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WEAKLY PRIME LEFT IDEALS IN GENERAL RINGS

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A.P.J. van der Walt introduced the concept of a weakly prime left ideal of an associative ring with unity. It is the purpose of the present paper to extend to general, that is not necessarily with unity associative rings, this concept as well as almost all results of van der Walt for rings with unity.

Let A be an associative ring, not necessarily with unity.

DEFINITION. (see van der Walt [1]). A left ideal P of A is called weakly prime if and only if $L_1L_2 \subseteq P$ implies $L_1 = P$ or $L_2 = P$ for any left ideals L_1, L_2 of A containing P.

PROPOSITION 1. *(compare van der Walt [1], Prop. 1.1). The following are equivalent for a left ideal P of A:*

(a) P is weakly prime.

(b) $(P+L_1)(P+L_2) \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$.

(c) $L_1 \supseteq P$ and $L_1L_2 \subseteq P$ imply $L_1 = P$ or $L_2 \subseteq P$.

(d) $(P+L_1)L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$.

(e) $(a+P)A\,(b+P) \subseteq P$ implies $a \in P$ or $b \in P$.

Here L_1, L_2 denote left ideals of A and a, b elements of A.

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Proof. The implications (a)$\to$(b)$\to$(c)$\to$(d) are immediate. So suppose (d) holds and let us prove (e). If $(a+P)A(b+P) \subseteq P$, then $aAb \subseteq P$ which implies $PAb \subseteq P$. Also $AaAb \subseteq P$. It follows that $(P+Aa)(Ab) \subseteq P$ and so by (d) $Aa \subseteq P$ or $Ab \subseteq P$. Suppose $Aa \subseteq P$. Let $(a\rangle = Za+Aa$. Then $(a\rangle$ is a left ideal of A containing a and $(P+(a\rangle)(a\rangle \subseteq P$. Again by (d) it follows that $(a\rangle \subseteq P$ and so $a \in P$. If $Ab \subseteq P$ then similarly one can prove that $b \in P$. Therefore (e) folds.

To conclude the proof suppose (e) holds and let $L_1, L_2 \supseteq P$ and $L_1L_2 \subseteq P$. If $L_1 \neq P$, choose $a \in L_1 \setminus P$. Then $((a\rangle+P)A((b\rangle+P) \subseteq P$ for every $b \in L_2$. This means that $(a+P)A(b+P) \subseteq P$, hence by (e), $b \in P$. So $L_2 \subseteq P$ which ends the proof.

PROPOSITION 2. *(compare van der Walt [1], Prop. 1.3). For a left ideal P of A which is not two-sided the following are equivalent:*

(a) P is weakly prime.

(b) $PL \subseteq P$ for a left ideal L of A implies $L \subseteq P$.

(c) $PAb \subseteq P$ implies $b \in P$.

(d) P is the largest left ideal of A which is not contained in the idealizer of P.

Proof. (a)$\to$(b): Suppose P is weakly prime and let $PL \subseteq P$. Then $(PA)L \subseteq P$ and so $(P+PA)L \subseteq P$. But since P is properly contained in $P+PA$ because it is not two-sided ideal, $L \subseteq P$ follows from Proposition 1 (c).

(b) $\to$ (c) : Suppose $PAb \subseteq P$. Then $PP(b \subseteq P$ and so by (b) $P(b\rangle \subseteq P$. This implies $(b\rangle \subseteq P$, also by (b). Hence $b \in P$. The implications (c) $\to$ (d) and (d) $\to$ (a) can be proved in the same way as the ones of van der Walt [1], Prop. 1.3.

Recall that an m-system is a nonempty subset M of A such that for any $m_1, m_2 \in M$ there is an $x \in A$ with $m_1xm_2 \in M$. As in the case of a two-sided ideal, a left ideal P of A is called prime *iff* its complement $A \setminus P$ is an m-system. It is easy to see that P is a prime left ideal *iff* $L_1L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$ for any left ideals L_1, L_2 of A.

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PROPOSITION 3. *(compare van der Walt [1], Prop. 1.6). Let P be a left ideal of A and $B(P)$ the largest two-sided ideal of A contained in P for example, $B(P) = \{x \in P \mid xA \subseteq P\}$. Then P is prime iff there is an m-system M such that P is a maximal left ideal of A not meeting M and $B(P)$ is a maximal two-sided ideal of A not meeting M.*

Proof. If P is prime then $M = A \setminus P$ is an m-system and of course P is a maximal left ideal not meeting M and $B(P)$ is a maximal two-sided ideal not meeting M.

