_i^k = 0, \quad i \neq j,$$

where $q_i = q^{d_i}$ and we used notation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad [n]_q! = [1]_q\cdot[2]_q\cdot,\dots,\cdot[n]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Comultiplication Δ, antipode S and counit ε in $\mathcal{U}_q\mathfrak{g}$ are given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(K_i) = K_i \otimes K_i.$$

$$S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \epsilon(K_i) = 1.$$

We define a $\mathbb{C}$-algebra involution ω of $\mathcal{U}_q\mathfrak{g}$ by

$$\omega(E_i) = -F_i, \qquad \omega(F_i) = -E_i, \qquad \omega(K_i) = K_i, \qquad \omega(q) = q^{-1}.$$

We have a decomposition of vector spaces $\mathcal{U}_q\mathfrak{g} = \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$, where $\mathcal{U}^-$ (resp. $\mathcal{U}^+$) is the subalgebra generated by F_i (resp. E_i), and $\mathcal{U}^0$ is generated by $K_i, K_i^{-1}, i = 1, \dots, r$.

Verma modules. For any $\lambda \in \mathfrak{h}^*$ we can introduce Verma module M_λ over $\mathcal{U}_q\mathfrak{g}$, i.e. $\mathcal{U}^-$-free module with a single generator v_λ and relations

$$E_iv_\lambda = 0, \qquad K_iv_\lambda = q^{\langle\alpha_i,\lambda\rangle}v_\lambda.$$

Remark. Here and below we work over the field $F = \mathbb{C}(\{q^a, a \in \mathbb{C}\})$. In this setting, $q^{\langle\mu,\lambda\rangle}$ is a function $\mathfrak{h}^* \to F$.

We have the decomposition

$$M_\lambda = \bigoplus_{\mu\in\mathbf{Q}_+} M_\lambda[\lambda - \mu],$$

of M_λ into direct sum of weight subspaces $M_\lambda[\lambda - \mu]$, where we say that a vector v has weight $\mu \in \mathfrak{h}^*$ if

$$K_iv = q^{\langle\alpha_i,\mu\rangle}v.$$

The restricted dual module M_λ^* is a $\mathcal{U}qg^+$-module with a lowest weight vector $v_{-\lambda}^*$ such that $\langle v_{-\lambda}^*, v_\lambda\rangle = 1$. By definition we have

$$\langle gv^*, v\rangle = \langle v^*, S(g)v\rangle, \quad v \in M_\lambda, v^* \in M_\lambda^*.$$

Introduce a symmetric form F on M_λ defined by

$$F(g_1v_\lambda, g_2v_\lambda) = \langle\omega(g_1)v_{-\lambda}^*, g_2v_\lambda\rangle, g_1, g_2 \in \mathcal{U}_-.$$

The weight subspaces are pairwise orthogonal with respect to this form. The restriction of F to weight subspaces $M_{\lambda-\mu}$ is proportional to the quantum Shapovalov form $\mathcal{F}$, introduced in [CK]:

$$F_\mu(\cdot,\cdot) = C_\mu q^{-\langle\lambda,\mu\rangle}\mathcal{F}_\mu(\cdot,\cdot),$$

for some constants C_μ.

Fix a basis $g_i^\mu \in \mathcal{U}^-[\mu]$. Let $F_\mu = (F_\mu)_{ij}, i,j = 1,2,\ldots,\dim M_\lambda[\lambda-\mu]$, denote the matrix of the restriction of form F to $M_\lambda[\lambda-\mu]$ with respect to the basis $g_i^\mu v_\lambda$ of $M_\lambda[\lambda-\mu]$. A variation of the quantum determinant formula [CK] asserts that

$$\det F_\mu = C \prod_{\alpha\in R^+} \prod_{n\in\mathbb{N}} \left(1 - q^{-2\langle\alpha,\lambda+\rho\rangle+n\langle\alpha,\alpha\rangle}\right)^{\mathrm{Par}(\mu-n\alpha)},$$

where Par is the generalized Kostant partition function, and C is a constant, depending on the choice of basis g_i^μ.

This determinant is a linear combination of terms $q^{-2\langle\mu,\lambda\rangle}$, where μ's belong to a finite subset $L \subset \mathbf{Q}$, with some coefficients from $\mathbb{C}(q)$. Motivated by this fact, we introduce:

DEFINITION. *Expressions of the form $\sum_{\mu\in L} a_\mu q^{-2\langle\mu,\lambda\rangle}$, $a_\mu \in \mathbb{C}(q)$, will be called q-polynomials with support L and coefficients a_μ.*

Verma modules are reducible when the form F is degenerate, i.e. $\det F_\mu = 0$ for some μ. This happens when λ satisfies one of the Kac-Kazhdan equations:

$$\langle\alpha,\lambda+\rho\rangle = \frac{n}{2}\langle\alpha,\alpha\rangle, \quad n = 1,2,\ldots. \tag{2.1}$$

For λ generic from Kac-Kazhdan hyperplanes, M_λ contains a unique submodule M_λ^1, isomorphic to $M_{\lambda-n\alpha}$.

Intertwining operators. Let U be an irreducible finite-dimensional $\mathcal{U}_q\mathfrak{g}$-module with non-trivial zero weight subspace $U[0]$. For $u \in U$ let $\Phi_\lambda^u: M_\lambda \to M_\lambda \otimes U$ be an intertwining operator such that $v_\lambda \to v_\lambda \otimes u+$ higher order terms, where 'higher order terms' mean terms of the form $v_{\lambda-\mu}\otimes u_\mu, \mu > 0$.

If M_λ is irreducible, then Φ_λ^u exists and is unique for any $u \in U[0]$. Indeed, we have a unique $\mathcal{U}^+$-intertwiner $\Omega: M_\lambda^* \to U$, such that $\Omega v_{-\lambda}^* = u$. Since $\mathrm{Hom}(M_\lambda^*, U) \cong M_\lambda^{**}\otimes U \cong M_\lambda \otimes U$, it corresponds to a singular (i.e. $\mathcal{U}^+$-invariant) vector $\phi \in M_\lambda \otimes U$. We now construct Φ_λ^u by putting $\Phi_\lambda^u v_\lambda = \phi$ and extending Φ_λ^u to the whole M_λ by the intertwining property.

For our purposes we need an explicit form for that singular vector.

PROPOSITION 2.1. *For any (homogeneous) basis $\{g_i^\mu\}$ of $\mathcal{U}^-$*

$$\phi = \sum_\mu \left(\sum_{i,j} \left(F_\mu^{-1}\right)_{ij} g_i^\mu v_\lambda \otimes \omega(g_j^\mu) u \right), \tag{2.2}$$

is a singular vector in $M_\lambda \otimes U$.

Note that since U has a highest weight, the summation is over the finite set of μ's such that $U[\mu] \neq 0$.

Proof. We check that the corresponding element $\Phi \in \mathrm{Hom}(M_\lambda^*, U)$ defined as the composition $M_\lambda^* \to M_\lambda^* \otimes M_\lambda \otimes U \to U$ is a $\mathcal{U}^+$-intertwiner. We have:

$$\begin{aligned}\Phi\left(\omega(g_n^\nu v_{-\lambda}^*)\right) &= \sum_\mu \left(\sum_{ij} \left(F_\mu^{-1}\right)_{ij} \langle \omega(g_n^\nu) v_{-\lambda}^*, g_i^\mu v_\lambda\rangle \omega(g_j^\mu) u\right) \\ &= \sum_{i,j} \left(F_\nu^{-1}\right)_{ij} \langle \omega(g_n^\nu) v_{-\lambda}^*, g_i^\nu v_\lambda\rangle \omega(g_j^\nu) u \\ &= \sum_j \left(\sum_i \left(F_\nu^{-1}\right)_{ij} (F_\nu)_{ni}\right) \omega(g_j^\nu) u \\ &= \sum_j \delta_{jn} \omega(g_j^\nu) u = \omega(g_n^\nu) u. \qquad \square\end{aligned}$$

Recall that in the classical case (i.e. $q = 1$) matrix elements of the inverse matrix F_μ^{-1} were rational functions of λ with at most simple poles on the Kac-Kazhdan hyperplanes given by (2.1). (see [ES]). A similar argument, also involving Jantzen filtration, proves that the same is true in the quantum case. Therefore, expression (2.2) for the singular vector ϕ is a ratio of two q-polynomials, with at most simple singularities on a finite collection of Kac-Kazhdan hyperplanes.

If we multiply the q-rational expression (2.2) by the least common denominator $\tilde{\chi}(\lambda)$, we will get a well-defined for all λ's formula for a singular vector $\tilde{\phi} \in M_\lambda \otimes U$. We are now going to show that in fact the least common denominator may only contain factors

$$\chi_n^\alpha(\lambda) = 1 - q^{-2\langle\alpha, \lambda+\rho\rangle + n\langle\alpha,\alpha\rangle},$$

corresponding to n, α such that $U[n\alpha] \neq 0$. Indeed, suppose that $\tilde{\chi}(\lambda)$ contained a factor $\chi_n^\alpha(\lambda)$, but $U[n\alpha] = 0$.

