COMPOSITIO MATHEMATICA

The affine part of the Picard scheme

Thomas Geisser

Compositio Math. **145** (2009), 415–422.

doi:10.1112/S0010437X08003710

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 145 (2009) 415–422
doi:10.1112/S0010437X08003710

The affine part of the Picard scheme

Thomas Geisser

Abstract

We describe the maximal torus and maximal unipotent subgroup of the Picard variety of a proper scheme over a perfect field.

1. Introduction

For a proper scheme $p: X \to k$ over a perfect field, the Picard scheme Pic_X representing the functor $T \mapsto H^0(T_{\mathrm{et}}, R^1 p_* \mathbb{G}_m)$ exists, and its connected component Pic^0_X is separated and of finite type [Mur64, II.15]. By Chevalley's structure theorem [Che60], the reduced connected component $\mathrm{Pic}^{0,\mathrm{red}}_X$ is an extension of an abelian variety A_X by a linear algebraic group L_X:

$$0 \to L_X \to \mathrm{Pic}^{0,\mathrm{red}}_X \to A_X \to 0. \qquad (1)$$

The commutative, smooth affine group scheme L_X is the direct product of a torus T_X and a unipotent group U_X. The following theorem completely characterizes T_X.

Theorem 1. *If X is proper over a perfect field, then the cocharactermodule $\mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T_X)$ of the maximal torus of Pic_X is isomorphic to $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z})$ as a Galois module.*

To analyze the unipotent part, we let $\mathrm{Pic}(X[t])_{[1]}$ be the typical part, that is, the subgroup of elements x of $\mathrm{Pic}(X[t])$ such that the map $X[t] \to X[t]$, $t \mapsto nt$ sends x to nx.

Theorem 2. *Let X be proper over a perfect field. Then $\mathrm{Pic}(X[t])_{[1]}$ is isomorphic to the group of morphisms of schemes $f: \mathbb{G}_a \to U_X$ satisfying $f(nx) = nf(x)$ for every $n \in \mathbb{Z}$. In particular, $\mathrm{Hom}_k(\mathbb{G}_a, U_X) \subseteq \mathrm{Pic}(X[t])_{[1]}$, and this is an equality in characteristic zero.*

To obtain another description of U_X, we assume that X is reduced (the map of Picard schemes induced by the map $X^{\mathrm{red}} \to X$ is well understood by the work of Oort [Oor62]). The seminormalization $X^+ \to X$ is the largest scheme between X and its normalization which is strongly universally homeomorphic to X in the sense that the map $X^+ \to X$ induces an isomorphism on all residue fields. A theorem of Traverso [Tra70] implies that $\mathrm{Pic}(X[t])_{[1]}$, hence U_X, vanishes if X is reduced and seminormal. We use this to show the following result.

Theorem 3. *Let X be reduced and proper over a perfect field.*

(a) *There is a short exact sequence*

$$0 \to U_X \to \mathrm{Pic}_X \to \mathrm{Pic}_{X^+} \to 0,$$

and $U_X = p_*(\mathbb{G}_{m,X^+}/\mathbb{G}_{m,X})$.

Received 17 April 2007, accepted in final form 21 April 2008, published online 25 March 2009.
2000 Mathematics Subject Classification 14K30.
Keywords: Picard scheme, torus, unipotent subgroup, semi-normalization, étale cohomology.
Supported in part by NSF grant No. 0556263.

(b) *The group scheme U_X represents the functor*

$$T \mapsto \{\mathcal{O}_{X\times T}\text{-line bundles } \mathcal{L} \subseteq \mathcal{O}_{X^+\times T} \text{ which are invertible in } \mathcal{O}_{X^+\times T}\}.$$

Notation. For a field k, we denote by $\bar{k}$ its algebraic closure, and for a scheme X over k we let $\bar{X} = X \times_k \bar{k}$. Unless specified otherwise, all extension and homomorphism groups are considered on the large étale site.