Conversely, Let M be any m-system, let P be a maximal left ideal not meeting M and let $B(P)$ be a maximal two-sided ideal not meeting M. We show that P is prime. Suppose $L_1L_2 \subseteq P$ with $L_1 \nsubseteq P$, $L_2 \nsubseteq P$, L_1 and L_2 left ideals. Then we have $(L_1+L_1A+B(P))A(P+L_2) \subseteq P$. Now $L_1+L_1A+B(P)$ is a two-sided ideal properly containing $B(P)$, so there is $m_1 \in (L_1+L_1A+B(P)) \cap M$. Also by the maximality of P there is $m_2 \in (P+L_2) \cap M$. However, this implies $(L_1+L_1A+B(P))A(P+L_2) \nsubseteq P$, a contradiction. Therefore $L_1L_2 \nsubseteq P$ and P is prime.

THEOREM. *Let A be a ring, $a \in A$ and $A^1a = \{y \in A \mid y = na+xa$ for some integer n and $x \in A\}$. Then every proper left ideal of A is weakly prime iff A is a simple ring and $Aa = A^1a$ for all $a \in A$.*

Proof. If A is a simple ring and $Aa = A^1a$ for each $a \in A$ then $\{0\}$ is certainly a prime ideal of A and so $\{0\}$ is weakly prime. Moreover every non-zero proper left ideal L of A is also weakly prime. For if $LAb \subseteq L$ then $Ab \subseteq L$, since LA is a non-zero two-sided ideal of A. This implies $b \in L$, for $Ab = A^1b$ and $b \in A^1b$ which means that L is weakly prime by Proposition 2.

Conversely, let every proper left ideal of A be weakly prime. If $Aa \neq A^1a$ for some $a \in A$ then $Aa \neq A$ and $A(A^1a) \subseteq Aa$ imply that Aa is not weakly prime, by Proposition 1. So $Aa = A^1a$ for each $a \in A$. This means that, first of all, $J(A) \neq A$, where $J(A)$ denotes the Jacobson radical of A. Furthermore, if B is any proper two-sided ideal of A and L any proper left ideal of A, then $(B \cap L) \cdot L \subseteq B \cap L$. Since $B \cap L$ is a weakly prime proper left ideal and $BL \subseteq B \cap L$ then, either $B = B \cap L$ or $L = B \cap L$, by Proposition 1. Moreover $B^2 = B$ and $L^2 = L$ as both B^2 and L^2 are proper weakly prime left ideals. Hence either $B = B^2 = B(B \cap L) \subseteq BL \subseteq B$ or $L = L^2 = (B \cap L)L \subseteq BL \subseteq L$. So whether $B = B \cap L$ or $L = B \cap L$, $B \cap L = BL$. Now suppose $J(A) \neq 0$ and let $0 \neq a \in J(A)$. Let P be a left ideal of A such that $a \notin P$ and P is maximal with respect to this property. Let $M = Aa + P$. Then M/P is a simple left A-module and so $J(A)M \subseteq P$. But $a \in J(A) \cap M$ and $J(A)M = J(A) \cap M$. Therefore $a \in P$ which is impossible. Hence $J(A) = \{0\}$. To end the proof consider a two-sided proper ideal I of A. Then for any maximal left ideal L of A, $IL \subseteq I \cap L$ and either $I = I \cap L$ or $L = I \cap L$. In any case $I \subseteq L$. Therefore $I \subseteq J(A) = \{0\}$. This implies that A is a simple ring which ends the proof.

COROLLARY. *(van der Walt [1], Prop. 2.5). If A is a ring with unity and every left ideal of A is weakly prime than A is simple.*

Remark. Notice that all the remaining results of van der Walt [1] except for Proposition 3.5 and Corollary 3.6 also hold for any, not necessarily with unity, associative ring and can be proved exactly in the same way as those of van der Walt [1].

References

[1] A.P.J. van der Walt, "Weakly prime one-sided ideals," *J. Austral. Math. Soc.* 38 (1985), 84-91.

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