Consider λ generic from the hyperplane $\langle\alpha, \lambda + \rho\rangle = \frac{n}{2}\langle\alpha,\alpha\rangle$. Then M_λ contains a unique maximal submodule $M_\lambda^1 \cong M_{\lambda - n\alpha}$, generated by the singular vector $v_{\lambda-n\alpha}$. Since the first term $\tilde{\chi}(\lambda) v_\lambda \otimes u$ in the expression for $\tilde{\phi}$ turns into zero on our hyperplane, the singular vector must have the form

$$\tilde{\phi} = v_{\lambda - n\alpha} \otimes \tilde{u} + \text{higher order terms}.$$

The intertwining property implies that $\tilde{u} \in U[n\alpha]$, and by assumption $\tilde{u} = 0$. Therefore, $\tilde{\phi}$ is zero for λ generic from the hyperplane, and by Bezout theorem is divisible by χ_n^α. This shows that $\tilde{\chi}(\lambda)$ was not the least common denominator – contradiction.

Denote $k_\alpha = \max\{n|U[n\alpha] \neq 0\}$,

$$L_\theta = \left\{ \nu | \nu = \sum_{\alpha \in R^+} m_\alpha \alpha, \quad 0 \leq m_\alpha \leq k_\alpha \text{ for all } \alpha \in R^+ \right\}.$$

We conclude this section with the following:

PROPOSITION 2.2. *If U is an irreducible finite-dimensional $\mathcal{U}_q\mathfrak{g}$-module with highest weight θ, the singular vector $\phi \in M_\lambda \otimes U$, given by (2.2), can be represented as*

$$\phi = \frac{\sum_l S_l(\lambda)\tilde{g}_l v_\lambda \otimes u_l}{\prod_{\alpha \in R^+} \prod_{m=1}^{k_\alpha} \left(1 - q^{-2\langle \alpha, \lambda+\rho \rangle + m\langle \alpha, \alpha \rangle}\right)},$$

where $g_l \in \mathcal{U}^-, u_l \in U$, and q-polynomials $S_l(\lambda)$ have supports, contained in L_θ.

Proof. We already proved that the least common denominator for the expression for ϕ may only contain factors $\chi_n^\alpha(\lambda), n = 1, \ldots, k_\alpha$.

The statement about the support of the polynomials $S_l(\lambda)$ follows from the fact that the support of the numerator must lie within the convex hull of the support of the denominator, which in this case is exactly L_θ. □

3. Matrix Trace, the Ψ-function and its properties

We now fix an irreducible $\mathcal{U}_q\mathfrak{g}$-module U with highest weight θ and nontrivial zero weight subspace. We use the notation

$$k_\alpha = \max\{n|U[n\alpha] \neq 0\}, \qquad \Theta = \sum_{\alpha \in R^+} k_\alpha \cdot \alpha \in \mathbf{Q}_+,$$

$$\chi_n^\alpha(\lambda) = 1 - q^{-2\langle \alpha, \lambda+\rho \rangle + n\langle \alpha, \alpha \rangle}, \qquad \chi(\lambda) = \prod_{\alpha \in R^+} \prod_{n=1}^{k_\alpha} \chi_n^\alpha(\lambda),$$

$$L_\theta = \left\{ \mu \in \mathbf{Q}_+ | \mu = \sum_{\alpha \in R^+} m_\alpha \alpha, \quad 0 \leq m_\alpha \leq k_\alpha \text{ for all } \alpha \in R^+ \right\}.$$

As in [ES], define a new intertwining operator

$$\tilde{\Phi}_\lambda^u = \chi(\lambda)\Phi_\lambda^u : M_\lambda \to M_\lambda \otimes U.$$

From Proposition 2.2 it follows that $\tilde{\Phi}_\lambda^u$ is well-defined even when λ belongs to Kac-Kazhdan hyperplanes, where Φ_λ^u did not always exist.

Introduce an End$(U[0])$-valued function $\Psi(\lambda, x), \lambda, x \in \mathfrak{h}^*$, by

$$\Psi(\lambda, x)u = \mathrm{Tr}\,|_{M_\lambda}(\tilde{\Phi}_\lambda^u e^x).$$

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PROPOSITION 3.1. *The function* $\Psi(\lambda, x)$ *defined above has the form*

$$\Psi(\lambda, x) = \mathrm{e}^{\langle\lambda,x\rangle} \sum_{\mu\in L_\theta} q^{-2\langle\mu,\lambda+\rho\rangle} P_\mu(x),$$

where $P_\mu(x) \in \mathrm{End}(U[0])$ *and* $P_\Theta(x)$ *is invertible for generic* x.

If $\langle\alpha, \lambda+\rho\rangle = \frac{n}{2}\langle\alpha,\alpha\rangle$ *for some* $\alpha \in R^+, n = 1, 2, \ldots, k_\alpha$ *then*

$$\Psi(\lambda, x) = \Psi(\lambda - n\alpha, x) B_{n\alpha}(\lambda). \tag{3.1}$$

for some (possibly infinite) sum

$$B_{n\alpha}(\lambda) = \sum_{\mu\in\mathbf{Q}_+} q^{-2\langle\mu,\lambda\rangle} B^\mu_{n\alpha}, \qquad B^\mu_{n\alpha} \in \mathrm{End}(U[0]).$$

(In fact, the matrix elements of $B_{n\alpha}(\lambda)$*'s are ratios of* q*-polynomials.)*

Remark. If we take U to be a trivial module, then the Ψ-function becomes the usual character of the Verma module. Therefore, we can regard the Ψ-function as a generalized (matrix-valued) character of the Verma module M_λ.

Proof of Proposition 3.1.

$$\mathrm{Tr}\,|_{M_\lambda} \left(\tilde{\Phi}^u_\lambda e^x\right) = \mathrm{e}^{\langle\lambda,x\rangle} \sum_{\mu\in\mathbf{Q}_+} \mathrm{e}^{-\langle\mu,x\rangle} \mathcal{B}_\mu(\lambda) u,$$

where ‘partial traces’ $\mathcal{B}_\mu(\lambda) \in \mathrm{End}(U[0])$, corresponding to weight subspaces $M_\lambda[\lambda-\mu]$, are defined by

$$\mathcal{B}_\mu(\lambda) u = \mathrm{Tr}\left(\mathrm{Proj}_{M_\lambda[\lambda-\mu]} \circ \tilde{\Phi}^u_\lambda \circ \mathrm{Proj}_{M_\lambda[\lambda-\mu]}\right).$$

By Proposition 2.2, $\mathcal{B}_\mu(\lambda)$ are $\mathrm{End}(U[0])$-valued q-polynomials with support L_θ. If we let $x \to \infty$ in such a way that $\langle\alpha, x\rangle \to +\infty, \alpha \in R_+$ (this just means that we are keeping only the highest weight terms of the series), we will get asymptotically

$$\Psi(\lambda, x) \sim \mathrm{e}^{\langle\lambda,x\rangle} \chi(\lambda) \cdot \mathbf{1}.$$

Therefore $P_\Theta(x) \sim \mathbf{1}$, and $P_\Theta(x)$ is invertible for generic x.

We now prove the second property.

If λ is generic from hyperplane $\langle\alpha, \lambda+\rho\rangle = \frac{n}{2}\langle\alpha,\alpha\rangle$, then M_λ is reducible and contains a unique submodule M^1_λ generated by singular vector $v_{\lambda-n\alpha}$. It is clear that for such λ there are no order μ terms in $\Phi^u_\lambda v_\lambda$ unless $\mu \geqslant n\alpha$. In other words, $\tilde{\Phi}^u_\lambda$ maps M_λ into $M^1_\lambda \otimes U$, and

$$\tilde{\Phi}^u_\lambda v_\lambda = v_{\lambda-n\alpha} \otimes u' + \text{higher order terms},$$

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$$\tilde{\Phi}_\lambda^u v_{\lambda-n\alpha} = v_{\lambda-n\alpha} \otimes u'' + \text{higher order terms.}$$

Clearly, both u' and u'' depend linearly on u. More precisely,

$$u'' = \mathcal{B}_{n\alpha}(\lambda)u. \tag{3.3}$$

Therefore we have

$$\chi(\lambda - n\alpha)\tilde{\Phi}_\lambda^u|_{M_\lambda^1} \cong \tilde{\Phi}_{\lambda-n\alpha}^{u''}.$$

Taking the traces of the operators from the last equation and using (3.3), we get

$$\chi(\lambda - n\alpha)\Psi(\lambda, x) = \Psi(\lambda - n\alpha, x)\mathcal{B}_{n\alpha}(\lambda).$$

Since $\chi(\lambda - n\alpha)$ is invertible in the 'Laurent series' completion of $\mathbb{C}[\mathbf{P}]$, we can introduce

$$B_{n\alpha}(\lambda) = \frac{\mathcal{B}_{n\alpha}(\lambda)}{\chi(\lambda - n\alpha)} = \sum_{\mu\in\mathbf{Q}_+} q^{2\langle\mu,\lambda\rangle} B_{n\alpha}^\mu$$

and (3.1) follows. Proposition 3.1 is proved. $\square$

We prove that property (3.1) of the Ψ-function determines it uniquely up to multiplication by a factor, depending only on x.