2. The torus

Lemma 4. *Let K and Q be abelian group schemes over a scheme S, and assume that $H^1_{\mathrm{et}}(Q, K|_Q) = 0$. Then every extension of Q by K of sheaves of abelian groups on the big étale site of S is representable, that is, the group of extensions of Q by K in the category of group schemes and in the category of sheaves agree.*

Proof. Given an extension of sheaves $K \xrightarrow{i} G \xrightarrow{p} Q$, it suffices to show that G is representable as a sheaf of sets. By hypothesis, we can find a section $\eta \in G(Q)$ with $p(\eta) = \mathrm{id}$. Define a (set-theoretic) splitting $s : Q \to G$ to p by sending a section $g : T \to Q$ of $Q(T)$ to $g^*(\eta) = \eta \circ g \in G(T)$. But, then $K \times Q \xrightarrow{(i,s)} G$ is an isomorphism of sheaves. $\square$

Proposition 5. *Let k be algebraically closed. Then $\mathrm{Ext}^1_k(\mathbb{G}_m, K)$ vanishes for every smooth, connected, commutative affine group scheme K over k. In particular, $\mathrm{Ext}^1_k(\mathbb{G}_m, p_*\mathbb{G}_{m,X}) = 0$ for every proper scheme $p : X \to k$.*

Proof. The group scheme K has a filtration consisting of schemes of the form $\mathbb{G}_m$ and $\mathbb{G}_a$. This is shown in [SGA3, Exposé XVII, proposition 4.1.1 and théorème 7.2.1] for the flat topology, and follows with [Mil80, Theorem 3.9] for the étale topology. Looking at long exact sequences, it suffices to show $\mathrm{Ext}^1_k(\mathbb{G}_m, \mathbb{G}_m) = \mathrm{Ext}^1_k(\mathbb{G}_m, \mathbb{G}_a) = 0$. Now $H^1_{\mathrm{et}}(\mathbb{G}_m, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{G}_m) = 0$, and $H^1_{\mathrm{et}}(\mathbb{G}_m, \mathbb{G}_a) = 0$ since higher cohomology of a quasi-coherent sheaf on an affine scheme vanishes. Thus, by Lemma 4, it suffices to show that the corresponding extension groups in the category of group schemes vanish. However, there are no extensions of groups schemes between $\mathbb{G}_m$ and $\mathbb{G}_m$, or $\mathbb{G}_m$ and $\mathbb{G}_a$ by [SGA3, Exposé XVII, théorème 5.1.1].

If X is proper over k, then by the Stein factorization, we can write p as the composition $X \xrightarrow{g} L \xrightarrow{h} k$, where L is the spectrum of an Artin k-algebra, and $g_*\mathcal{O}_X = \mathcal{O}_L$. In particular, $g_*\mathbb{G}_m = \mathbb{G}_m$, and we can assume that X is finite of some degree m over k. In this case, the Weil-restriction $p_*\mathbb{G}_m$ is an open subscheme of $p_*\mathbb{A}^1 \cong \mathbb{A}^m$ by [BLR90, Propostion 7.6/2], hence smooth, connected and affine, and we can apply the above. $\square$

Remark. The hypothesis that k is algebraically closed is necessary. For example, if S^1 is the anisotropic form of $\mathbb{G}_m$, then $\mathrm{Ext}^1_{\mathbb{R}}(\mathbb{G}_m, S^1) = \mathbb{Z}/2$, generated by the Weil restriction $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$.

Proof of Theorem 1. Since the maps defined below are natural, we can assume that k is algebraically closed. We can also assume that X is reduced, because $H^1_{\mathrm{et}}(X, \mathbb{Z}) \xrightarrow{\sim} H^1_{\mathrm{et}}(X^{\mathrm{red}}, \mathbb{Z})$, and because the map $\mathrm{Pic}_X \to \mathrm{Pic}_{X^{\mathrm{red}}}$ has unipotent kernel and cokernel [Oor62, p. 9, Corollary]. It suffices to calculate $\mathrm{Hom}_k(\mathbb{G}_m, \mathrm{Pic}_X)$, because there are no homomorphisms from $\mathbb{G}_m$ to commutative group schemes other than tori [Oor66]. By Yoneda's lemma, the latter group is isomorphic to the group of homomorphisms of sheaves on the big (étale) site $\mathrm{Hom}_k(\mathbb{G}_m, R^1p_*\mathbb{G}_m)$. From the duality of diagonal group schemes and locally constructible sheaves, we obtain