PROPOSITION 3.2. *Suppose we have an* $\mathrm{End}(U[0])$*-valued function*

$$\Psi'(\lambda, x) = \mathrm{e}^{\langle\lambda,x\rangle} \sum_{\mu\in L} q^{-2\langle\mu,\lambda\rangle} Q_\mu(x),$$

where $L \subset \mathbf{Q}_+$ *and* $Q_\mu(x) \in \mathrm{End}(U[0])$, *satisfying condition* (3.1). *Then the* L *contains at least one weight* $\mu \geqslant \Theta$.

Proof. (cf. [ES]). Let us rewrite the condition (3.1). We have:

$$\begin{aligned}
&\mathrm{e}^{(n/2)\langle\alpha,x\rangle} \sum_{\mu\in\mathbf{Q}} q^{-2\langle\mu,\lambda+(n\alpha/2)\rangle} P_\mu(x) \\
&\quad = \mathrm{e}^{-(n/2)\langle\alpha,x\rangle} \sum_{\mu\in\mathbf{Q}} q^{-2\langle\mu,\lambda-(n\alpha/2)\rangle} P_\mu(x) \sum_{\nu\in\mathbf{Q}} q^{-2\langle\nu,\lambda\rangle} B_{n\alpha}^\nu.
\end{aligned}$$

For every $\mu \in \mathbf{Q}$ consider the set

$$Z_\alpha(\mu) = \mu + \mathbb{Z}\alpha = \{\nu \in \mathbf{Q} | \nu = \mu + m\alpha \quad \text{for some } m \in \mathbb{Z}\}.$$

Comparing coefficients for $q^{\langle\mu,\lambda\rangle}$, we get for $n = 1, 2, \ldots, k_\alpha$:

$$\sum_{\nu\in Z_\alpha(\mu)} \left(\mathrm{e}^{n\langle\alpha,x\rangle} q^{-n\langle\nu,\alpha\rangle} P_\nu(x) - \sum_{\beta\in\mathbf{Q}} q^{-n\langle\beta,\alpha\rangle} P_{\nu-\beta}(x) B^\beta_{n\alpha} \right) = 0.$$

This is a system of linear equations on unknown functions $P_\nu(x)$. Note that the summation over β is finite, since $P_{\nu-\beta} = 0$ if $\nu - \beta \notin \mathbf{Q}_+$. The matrix of this system has a block-upper-triangular form, blocks corresponding to subsets $Z_\alpha(\mu)$ for different μ. The determinant of this matrix is an entire function of x, and asymptotically when $\langle\alpha, x\rangle \to +\infty, \alpha \in R_+$

$$\text{LHS} \sim \sum_{\nu\in Z_\alpha(\mu)} \mathrm{e}^{n\langle\alpha,x\rangle} q^{-n\langle\nu,\alpha\rangle} P_\nu(x).$$

Therefore, asymptotically the determinant of the matrix of this system is equal to a Vandermonde-type determinant, which is nonzero. It follows that the determinant of the system of equations is nonzero for generic x.

Suppose μ is such that not all $P_\mu(x)$ are identically zero (i.e. we have a nontrivial solution of the system of equations). Then for such μ we need to have $\mathrm{Card}(Z_\alpha(\mu)) > k_\alpha$ for all α.

Below we will prove the following:

LEMMA 3.3. *Let $S = \{v_1, \ldots, v_m\}$ be a system of pairwise noncollinear vectors in $\mathbb{R}^n$; assume that all v_i lie in the halfspace $\langle\mathbf{n}, v_i\rangle > 0$ for some vector $\mathbf{n} \in \mathbb{R}^n$.*

Let B be a closed bounded convex polytope in $\mathbb{R}^n$, such that the origin 0 is a vertex of B, and moreover $B \setminus 0$ lies in the halfspace $\langle\mathbf{n}, x\rangle < 0$.

Suppose that for any $x \in B, v_i \in S$ we can draw a line segment I through x, parallel to v_i and of length at least $|v_i|$, such that $I \subset B$.

Then $B \supset B_0$, where the polytope B_0 is defined by

$$B_0 = \left\{ -\sum_{i=1}^{k} s_i v_i | 0 \le s_i \le 1 \right\}.$$

Let $\mathfrak{L}$ be the convex hull of the set L, $S = \Delta_+, v_\alpha = k_\alpha\alpha$. Take $\mathbf{n}$ from the positive Weyl alcove, so that it is not orthogonal to any edge of $\mathfrak{L}$, and let μ_0 be the vertex of $\mathfrak{L}$ such that the product $\langle\mathbf{n}, \mu_0\rangle$ is maximal. Then $B = \mathfrak{L} - \mu_0$ satisfies all the conditions of Lemma 3.3, and we conclude that $\mathfrak{L}$ contains all vectors of the form

$$\mu = \mu_0 - \sum_{\alpha\in\Delta_+} s_\alpha \alpha, \quad 0 \le s_\alpha \le k_\alpha.$$

Since $\mathfrak{L}$ contains only positive weights, we see in particular that

$$\mu_0 - \Theta = \mu_0 - \sum_{\alpha\in\Delta_+} k_\alpha \alpha \ge 0.$$

Being a vertex of $\mathfrak{L}$, the weight μ_0 also belongs to the (discrete) set L, which completes the proof of Proposition 3.2. □

Proof of Lemma 3.3. We use induction on n.

Base of induction: n=2. Note that the polygon B has exactly two (opposite) edges, parallel to v_1, both of length at least $|v_1|$. It can be easily seen that the polygon $\bar{B}$, defined by

$$\bar{B} = \{x \in B | x + v_1 \in B\},$$

satisfies the condition of the Lemma for the family of vectors $\bar{S} = \{v_2, \ldots, v_m\}$. We can therefore use induction on m to prove the statement, which is obvious for $m = 1$. The technical details are left to the reader.

Induction step. Let $n \geq 3$. Consider orthogonal projections of our data in the directions $\mathbf{u}$ from the hyperplane $\langle \mathbf{u}, \mathbf{n} \rangle = 0$. Projections B', B'_0 of B, B_0 are convex closed $(n-1)$-dimensional polytopes, lying in the halfspace $\langle \mathbf{n}, x \rangle < 0$; the projections v'_i of v_i lie in the halfhyperplane $\langle \mathbf{n}, v'_i \rangle > 0$. For generic $\mathbf{u}$, vectors v'_i will be pairwise noncollinear, and by induction hypothesis we will conclude that $B'_0 \subset B'$. By continuity, this is true for all $\mathbf{u}$.

Suppose now that there exists a point x such that $x \in B_0, x \notin B$. Consider a (generic) hyperplane separating x from B. Its intersection with the hyperplane $\langle \mathbf{u}, \mathbf{n} \rangle = 0$ has codimension 2, and therefore is nonzero. Take u from this intersection; then for projection x' of x we will have $x' \in B'_0, x' \notin B'$ – contradiction. Therefore, we have proven the induction step.

Lemma 3.3 is proven. □

COROLLARY 3.4. *The Ψ-function, satisfying* (3.1)*, is unique up to a factor, depending on x.*

Proof. If we have another function $\Psi'(\lambda, x)$ with highest coefficient $P'_\Theta(x)$, satisfying (3.1), then the function

$$\phi(\lambda, x) = \Psi'(\lambda, x) - P'_\Theta(x) \left(P_\Theta(x)\right)^{-1} \Psi(\lambda, x),$$

will still satisfy (3.1), but its support will only contain weights $\mu < \Theta$. By Proposition 3.2, $\phi(\lambda, x) \equiv 0$, and the statement follows. □

Remark. We use this opportunity to correct some errors in our paper [ES].

1. Corollary 5.3 in [ES], which is used to prove the uniqueness of the classical ψ-function, is incorrect (a counterexample is the function $e^{\langle \lambda, x \rangle} q(\lambda)$, where q is any polynomial vanishing on the hyperplanes involved in (4–11)). The mistake is that the polynomial $q_1(\lambda)$ introduced in the proof does not have to satisfy any invariance condition, so $\phi_1(\lambda, x)$ does not have to satisfy (4–11). The statement

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and the proof of Corollary 5.3 are valid only if $\phi(\lambda, x) = e^{\langle\lambda,x\rangle}Q(\lambda, x)$ with $\deg Q = \sum_\alpha n_\alpha$, which still implies the uniqueness property (Corollary 5.4).

2. Corollary 5.3 is implicitly used in the proof of Theorem 6.1. The theorem is correct, but the proof has to be changed. Namely, one should prove the following

PROPOSITION. *Any* $\mathrm{End}(U[0])$*-valued function* $\phi(\lambda, x) = e^{\langle\lambda,x\rangle}Q(\lambda, x)$*, satisfying* (4–11)*, can be represented as* $\phi(\lambda, x) = \mathcal{D}\psi(\lambda, x)$ *for a unique differential operator* $\mathcal{D}$ *with coefficients, depending on* x *but not on* λ*.*

This can be easily proved by induction (cf. [CV], and also Proposition 4.1 below). Theorem 6.1 is the special case of this Proposition.

3. We would also like to point out a misprint in the definition of η_μ^α in Section 3 of [ES]; it should read $\eta_\mu^\alpha = \max_n, \{n \in \mathbb{N} | K(\mu - n\alpha) \neq 0\} = \max_n\{n \in \mathbb{N} | n\alpha \leqslant \mu\}$.

4. Existence of difference operators

Introduce a family of difference operators T_Λ, corresponding to weights $\Lambda \in \mathbf{P}$, acting on $\mathfrak{h}^*$, by

$$T_\Lambda(x) = x + \Lambda \log q^2, \quad x \in \mathfrak{h}^*.$$

They naturally act on functions on $\mathfrak{h}^*$; for example, for $f(x) = \mathrm{e}^{\langle\alpha,x\rangle}$ we have

$$(T_\Lambda f)(x) = \mathrm{e}^{\langle\alpha,x+\Lambda\log q^2\rangle} = q^{2\langle\alpha,\Lambda\rangle}\,\mathrm{e}^{\langle\alpha,x\rangle} = q^{2\langle\alpha,\Lambda\rangle} f(x).$$

PROPOSITION 4.1. *Any function* $\phi(\lambda, x)$*, satisfying* (3.1)*, which has the form*

$$\phi = e^{\langle\lambda,x\rangle}P(\lambda, x),$$

for some q*-polynomial* $P(\lambda, x)$*, can be represented as*

$$\phi(\lambda, x) = D\Psi(\lambda, x),$$

for a unique difference operator D *with depending on* x *coefficients.*

Proof. Uniqueness is obvious, since otherwise the Ψ-function would be annihilated by a nontrivial difference operator for all λ, which is impossible. (See, for example, [EK]).

To prove existence of D we use induction on the support of $P(\lambda, x)$. Consider the family of all finite subsets $L \subset \mathbf{Q}_+$ such that if $\mu \in L, \nu \in \mathbf{Q}_+$ and $\nu < \mu$, then also $\nu \in L$.

We prove that if our statement is true for all q-polynomials $P(\lambda, x)$ whose support is strictly contained in L, then it is also true for q-polynomials with support L.

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If L does not contain weights $\mu \geqslant \Theta$ then by Proposition 3.2 we have $P(\lambda, x) \equiv 0$, and the operator $D \equiv 0$.

Suppose there is a $\nu \in L$ such that $\nu \geqslant \Theta$. Consider the set of all such ν's and let μ be a maximal element from this set.

Consider the function

$$\phi'(\lambda, x) = \phi(\lambda, x) - P_\mu(x) T_{\mu-\Theta} \Psi(\lambda, x).$$

It satisfies (3.1) and has support strictly contained in L. By induction hypothesis we can represent ϕ' as

$$\phi'(\lambda, x) = D' \Psi(\lambda, x).$$

The operator

$$D = P_\mu(x) T_{\mu-\Theta} + D',$$

satisfies the required properties. $\square$

We now prove a simple technical

LEMMA 4.2. *Let a q-polynomial $c(\lambda)$ be represented as*

$$c(\lambda) = \sum_{\pi \in \mathbf{P}/\mathbf{Q}} c_\pi(\lambda),$$

where $c_\pi(\lambda)$ are q-polynomials with support in the coset $\pi + \mathbf{Q}$.

Suppose for some n, α we have that

$$c\left(\lambda + \frac{n\alpha}{2}\right) = c\left(\lambda - \frac{n\alpha}{2}\right), \tag{4.1}$$

whenever $\langle \alpha, \lambda \rangle = 0$. Then (4.1) is also satisfied for each $c_\pi(\lambda)$.

Proof. Property (4.1) is equivalent to divisibility by $q^{2\langle \alpha, \lambda \rangle} - 1$ of the q-polynomial

$$\tilde{c}(\lambda) = c\left(\lambda + \frac{n\alpha}{2}\right) - c\left(\lambda - \frac{n\alpha}{2}\right).$$

On the other hand, one can see that

$$\tilde{c}(\lambda) = \sum_{\pi \in \mathbf{P}/\mathbf{Q}} \tilde{c}_\pi(\lambda),$$

where

$$\tilde{c}_\pi(\lambda) = c_\pi\left(\lambda + \frac{n\alpha}{2}\right) - c_\pi\left(\lambda - \frac{n\alpha}{2}\right).$$

Clearly, $\tilde{c}(\lambda)$ is divisible by $q^{2\langle \alpha, \lambda \rangle} - 1$ if and only if all $\tilde{c}_\pi(\lambda)$'s are divisible by $q^{2\langle \alpha, \lambda \rangle} - 1$, and the Lemma follows. $\square$

THEOREM 4.3. *For any q-polynomial $c(\lambda)$ such that for any $\alpha \in R^+$*

$$c\left(\lambda + \frac{n\alpha}{2}\right) = c\left(\lambda - \frac{n\alpha}{2}\right), \quad n = 1, 2, \ldots, k_\alpha, \tag{4.2}$$

whenever $\langle \alpha, \lambda \rangle = 0$, there exists a difference operator D_c with coefficients in $\mathrm{End}(U[0])$ *such that*

$$D_c \Psi(\lambda, x) = \Psi(\lambda, x) c(\lambda + \rho).$$

The correspondence $c(\lambda) \to D_c$ is a homomorphism of rings.

Remark. We put $c(\lambda + \rho)$ on the right since in that form it admits generalization to the matrix case (see Theorem 4.4). Of course, for scalar q-polynomial $c(\lambda)$ we could write it in a more traditional form $D_c \Psi(\lambda, x) = c(\lambda + \rho) \Psi(\lambda, x)$.

Proof. By Lemma 4.2, it suffices to prove the theorem for $c_\pi(\lambda)$ of the form

$$c(\lambda) = q^{2\langle \mu_0, \lambda \rangle} \sum_{\mu \in \mathbf{Q}_+} c_\mu q^{-2\langle \mu, \lambda \rangle},$$

for some $\mu_0 \in \mathbf{P}$. Consider the function

$$\phi(\lambda, x) = T_{-\mu_0} \Psi(\lambda, x) c(\lambda + \rho).$$

It satisfies (3.1), and it has the form

$$\phi(\lambda, x) = \mathrm{e}^{\langle \lambda, x \rangle} \sum_{\mu \in \mathbf{Q}_+} q^{-\langle \mu, \lambda \rangle} Q_\mu(x).$$

By Proposition 4.1 it can be represented as

$$\phi(\lambda, x) = D \Psi(\lambda, x),$$

for some difference operator D. Put $D_c = T_{\mu_0} D$. Then we have

$$\begin{aligned} D_c \Psi(\lambda, x) &= T_{\mu_0} D \Psi(\lambda, x) = T_{\mu_0} \phi(\lambda, x) \\ &= T_{\mu_0} T_{-\mu_0} \Psi(\lambda, x) c(\lambda + \rho) = \Psi(\lambda, x) c(\lambda + \rho), \end{aligned}$$

We now prove the homomorphism property. Suppose we have two polynomials $c(\lambda)$ and $c'(\lambda)$. It is easily checked that operator $D_{cc'} - D_c D_{c'}$ annihilates the Ψ-function for any λ; therefore it has to be identically zero. Hence our correspondence is a homomorphism of rings. □

We have big supply of (scalar) q-polynomials, satisfying (4.2), arising from the algebra of Weyl group invariant q-polynomials, which is freely generated by the

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Casimir elements $c_1, \ldots, c_r$. This give us n algebraically independent difference operators $D_1, \ldots, D_r$. However, there exist other q-polynomials, with this property. For instance, any polynomial divisible by

$$c_0(\lambda) = \prod_{\alpha \in R^+} \prod_{n=-k_\alpha}^{k_\alpha} \left(q^{2\langle \alpha, \lambda \rangle + n\langle \alpha, \alpha \rangle} - 1 \right),$$

also satisfies (4.2). It gives rise to a difference operator, commuting with all those generated by the Casimir elements, but not necessarily lying in the ring generated by them. This procedure gives examples of what we called algebraically integrable commutative rings of difference operators.

Theorem 4.3 can be slightly generalized to the matrix case.

THEOREM 4.4. *For any* $\mathrm{End}(U[0])$*-valued* q*-polynomial* $C(\lambda)$ *such that*

$$C\left(\lambda + \frac{n\alpha}{2}\right) B_{n\alpha}(\lambda) = B_{n\alpha}(\lambda) C\left(\lambda - \frac{n\alpha}{2}\right), \quad \alpha \in R^+, n = 1, \ldots, k_\alpha,$$

whenever $\langle \alpha, \lambda \rangle = 0$, *there exists a unique difference operator* D_C *with coefficients in* $\mathrm{End}(U[0])$, *such that*

$$D_C \Phi(\lambda, x) = \Psi(\lambda, x) C(\lambda + \rho).$$

The correspondence $\xi \colon C(\lambda) \mapsto D_C$ *is a homomorphism of rings.*

Proof. The argument used in proof of Theorem 4.3 in the obvious way extends to the matrix case. □

Remark. Operators $D_1, \ldots, D_r$ act on Ψ-function as scalars, and therefore commute with all operators D_C constructed as above. In fact, one can show that the centralizer of the subring generated by operators $D_1, \ldots, D_r$ in $\mathfrak{D}_q^r(U[0])$ coincides with the image of ξ. We do not include the proof of this statement here.