$\mathcal{H}\mathfrak{om}_X(\mathbb{G}_m,\mathbb{G}_m)\cong\mathcal{H}\mathfrak{om}_X(\mathbb{Z},\mathbb{Z})\cong\mathbb{Z}$ and $\mathcal{E}\mathfrak{xt}^1_X(\mathbb{G}_m,\mathbb{G}_m)=\mathcal{E}\mathfrak{xt}^1_X(\mathbb{Z},\mathbb{Z})=0$, hence the spectral sequence [Mil80, III, Theorem 1.22]

$$E_2^{s,t}=H^s_{\mathrm{et}}(X,\mathcal{E}\mathfrak{xt}^t_X(\mathbb{G}_m,\mathbb{G}_m))\Rightarrow\operatorname{Ext}^{s+t}_X(\mathbb{G}_m,\mathbb{G}_m)$$

gives an isomorphism $H^1_{\mathrm{et}}(X,\mathbb{Z})\cong\operatorname{Ext}^1_X(\mathbb{G}_m,\mathbb{G}_m)$. The Leray spectral sequence

$$E_2^{s,t}=\operatorname{Ext}^s_k(\mathbb{G}_m,R^tp_*\mathbb{G}_m)\Rightarrow\operatorname{Ext}^{s+t}_X(\mathbb{G}_m,\mathbb{G}_m)\tag{2}$$

gives an exact sequence

$$0\to\operatorname{Ext}^1_k(\mathbb{G}_m,p_*\mathbb{G}_m)\to\operatorname{Ext}^1_X(\mathbb{G}_m,\mathbb{G}_m)\xrightarrow{\beta_X}\operatorname{Hom}_k(\mathbb{G}_m,R^1p_*\mathbb{G}_m)\xrightarrow{\delta_X}\operatorname{Ext}^2_k(\mathbb{G}_m,p_*\mathbb{G}_m).$$

By Proposition 5, β_X is injective. If X is normal, then $\operatorname{Hom}_k(\mathbb{G}_m,R^1p_*\mathbb{G}_m)=0$ by [Gro62, théorème 2.1], hence δ_X is the zero-map. We will show by induction on the dimension of X that δ_X is the zero-map in general. Let $X'\xrightarrow{f}X$ be the normalization of X, and consider the conductor square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}\tag{3}$$

where Z is the closed subscheme (of smaller dimension) where f is not an isomorphism. It suffices to show that the lower horizontal map in the following commutative diagram is injective.

$$\begin{tikzcd} \operatorname{Hom}_k(\mathbb{G}_m,R^1p_*\mathbb{G}_{m,X}) \arrow[r] \arrow[d, "\delta_X"'] & \operatorname{Hom}_k(\mathbb{G}_m,R^1p_*i_*\mathbb{G}_{m,Z})\oplus\operatorname{Hom}_k(\mathbb{G}_m,R^1p_*f_*\mathbb{G}_{m,X'}) \arrow[d, "0"'] \\ \operatorname{Ext}^2_k(\mathbb{G}_m,p_*\mathbb{G}_{m,X}) \arrow[r] & \operatorname{Ext}^2_k(\mathbb{G}_m,p_*i_*\mathbb{G}_{m,Z})\oplus\operatorname{Ext}^2_k(\mathbb{G}_m,p_*f_*\mathbb{G}_{m,X'}) \end{tikzcd}$$

This will follow if $\operatorname{Ext}^1_k(\mathbb{G}_m,C)=0$, where C is the cokernel of the injection $p_*\mathbb{G}_{m,X}\to p_*i_*\mathbb{G}_{m,Z}\oplus p_*f_*\mathbb{G}_{m,X'}$. However, this cokernel is a quotient of a commutative, smooth, connected linear algebraic group, hence is a commutative, smooth, connected linear group itself, and we can apply Proposition 5. □

Remark. The example in [Gei06, Proposition 8.2] shows that the map $H^i_{\mathrm{et}}(\bar X,\mathbb{Z})\to\operatorname{Ext}^i_{\bar X}(\mathbb{G}_m,\mathbb{G}_m)$ is not an isomorphism for $i\geqslant 2$. One can ask whether it is an isomorphism if one replaces $H^i_{\mathrm{et}}(\bar X,\mathbb{Z})$ by the eh-cohomology group $H^i_{\mathrm{eh}}(\bar X,\mathbb{Z})$ of [Gei06].