In the next section we explain how our construction is related to Macdonald theory.

5. Root system A_n and Macdonald theory

Consider a special case of our construction for $\mathfrak{g} = \mathfrak{sl}_N$, $U = U_k = S^{kN}V$, where V is the fundamental representation. It is well-known that the zero weight subspace $U[0]$ is one-dimensional, and we can regard $\Psi(\lambda, x)$ as a scalar-valued function. Note also that in this case $k_\alpha = k$ for all $\alpha \in R^+$, and $\Theta = k \sum_{\alpha \in R^+} \alpha = 2k\rho$. We have $d_i = 1$, and therefore $q_i = q$ for all $i = 1, \ldots, r$.

We will use the notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_+ = \frac{q^{2n} - 1}{q^2 - 1}, \qquad [n]_- = \frac{1 - q^{-2n}}{1 - q^{-2}}.$$

We first prove an important property of partial traces

$$\mathcal{B}_\mu(\lambda) = \mathrm{Tr}\left(\mathrm{Proj}_{M_\lambda[\lambda-\mu]} \circ \tilde{\Phi}^u_\lambda \circ \mathrm{Proj}_{M_\lambda[\lambda-\mu]}\right),$$

introduced in Section 3.

PROPOSITION 5.1. *Given any* $\alpha \in R^+, n = 1, \dots, k$, *we have for all* $\mu \in \mathbf{Q}$

$$\mathcal{B}_\mu(\lambda) = q^{-\langle n\alpha,\Theta\rangle} \cdot \mathcal{B}_{\mu-n\alpha}(\lambda - n\alpha), \tag{5.1}$$

whenever $\langle \alpha, \lambda+\rho\rangle = n$.

COROLLARY 5.2. *For* $\alpha \in R^+, n = 1, \dots, k$, *the function* $\Psi(\lambda, x)$ *satisfies*

$$\Psi\left(\lambda + \frac{n\alpha}{2}, x\right) = q^{-n\langle\alpha,\Theta\rangle}\Psi\left(\lambda - \frac{n\alpha}{2}, x\right).$$

whenever $\langle\alpha,\lambda\rangle = n$.

Proof of Proposition 5.1. For given n, α it is sufficient to prove a special case of (5.1), corresponding to $\mu = n\alpha$:

$$\mathcal{B}_{n\alpha}(\lambda) = q^{-\langle n\alpha,\Theta\rangle} \cdot \mathcal{B}_0(\lambda - n\alpha) = q^{-\langle n\alpha,\Theta\rangle}\chi(\lambda - n\alpha). \tag{5.2}$$

Indeed, let λ be such that $\langle\alpha, \lambda+\rho\rangle = n$. Then the image of $\tilde{\Phi}^u_\lambda$ is contained in $M_{\lambda-n\alpha} \otimes U$. From (5.2) we see that

$$\tilde{\Phi}^u_\lambda|_{M_{\lambda-n\alpha}} = q^{-\langle n\alpha,\Theta\rangle} \cdot \tilde{\Phi}^u_{\lambda-n\alpha}$$

and the more general formula (5.1) follows. We now use induction on the height of root α to prove formula (5.2), and thus Proposition 5.1.

Base of induction. Consider the case when $\alpha = \alpha_i$ is a simple root. Let λ be generic in the hyperplane $\langle\alpha_i, \lambda+\rho\rangle = n$. Then M_λ contains a unique nonzero proper submodule M^1_λ, generated by the singular vector $v_{\lambda-n\alpha_i} = F_i^n v_\lambda$. One can check that

$$\begin{aligned}\tilde{\Phi}^u_\lambda v_\lambda &= \frac{(q^{-1}-q)^n}{[n]_+!}\left(\prod_{m=n+1}^{k}\chi_m^{\alpha_i}(\lambda)\right) \\ &\quad\times\left(\prod_{\beta\neq\alpha_i}\prod_{m=1}^{k}\chi_m^{\beta}(\lambda)\right)v_{\lambda-n\alpha}\otimes E_i^n u + \text{t.o.w},\end{aligned}$$

where t.o.w. denotes ‘terms of other weights’ (in the first component).

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It follows that

$$\mathcal{B}_{n\alpha_i}(\lambda) = \frac{(q^{-1}-q)^n}{[n]_+!}\left(\prod_{m=n+1}^{k}(1-q^{-2(n-m)})\right) \times \left(\prod_{\beta\neq\alpha_i}\prod_{m=1}^{k}\chi_m^\beta(\lambda)\right) F_i^n E_i^n.$$

It is known that $F_i^n E_i^n$ acts as multiplication by $\frac{[k+n]!}{[k-n]!}$ in $U[0]$. Since

$$\frac{(q^{-1}-q)^n}{[n]_+!}\left(\prod_{m=n+1}^{k}(1-q^{-2(n-m)})\right)\frac{[k+n]!}{[k-n]!} = q^{-2kn}\prod_{m=1}^{k}\left(1-q^{2(n+m)}\right),$$

for λ from the hyperplane $\langle\alpha_i, \lambda+\rho\rangle = n$ we have

$$\begin{aligned}\mathcal{B}_{n\alpha_i}(\lambda) &= q^{-2kn}\prod_{m=1}^{k}\left(1-q^{2(n+k)}\right)\left(\prod_{\beta\neq\alpha_i}\prod_{m=1}^{k}\chi_m^\beta(\lambda)\right)\\ &= q^{-2kn}\left(\prod_{m=1}^{k}\chi_m^{\alpha_i}(\lambda-n\alpha_i)\right)\left(\prod_{\gamma\neq\alpha_i}\prod_{m=1}^{k}\chi_m^{\gamma}(\lambda-n\alpha_i)\right)\\ &= q^{-n\langle\alpha_i,\Theta\rangle}\chi(\lambda-n\alpha_i).\end{aligned}$$

Induction step. Suppose (5.1) is true for all roots β such that height $\beta <$ height α. We are going to prove that (5.2) is true also for α.

We first show that $B_{n\alpha}(\lambda)$ is divisible by factors $\chi_m^\beta(\lambda - n\alpha)$ for all $\beta \neq \alpha, m = 1, \ldots, k$. It suffices to prove that $B_{n\alpha}(\lambda)$ vanishes whenever $\langle\beta, \lambda - n\alpha + \rho\rangle = m$.

Consider two cases:

(1) If $s_\alpha(\beta) \in R^+$, put $\gamma = s_\alpha(\beta)$. Then $\langle\gamma, \lambda+\rho\rangle = \langle\beta, \lambda - n\alpha + \rho\rangle = m$.

If λ is generic from hyperplane $\langle\gamma, \lambda+\rho\rangle = m$, then the image of $\tilde{\Phi}_\lambda^u$ is contained in $M_{\lambda-m\gamma}$. Since $n\alpha - m\gamma \notin \mathbf{Q}_+$, there will be no terms contributing to $\mathcal{B}_{n\alpha}(\lambda) = \mathrm{Tr}\,|_{M_\lambda[\lambda-n\alpha]}\tilde{\Phi}_\lambda^u$, and $\mathcal{B}_{n\alpha}(\lambda) = 0$ generically (and, therefore, identically) in the hyperplane $\langle\gamma, \lambda+\rho\rangle = m$.

(2) If $s_\alpha(\beta) \notin R^+$, put $\gamma = -s_\alpha(\beta) \in R^+$. Then $\alpha = \beta + \gamma$, and we can assume the induction hypothesis true for β and γ.

We have:

$$\langle\gamma, \lambda+\rho\rangle = -\langle\beta, \lambda - n\alpha + \rho\rangle = -m,$$

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$$\langle\beta, \lambda + m\gamma + \rho\rangle = \langle\beta, \lambda - n\alpha + \rho\rangle + n\langle\beta, \alpha\rangle + m\langle\gamma, \beta\rangle$$
$$= m + n - m = n.$$

Therefore,

$$\mathcal{B}_{n\alpha}(\lambda) = q^{\langle m\gamma,\Theta\rangle}\mathcal{B}_{n\alpha+m\gamma}(\lambda + m\gamma)$$
$$= q^{\langle m\gamma-n\beta,\Theta\rangle}\mathcal{B}_{(n+m)\gamma}(\lambda + m\gamma - n\beta).$$

Also,

$$\langle\alpha, \lambda + m\gamma - n\beta + \rho\rangle = n + m - n = m,$$

so the image of $\tilde{\Phi}^u_\lambda$ is contained in $M_{\lambda-(m+n)\beta}$, and

$$\mathcal{B}_{(n+m)\gamma}(\lambda + m\gamma - n\beta) = 0,$$

because $m\alpha - (m + n)\beta \notin \mathbf{Q}_+$ and there are no terms contributing to $\mathcal{B}_{(n+m)\gamma}(\lambda + m\gamma - n\beta)$.