Example. If X is the node over an algebraically closed field, then $H^1_{\mathrm{et}}(X,\mathbb{Z})\cong\mathbb{Z}$, and $T_X\cong\mathbb{G}_m$. Let X be a node with non-rational tangent slopes at the singular point. Base changing to the algebraic closure, one sees that $H^1_{\mathrm{et}}(\bar X,\mathbb{Z})\cong\mathbb{Z}$, with Galois group acting as multiplication by -1, hence T_X is an anisotropic torus.

Using the theorem, we are able to recover the torsion of T_X, A_X and the diagonalizable part of $NS_X:=\operatorname{Pic}_X/\operatorname{Pic}^{0,\mathrm{red}}_X$ in terms of étale cohomology.

COROLLARY 6. *Let X be proper over a perfect field k. Then we have canonical isomorphisms:*

$$H^1_{\mathrm{et}}(\bar X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z}\cong\operatorname{colim}\operatorname{Hom}_{\bar k}(\mu_m,T_X);$$
$$\operatorname{Div}({}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar X,\mathbb{Z}))\cong\operatorname{colim}\operatorname{Hom}_{\bar k}(\mu_m,A_X);$$
$${}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar X,\mathbb{Z})/\operatorname{Div}\cong\operatorname{colim}\operatorname{Hom}_{\bar k}(\mu_m,NS_X).$$

Proof. Taking the colimit of the isomorphism $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}/m) \cong \mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X)$ of [Mil80, Proposition 4.16] or [Ray70, § 6.2], we obtain $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X)$. Since there are no extensions of group schemes of μ_m by a smooth commutative group scheme G, and since $H^1_{\mathrm{et}}(\mu_{m,\bar{k}}, G) = 0$, we can apply Lemma 4 to conclude that $\mathrm{Ext}^1_{\bar{k}}(\mu_m, G) = 0$. In particular, Theorem 1 implies that $\mathrm{Hom}_{\bar{k}}(\mu_m, T_X) \cong \mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, T_X)/m \cong H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z})/m$. Consider the following commutative diagram.

$$\begin{tikzcd}
\mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, T_X) \arrow[r, equal] \arrow[d] & H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[d] & \\
\mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}^{0,\mathrm{red}}_X) \arrow[r] \arrow[d] & \mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, \mathrm{Pic}_X) \arrow[r] \arrow[d] & \mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, NS_X) \arrow[d, equal] \\
\mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, A_X) \arrow[r, "f"] & {}_{\mathrm{tor}}H^2_{\mathrm{et}}(\bar{X}, \mathbb{Z}) \arrow[r] & \mathrm{coker}\, f
\end{tikzcd}$$

The middle column is the short exact coefficient sequence, and the left column and middle row are short exact by the above. A diagram chase shows that f is injective, and the right vertical map is an isomorphism. The corollary follows because the group $\mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, A_X)$ is divisible and the group $\mathrm{colim}\,\mathrm{Hom}_{\bar{k}}(\mu_m, NS_X)$ is finite. □

The above result should be compared with [Gei09, Proposition 6.2], where we show that, for every proper scheme over an algebraically closed field, the higher Chow group of zero-cycles $CH_0(X, 1, \mathbb{Z}/m)$ is the Pontrjagin dual of $H^1_{\mathrm{et}}(X, \mathbb{Z}/m)$. This implies a short exact sequence

$$0 \to {}_{\mathrm{tor}}A^t_X(k) \to CH_0(X, 1, \mathbb{Q}/\mathbb{Z}) \to \chi(T_X) \otimes \mathbb{Q}/\mathbb{Z} \to 0,$$

for A^t_X the dual abelian variety of A_X, and $\chi(T_X)$ the character module of T_X. However, in this case the contribution from the torus and from the abelian variety are not compatible with the coefficient sequence

$$0 \to CH_0(X, 1) \otimes \mathbb{Q}/\mathbb{Z} \to CH_0(X, 1, \mathbb{Q}/\mathbb{Z}) \to {}_{\mathrm{tor}}CH_0(X) \to 0$$

as in Corollary 6.

Looking at tangent spaces, the previous corollary gives a dimension formula.