We have proved that $\mathcal{B}_{n\alpha}(\lambda)$ vanishes on the required hyperplanes, and is therefore divisible by all the required factors. Thus, in the hyperplane $\langle\alpha, \lambda+\rho\rangle = n$ we get

$$\mathcal{B}_{n\alpha}(\lambda) = C(\lambda) \prod_{\beta\neq\alpha} \prod_{j=1}^{k} \chi^\beta_n(\lambda - n\alpha),$$

for some q-polynomial $C(\lambda)$. It is easy to see by comparing highest terms that $C(\lambda)$ is constant on the hyperplane $\langle\alpha, \lambda + \rho\rangle = n$. To compute this constant, take λ generic such that $\langle\alpha, \lambda+\rho\rangle = \langle\beta, \lambda+\rho\rangle = n$. Then automatically $\langle\gamma, \lambda+\rho-n\beta\rangle = n$. We have:

$$\Psi(\lambda, x) = q^{-\langle n\beta,\Theta\rangle}\Psi(\lambda - n\beta, x) = q^{-\langle n\beta,\Theta\rangle}q^{-\langle n\gamma,\Theta\rangle}\Psi(\lambda - n\beta - n\gamma, x).$$

But $\lambda-n\beta-n\gamma = \lambda-n\alpha$ does not lie on any Kac-Kazhdan hyperplanes, therefore $\Psi(\lambda - n\alpha, x) \neq 0$, and

$$C(\lambda) = q^{-\langle n\beta,\Theta\rangle}q^{-\langle n\gamma,\Theta\rangle} = q^{-n\langle\alpha,\Theta\rangle}.$$

Proposition 5.1 is now proved. □

Let λ be a dominant integral weight, and V_λ is the irreducible $\mathcal{U}_q\mathfrak{g}$-module with highest weight λ.

PROPOSITION 5.3 (Generalized Weyl formula). *The operator* $\tilde{\Phi}^u_\lambda: M_\lambda \to M_\lambda \otimes U$ *descends to a homomorphism* $V_\lambda \to V_\lambda \otimes U$. *The function*

$$\tilde{p}_\lambda(x) = \mathrm{Tr}|_{V_\lambda}(\tilde{\Phi}^u_\lambda\ \mathrm{e}^x),$$

is expressed in terms of the functions $\Psi(\lambda, x)$ *by*

$$q^{\langle\Theta,\lambda\rangle}\tilde{p}_\lambda(x) = \sum_{w\in W}(-1)^{l(w)}q^{\langle\Theta,w^\rho\lambda\rangle}\Psi(w^\rho\lambda, x). \tag{5.3}$$

Proof. The operator $\tilde{\Phi}^u_\lambda$ defines an operator $M_\lambda \to V_\lambda \otimes U$. This operator has to factor through V_λ because it lands in a finite dimensional representation. Thus, $\tilde{\Phi}^u_\lambda$ in fact defines an operator $V_\lambda \to V_\lambda \otimes U$.

Recall that for $\lambda \in \mathbf{P}_{++}$ we have a resolution

$$0 \leftarrow V_\lambda \leftarrow M^0_\lambda \leftarrow M^1_\lambda \leftarrow M^2_\lambda \leftarrow \cdots,$$

where

$$M^0_\lambda = M_\lambda, \qquad M^i_\lambda = \bigoplus_{l(w)=i} M_{w^\rho\lambda}.$$

For matrix traces we have as for usual characters

$$\tilde{p}_\lambda(x) = \mathrm{Tr}|_{V_\lambda}(\tilde{\Phi}^u_\lambda\ \mathrm{e}^x) = \sum_i(-1)^i\ \mathrm{Tr}|_{M^i_\lambda}(\tilde{\Phi}^u_\lambda\ \mathrm{e}^x).$$

When λ is generic from hyperplane $\langle\alpha_i, \lambda+\rho\rangle = \frac{n}{2}\langle\alpha_i,\alpha_i\rangle,\ \ n > k$, then M_λ contains a submodule $M^1_\lambda \cong M_{\lambda-n\alpha_i}$, generated by a singular vector $v_{\lambda-n\alpha_i} = F^n_i v_\lambda$. Since $M_\lambda[\lambda - n\alpha_i]$ is one-dimensional, we can write

$$\tilde{\Phi}^u_\lambda v_{\lambda-n\alpha_i} = v_{\lambda-n\alpha_i} \otimes u'' + \cdots.$$

Then we will have $\tilde{\Phi}^u_\lambda|_{M^1_\lambda} = \tilde{\Phi}^{u''}_{\lambda-n\alpha_i}$. To compute u'' we use the formula

$$\begin{aligned}\tilde{\Phi}^u_\lambda v_\lambda \ &= \ \sum_{m=0}^{k}\frac{(q^{-1}-q)^m}{[m]_+!}\left(\prod_{l=m+1}^{k}\chi^{\alpha_i}_l(\lambda)\right)\\ &\quad\times\left(\prod_{\beta\neq\alpha_i}\prod_{l=1}^{k}\chi^\beta_l(\lambda)\right)F^m_i v_\lambda \otimes E^m_i u + \cdots,\end{aligned}$$

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where we retained only terms, which will contribute to the expression for u''. Applying $\Delta(F_i^n)$ to the RHS, and collecting terms involving $v_{\lambda-n\alpha_i}$, we deduce that

$$\begin{aligned}
u'' &= \sum_{m=0}^{k} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(q^{-1}-q)^m}{[m]_+!} \left(\prod_{l=m+1}^{k} \chi_l^{\alpha_i}(\lambda) \right) \\
&\quad \times \left(\prod_{\beta\neq\alpha_i} \prod_{l=1}^{k} \chi_l^{\beta}(\lambda) \right) F_i^m K_i^{n-m} E_i^m u \\
&= \sum_{m=0}^{k} (-1)^m q^{m-k} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(q^{-1}-q)^k}{[m]_+!} \frac{[n-m-1]_-!}{[n-k-1]_-!} \\
&\quad \times \left(\prod_{\beta\neq\alpha_i} \prod_{l=1}^{k} \chi_l^{\beta}(\lambda) \right) q^{2m(n-m)} F_i^m E_i^m u.
\end{aligned}$$

The operator $F_i^m E_i^m$ acts in $U[0]$ as multiplication by $\frac{[k+m]!}{[k-m]!}$.

Also, we need to use the identity

$$\sum_{m=0}^{k} (-1)^m \frac{[n]!}{[m]![n-m]!} \frac{[n-m-1]!}{[m]![n-k-1]!} \frac{[k+m]!}{[k-m]!} = (-1)^k \frac{[n+k]!}{[n]!}.$$

It can be interpreted as equality of two polynomials in $z = q^{2n}$ of degree k. To prove this identity it suffices to check it for $n = 0, \dots, k$, when there is only one nonzero term in the LHS of the equation.

After easy transformations, we conclude that

$$\begin{aligned}
u'' &= q^{-2kn} \left(\prod_{m=1}^{k} (1-q^{-2(n+m)}) \right) \left(\prod_{\beta\neq\alpha_i} \prod_{m=1}^{k} \chi_m^{\beta}(\lambda) \right) u \\
&= q^{-2kn} \left(\prod_{m=1}^{k} \chi_m^{\alpha_i}(\lambda - n\alpha_i) \right) \left(\prod_{\gamma\neq\alpha_i} \prod_{m=1}^{k} \chi_m^{\gamma}(\lambda - n\alpha_i) \right) u \\
&= q^{-n\langle\alpha_i,\Theta\rangle} \chi(\lambda - n\alpha_i) u.
\end{aligned}$$

It follows that the restriction of $\tilde{\Phi}_\lambda^u$ to the submodule $M_\lambda^1 \cong M_{\lambda-n\alpha_i}$ coincides with $q^{-n\langle\alpha_i,\Theta\rangle}\tilde{\Phi}_{\lambda-n\alpha_i}^u$ for λ generic from hyperplane $\langle\alpha_i, \lambda+\rho\rangle = \frac{n}{2}\langle\alpha_i,\alpha_i\rangle$. Therefore, it is true for all λ from that hyperplane. Even more generally, for $\lambda \in \mathbf{P}_{++}$ we have $q^{\langle\Theta,\lambda\rangle}\tilde{\Phi}_\lambda^u|_{M_{w^\rho\lambda}} = q^{\langle\Theta,w^\rho\lambda\rangle}\tilde{\Phi}_{w^\rho\lambda}^u$. Thus

$$\begin{aligned}
q^{\langle\Theta,\lambda\rangle}\ \mathrm{Tr}|_{M^i}(\tilde{\Phi}_\lambda^u\ \mathrm{e}^x) &= \sum_{l(w)=i} q^{\langle\Theta,\lambda\rangle}\ \mathrm{Tr}|_{M_{w^\rho\lambda}}(\tilde{\Phi}_\lambda^u\ \mathrm{e}^x) \\
&= \sum_{l(w)=i} q^{\langle\Theta,w^\rho\lambda\rangle}\Psi\left(w^\rho\lambda, x\right).
\end{aligned}$$