Corollary 7. *Let l be a prime different from* char k. *Then*

$$\dim_k H^1(X, \mathcal{O}_X) = \dim U_X + \dim_k Lie(NS^0_X) + \tfrac{1}{2}\,\mathrm{rank}\, H^1_{\mathrm{et}}(X, \mathbb{Z}) + \tfrac{1}{2}\,\mathrm{corank}_l\, H^1_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l).$$

3. The unipotent part

Let $N\,\mathrm{Pic}(X) := \ker\big(\mathrm{Pic}(X[t]) \xrightarrow{0^*} \mathrm{Pic}(X)\big)$. Since $t \mapsto 0t$ induces $x \mapsto 0x$ on the typical part, $\mathrm{Pic}(X[t])_{[1]}$ is a subgroup of $N\,\mathrm{Pic}(X)$. In [Wei91], Weibel shows that for every scheme there is a direct sum decomposition

$$\mathrm{Pic}(X[t, t^{-1}]) \cong \mathrm{Pic}(X) \oplus N\,\mathrm{Pic}(X) \oplus N\,\mathrm{Pic}(X) \oplus H^1_{\mathrm{et}}(X, \mathbb{Z}).$$

Proof of Theorem 2. We show first that $N\,\mathrm{Pic}(X) = \ker(U_X(\mathbb{A}^1) \to U_X(k))$. Since there are no non-trivial morphisms of schemes from $\mathbb{A}^1_k$ to an abelian variety, a torus, an infinitesimal group, or a discrete group, we see that the kernel of $U_X(\mathbb{A}^1_k) \to U_X(k)$ agrees with the kernel of $\mathrm{Pic}_X(\mathbb{A}^1_k) \to \mathrm{Pic}_X(k)$. Let $p : X \to k$ and $p' : X \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the structure morphisms. Then the Leray spectral sequence gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m) \arrow[r] \arrow[d] & \mathrm{Pic}(X \times \mathbb{A}^1_k) \arrow[r] \arrow[d] & \mathrm{Pic}_X(\mathbb{A}^1_k) \arrow[r] \arrow[d] & H^2_{\mathrm{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m) \arrow[d] \\ 0 \arrow[r] & H^1_{\mathrm{et}}(k, p_*\mathbb{G}_m) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}_X(k) \arrow[r] & H^2_{\mathrm{et}}(k, p_*\mathbb{G}_m) \end{tikzcd}$$

and it suffices to show that the outer vertical maps are isomorphisms. Let $X \xrightarrow{g} L \to k$ be the Stein factorization of p, such that $\mathcal{O}_L \cong g_*\mathcal{O}_X$ and L is the spectrum of an Artinian k-algebra. Since $\mathbb{A}^1_k \to k$ is flat, $p'_*\mathcal{O}_{X\times\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k} \otimes_k p_*\mathcal{O}_X$, and $X \times \mathbb{A}^1_k \xrightarrow{g'} \mathbb{A}^1_L \to \mathbb{A}^1_k$ is the Stein factorization of p'. We obtain

$$H^i_{\mathrm{et}}(\mathbb{A}^1_k, p'_*\mathbb{G}_m) \cong H^i_{\mathrm{et}}(\mathbb{A}^1_L, g'_*\mathbb{G}_m) \cong H^i_{\mathrm{et}}(\mathbb{A}^1_L, \mathbb{G}_m),$$

and $H^i_{\mathrm{et}}(k, p_*\mathbb{G}_m) \cong H^i_{\mathrm{et}}(L, \mathbb{G}_m)$. Hence, the terms on the left vanish because $\mathrm{Pic}(L) = \mathrm{Pic}(\mathbb{A}^1_L) = 0$. To show that $H^2_{\mathrm{et}}(\mathbb{A}^1_L, \mathbb{G}_m) \to H^2_{\mathrm{et}}(L, \mathbb{G}_m)$ is an isomorphism, we can assume that L is a local Artinian k-algebra with (perfect) residue field k'. By [Mil80, III, Remark 3.11] we are reduced to showing that $H^2_{\mathrm{et}}(\mathbb{A}^1_{k'}, \mathbb{G}_m) \to H^2_{\mathrm{et}}(k', \mathbb{G}_m)$ is an isomorphism, and this can be found in [Mil80, IV, Exercise 2.20].