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Therefore

$$q^{\langle \Theta,\lambda\rangle}\tilde{p}_\lambda(x) = \sum_{w\in W}(-1)^{l(w)}q^{\langle\Theta,w^\rho\lambda\rangle}\Psi\left(w^\rho(\lambda),x\right). \qquad \square$$

Introduce a normalized matrix trace by

$$\psi(\lambda,x) = \frac{q^{\langle\Theta,\lambda-\rho\rangle}\Psi(\lambda-\rho,x)}{\prod_{\alpha\in R^+}\prod_{i=1}^{k}(q^i\ \mathrm{e}^{\langle\alpha,x\rangle/2\rangle} - q^{-i}\ \mathrm{e}^{-\langle\alpha,x\rangle/2\rangle})}.$$

Condition (3.1) can be rewritten for the function $\psi(\lambda,x)$ as

$$\psi\left(\lambda+\frac{n\alpha}{2}\right) = \psi\left(\lambda-\frac{n\alpha}{2}\right), \tag{5.4}$$

for all $\alpha\in R^+,\ n=1,\dots,k$ and for all λ such that $\langle\alpha,\lambda\rangle=0$. Combining (5.3) with results in $[EK]$, we get:

COROLLARY (Macdonald polynomials). *Macdonald polynomials are equal to*

$$p_\lambda(x) = \sum_{w\in W}(-1)^{l(w)}\psi(w(\lambda+(k+1)\rho),x). \tag{5.5}$$

up to a factor

PROPOSITION 5.4 (Macdonald operators). *The function $\psi(\lambda,x)$ is a common eigenfunction for Macdonald operators, corresponding to $t=q^{k+1}$:*

$$\mathcal{M}_i = \sum_{w\in W} w\left(\prod_{\langle\alpha,\Lambda_i\rangle=1}\frac{q^{k+1}\ \mathrm{e}^{\langle\alpha,x\rangle/2} - q^{-k-1}\ \mathrm{e}^{-\langle\alpha,x\rangle/2}}{\mathrm{e}^{\langle\alpha,x\rangle/2}-\mathrm{e}^{-\langle\alpha,x\rangle/2}}T_{\Lambda_i}\right), \quad i=1,\dots,r.$$

The corresponding eigenvalues are W-invariant q-polynomials $c_i(\lambda) = \sum_{w\in W}q^{\langle\lambda,w\Lambda_i\rangle}$:

$$\mathcal{M}_i\psi(\lambda,x) = c_i(\lambda)\psi(\lambda,x).$$

Proof. We already know that the Ψ-function is the common eigenfunction for a family of commuting difference operators, corresponding to $c(\lambda)$, satisfying (4.2). It is shown in [EK] by using central elements in $\mathcal{U}_q\mathfrak{g}$, that for the Ψ-function normalized as above, the operators, corresponding to elementary symmetric functions $c_i(\lambda)$, are exactly Macdonald operators. $\square$

Now we study relations between ψ-functions for different values of k. We use notation $\psi_k(\lambda,x)$ for the ψ-function constructed from representation U_k.

THEOREM 5.5 (Shift operators). *There exist difference operators G_k such that*

$$\psi_{k+1}(\lambda, x) = G_k \psi_k(\lambda, x).$$

These operators are W invariant, and their action in the basis of Macdonald polynomials is given by the formula

$$G_k p_{k,\lambda}(x) = p_{k+1,\lambda-\rho}(x), \quad \lambda - \rho \in \mathbf{P}_+; \qquad G_k p_{k,\lambda} = 0, \quad \lambda - \rho \notin \mathbf{P}_+.$$

Remark. Shift operators in the q-deformed case were introduced by Cherednik, [Ch1].

Proof. The argument from the proof of Proposition 4.1 can be used to prove that any function, satisfying (5.4), can be represented as

$$\phi(\lambda, x) = D\psi_k(\lambda, x),$$

for some difference operator D. Applying this to the function $\psi_{k+1}(\lambda, x)$, we prove the existence of an operator G_k such that

$$\psi_{k+1}(\lambda, x) = G_k \psi_k(\lambda, x).$$

From the generalized Weyl formula we get

$$\begin{aligned} G_k p_{k,\lambda} &= \sum_{w \in W} (-1)^{l(w)} G_k \psi_k(w(\lambda + (k+1)\rho), x) \\ &= \sum_{w \in W} (-1)^{l(w)} \psi_{k+1}(w(\lambda + (k+1)\rho), x) = p_{k,\lambda-\rho}(x). \end{aligned}$$

If $\lambda - \rho$ is not dominant then the right-hand side of the last formula is zero by (5.4).

We see that G_k maps Macdonald polynomials to W-invariant functions. This implies that G_k is itself W-invariant. □

Remark. We saw that shift operators relate eigenfunctions for Macdonald operators, corresponding to different (integral) values of parameter k. One can write it in the form

$$G_k \mathcal{M}_i^{(k)} = \mathcal{M}_i^{(k+1)} G_k. \tag{5.6}$$

One can check that G_k analytically depends on k, therefore one can extend equality (5.6) to the case of arbitrary k. This implies the existence of shift operators in the general case, which is proven in [Ch1] using representation theory of double affine Hecke algebras.

Denote

$$\delta_k(x) = \prod_{\alpha\in R^+} \prod_{i=-k}^{k} \left(q^i \ \mathrm{e}^{\langle\alpha,x\rangle/2} - q^{-i} \ \mathrm{e}^{-\langle\alpha,x\rangle/2}\right).$$

THEOREM 5.6 (Duality). *The function* $\varphi_k(\lambda,x) = \delta_k(x)\psi_k(\lambda,x)$ *is symmetric with respect to transformation* $q^{\langle\alpha,\lambda\rangle} \leftrightarrow \mathrm{e}^{\langle\alpha,x\rangle}$.

Proof. The idea of the proof is the same as in [VSC]. First we prove that φ_k as a function of x satisfies condition (4.2), and then duality will follow from the uniqueness property.

We already know that ψ_k is the eigenfunction for the Macdonald operators. Therefore, $\varphi_k(\lambda,x)$ is the eigenfunction for the operators, obtained from Macdonald operators by conjugation by $\delta_k(x)$. Such an operator, corresponding to a minuscule weight Λ, is also W-invariant and has the form

$$\widetilde{\mathcal{M}}_\Lambda = \sum_{w\in W} f_\Lambda(wx)T_{w(\Lambda)},$$

where

$$f_\Lambda(x) = \prod_{\langle\beta,\Lambda\rangle=1} \frac{q^{-k} \ \mathrm{e}^{\langle\beta,x\rangle/2} - q^k \ \mathrm{e}^{-\langle\beta,x\rangle/2}}{\mathrm{e}^{\langle\beta,x\rangle/2} - \mathrm{e}^{-\langle\beta,x\rangle/2}}.$$

Fix a root α and denote $W_\alpha = \{w \in W | \langle\alpha, w\Lambda\rangle = 1\}$. Let x be such that $\langle\alpha,x\rangle = 0$. The ϕ-function does not have singularities along that hyperplane. Collecting all the singular terms in the equation

$$\widetilde{\mathcal{M}}_\Lambda\varphi_k(\lambda,x) = c_\Lambda(\lambda)\varphi_k(\lambda,x),$$

which occur for $w \in W$ such that $\langle\alpha, w\Lambda\rangle = \pm1$, we obtain

$$\sum_{w\in W_\alpha} f_\Lambda(wx)T_{w(\Lambda)}\varphi_k(\lambda,x) + \sum_{w\in W_\alpha} f_\Lambda(s_\alpha wx)T_{(s_\alpha w)(\Lambda)}\varphi_k(\lambda,x) = 0. \tag{5.7}$$

When x belongs to hyperplane $\langle\alpha,x\rangle = 0$ we have

$$w\left(\prod_{\langle\beta,\Lambda\rangle=1} \frac{q^{-k-1} \ \mathrm{e}^{\langle\beta,x\rangle/2} - q^{k+1} \ \mathrm{e}^{-\langle\beta,x\rangle/2}}{\mathrm{e}^{\langle\beta,x\rangle/2} - \mathrm{e}^{-\langle\beta,x\rangle/2}}\right) = -(s_\alpha w)\left(\prod_{\langle\beta,\Lambda\rangle=1} \frac{q^{-k} \ \mathrm{e}^{\langle\beta,x\rangle/2} - q^k \ \mathrm{e}^{-\langle\beta,x\rangle/2}}{\mathrm{e}^{\langle\beta,x\rangle/2} - \mathrm{e}^{-\langle\beta,x\rangle/2}}\right),$$

therefore (5.7) simplifies to

$$\sum_{w\in W_\alpha} f_\Lambda(wx)\left(\varphi_k\left(\lambda, x_w+\frac{\alpha}{2}\right)-\varphi_k\left(\lambda, x_w-\frac{\alpha}{2}\right)\right)=0,$$

where

$$x_w = x + w(\Lambda)^\perp = x + w(\Lambda) - \frac{\alpha}{2},$$

also belongs to hyperplane $\langle\alpha, x_w\rangle = 0$. It follows that

$$\varphi_k\left(\lambda, x+\frac{\alpha}{2}\right)-\varphi_k\left(\lambda, x-\frac{\alpha}{2}\right)=0,$$

identically when $\langle\alpha, x\rangle = 0$, i.e. that condition (5.4) is satisfied for $n = 1$. Taking n-th power of operators $\widetilde{\mathcal{M}}_\Lambda$, and repeating the same argument, we can prove that it is satisfied for $n = 1, \ldots, k$.

From the obvious modification of the theorem about uniqueness of the Ψ-function, we conclude that the function $\varphi_k(\lambda, x)$ transforms into itself when we interchange $q^{\langle\alpha,\lambda\rangle} \leftrightarrow \mathrm{e}^{\langle\alpha,x\rangle}$. □

Remark. This duality result is closely related to the symmetry of the difference Fourier pairing defined recently by Cherednik [Ch2].

Appendix. Example: $\mathfrak{g} = \mathfrak{sl}_2$

In this case we have only one root α, and we can make identifications

$$\mathrm{e}^{\langle\lambda,x\rangle} \leftrightarrow \mathrm{e}^{\lambda x}, \qquad \mathrm{e}^{\langle\alpha,x\rangle} \leftrightarrow \mathrm{e}^{2x}, \qquad q^{\langle\alpha,\lambda\rangle} \leftrightarrow q^{\lambda}.$$

Operators T, T^{-1} act on a function $f(x)$ by

$$(Tf)(x) = f(x+1), \qquad (T^{-1}f)(x) = f(x-1).$$

Case $k = 0$. This is the simplest example, corresponding to the case U- trivial representation. In that case

$$\tilde{\Phi}^u_\lambda v_\lambda = \Phi^u_\lambda v_\lambda = v_\lambda \otimes u,$$

$$\Psi(\lambda, x) = \frac{\mathrm{e}^{\lambda x}}{1-\mathrm{e}^{-2x}} = \frac{\mathrm{e}^{(\lambda+1)x}}{\mathrm{e}^{x}-\mathrm{e}^{-x}},$$

$$\psi_0(\lambda, x) = \frac{\mathrm{e}^{\lambda x}}{\mathrm{e}^{x}-\mathrm{e}^{-x}}.$$

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Formula (5.5) is the usual Weyl formula

$$\mathrm{char}\ V_\lambda = \frac{\mathrm{e}^{(\lambda+1)x} - \mathrm{e}^{-(\lambda+1)x}}{\mathrm{e}^{x} - \mathrm{e}^{-x}}.$$

The operator, corresponding to W-invariant polynomial $c_1(\lambda) = q^\lambda + q^{-\lambda}$, is the Macdonald operator

$$\mathcal{M}_1 = \frac{q\ \mathrm{e}^{x} - q^{-1}\ \mathrm{e}^{-x}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} T + \frac{q\ \mathrm{e}^{-x} - q^{-1}\ \mathrm{e}^{x}}{\mathrm{e}^{-x} - \mathrm{e}^{x}} T^{-1}$$

and we have

$$\mathcal{M}_1 \psi_1(\lambda, x) = c_1(\lambda)\psi_1(\lambda, x).$$

Condition (4.2) gives no restriction on $c(\lambda)$, and we can set $c_0(\lambda) = q^\lambda$. The corresponding operator $\mathcal{M}_0$, whose existence is predicted by Theorem 4.1, is equal to

$$\mathcal{M}_0 = \frac{q\ \mathrm{e}^{x} - q^{-1}\ \mathrm{e}^{-x}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} T$$

and $\mathcal{M}_1 = \mathcal{M}_0 + \mathcal{M}_0^{-1}$.

Case $k = 1$. Let U now be the 3-dimensional representation. We have:

$$\Phi_\lambda^u v_\lambda = v_\lambda \otimes u - \frac{q - q^{-1}}{1 - q^{-2\lambda}} F v_\lambda \otimes E u,$$

$$\tilde{\Phi}_\lambda^u v_\lambda = (1 - q^{-2\lambda}) v_\lambda \otimes u - (q - q^{-1}) F v_\lambda \otimes E u,$$

$$\begin{aligned}\Psi(\lambda, x) &= \frac{\mathrm{e}^{\lambda x}}{1 - \mathrm{e}^{-2x}} \left(1 - q^{-2\lambda} - (q^2 - q^{-2}) \frac{\mathrm{e}^{-2x}}{1 - q^{-2}\ \mathrm{e}^{-2x}}\right) \\ &= \frac{\mathrm{e}^{\lambda x}}{1 - \mathrm{e}^{-2x}} \left(\frac{1 - q^2\ \mathrm{e}^{-2x}}{1 - q^{-2}\ \mathrm{e}^{-2x}} - q^{-2\lambda}\right),\end{aligned}$$

$$\psi_1(\lambda, x) = \frac{\mathrm{e}^{\lambda x}}{\mathrm{e}^{x} - \mathrm{e}^{-x}} \left(\frac{q^\lambda}{q\ \mathrm{e}^{x} - q^{-1}\ \mathrm{e}^{-x}} - \frac{q^{-\lambda}}{q^{-1}\ \mathrm{e}^{x} - q\ \mathrm{e}^{-x}}\right).$$

It is easy to check that indeed

$$\Psi(0, x) = q^{-2}\Psi(-2, x), \qquad \psi_1(1, x) = \psi_1(-1, x).$$

Function $\psi_1(\lambda, x)$ is related to $\psi_0(\lambda, x)$ by

$$\psi_1(\lambda, x) = G_0 \psi_0(\lambda, x),$$

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where the shift operator G_0 is equal to

$$G_0 = \frac{1}{\mathrm{e}^x - \mathrm{e}^{-x}} \left(T - T^{-1}\right).$$

The Macdonald operator $\mathcal{M}_1$ is equal to

$$\mathcal{M}_1 = \frac{q^2 \ \mathrm{e}^x - q^{-2} \ \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} T + \frac{q^2 \ \mathrm{e}^{-x} - q^{-2} \ \mathrm{e}^x}{\mathrm{e}^{-x} - \mathrm{e}^x} T^{-1},$$

$$\mathcal{M}_1 \psi_1(\lambda, x) = c_1(\lambda) \psi_1(\lambda, x).$$

The operator $\mathcal{M}_0$, corresponding to the eigenvalue $c_0(\lambda) = q^{3\lambda} - [3]_q q^\lambda$, is equal to

$$\mathcal{M}_0 = \frac{q^4 \ \mathrm{e}^x - q^{-4} \ \mathrm{e}^{-x}}{q \ \mathrm{e}^x - q^{-1} \ \mathrm{e}^{-x}} \frac{q^3 \ \mathrm{e}^x - q^{-3} \ \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} T^3 - [3]_q \frac{q \ \mathrm{e}^x - q^{-1} \ \mathrm{e}^{-x}}{q^{-1} \ \mathrm{e}^x - q \ \mathrm{e}^{-x}} T.$$

Classical limit $q \to 1$. Set $\epsilon = \log(q)$. We have the expansion

$$\psi_1(\lambda, x) = 2\varepsilon \cdot \psi_1^{(0)}(\lambda, x) + O(\varepsilon^2),$$

where

$$\psi_1^{(0)}(\lambda, x) = \frac{\mathrm{e}^{\lambda x}}{(\mathrm{e}^x - \mathrm{e}^{-x})^2} \left(\lambda - \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}\right).$$

is the ψ-function for the classical case (cf. [ES]). The difference operators become

$$\mathcal{M}_1 = 2 + \varepsilon^2(\mathcal{D}_2 + 4) + O(\varepsilon^3),$$

$$\mathcal{M}_0 = -2 + \varepsilon^2(3\mathcal{D}_2 + 8) + 4\varepsilon^3(\mathcal{D}_3 + 1) + O(\varepsilon^4),$$

where commuting differential operators $\mathcal{D}_2, \mathcal{D}_3$ are equal

$$\mathcal{D}_2 = \frac{\partial^2}{\partial x^2} + 4 \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} \frac{\partial}{\partial x},$$

$$\begin{aligned} \mathcal{D}_3 \ &= \ \frac{\partial^3}{\partial x^3} + 6 \left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}\right) \frac{\partial^2}{\partial x^2} + \left(11 + \frac{12}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}\right) \frac{\partial}{\partial x} \\ &\quad + 6 \left(\frac{\mathrm{e}^{3x} - 3 \ \mathrm{e}^x - 3 \ \mathrm{e}^{-x} + \mathrm{e}^{-3x}}{(\mathrm{e}^x - \mathrm{e}^{-x})^3}\right). \end{aligned}$$

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