Given an element x of $N\,\mathrm{Pic}(X)$, the condition $x \in \mathrm{Pic}(X[t])_{[1]}$ implies that the corresponding $f \in \mathrm{Hom}_{\mathrm{Sch}}(\mathbb{A}^1, U_X)$ satisfies $f(nx) = nf(x)$ for all n. If k has characteristic zero, then $U_X \cong \mathbb{G}_a^r$ for some r, and the map $f : \mathbb{G}_a \to U_X$ corresponds to a morphism of Hopf algebras $f^* : k[x_1, \ldots, x_r] \to k[t]$. If $f^*(x_i) = \sum_j a_j t^j$, then

$$\sum_j a_j (nt)^j = f^*(nx_i) = nf^*(x_i) = n \sum_j a_j t^j$$

only if $n^j = n$ for all n, hence $j = 1$. □

Example. If k has characteristic p, then $t \mapsto t^{2p-1}$ induces a map $\mathbb{G}_a \to \mathbb{G}_a$ which is compatible with multiplication by n, but not a homomorphism of group schemes.

Corollary 8. *We have $U_X = 0$ if and only if $N\,\mathrm{Pic}(X) = 0$.*

Proof. This follows from $N\,\mathrm{Pic}(X) = \ker\big(U_X(\mathbb{A}^1) \to U_X(k)\big)$, because any unipotent, connected, smooth affine group is an affine space as a scheme, hence admits a non-trivial morphism from $\mathbb{A}^1$ which sends 0 to 0 if it is non-trivial. □

The kernel and cokernel of $\mathrm{Pic}_X \to \mathrm{Pic}_{X^{\mathrm{red}}}$ has been described in [Oor62], hence from now we will assume that X is reduced. If X^+ is the semi-normalization of X, then the map $\mathcal{O}_X \to \mathcal{O}_{X^+}$ is an injection of sheaves on the same topological space. For X^+ reduced and semi-normal, $N\,\mathrm{Pic}(X^+) = 0$ by Traverso's theorem [Tra70] together with [Wei91, Theorem 4.7]. Hence, the corollary implies that $U_{X^+} = 0$, and that

$$U_X = \ker(\mathrm{Pic}^{0,\mathrm{red}}_X \to \mathrm{Pic}^{0,\mathrm{red}}_{X^+}).$$

(For curves, this recovers [BLR90, Proposition 9.2/10].) Indeed, by Corollary 6, the map $\mathrm{Pic}^{0,\mathrm{red}}_X \to \mathrm{Pic}^{0,\mathrm{red}}_{X^+}$ induces an isomorphism on the torus and abelian variety part, because it induces an isomorphism on étale cohomology.

https://doi.org/10.1112/S0010437X08003710 Published online by Cambridge University Press

Proposition 9. *Let $p : X \to k$ be reduced and proper, and $f : X' \to X$ be a bijection which induces an isomorphism on residue fields.*

(a) *If X' is reduced, then there is an isomorphism $p_*\mathbb{G}_{m,X} \cong p_*\mathbb{G}_{m,X'}$.*

(b) *We have that $p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$ is a smooth, connected, unipotent commutative group scheme, and for $j > 0$, $R^j p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) = 0$.*

Proof. (a) Since any scheme T over k is flat, we have by flat base change $R^j q_*\mathcal{O}_{X_T} = H^j(X, \mathcal{O}_X) \otimes_k \mathcal{O}_T$, where $q : X_T \to T$ is the projection. In particular,

$$p_*\mathbb{G}_{m,X}(T) := \Gamma(X \times T, \mathcal{O}_{X\times T})^\times = (\Gamma(X, \mathcal{O}_X) \otimes \Gamma(T, \mathcal{O}_T))^\times,$$

and it suffices to show that $\Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'})$. Since $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}})^{\mathrm{Gal}(\bar{k}/k)} = \Gamma(X, \mathcal{O}_X)$, we can assume that k is algebraically closed and that X is connected. But, $X' \to X$ is a universal homeomorphism, so that $\Gamma(X, \mathcal{O}_X) \cong \Gamma(X', \mathcal{O}_{X'}) \cong k$ because X and X' are reduced, proper, connected, and have a k-rational point.

(b) We proceed by induction on the dimension of X. If $\dim X = 0$, then $R^j p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) = 0$ for $j > 0$ because p_* is exact for finite maps, and to show that $p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X})$ is a smooth, connected, unipotent group scheme, we can assume that $X = \operatorname{Spec} L$ is a finite field extension of k, and that $X' = \operatorname{Spec} k'$ is a local Artin algebra over L with residue field L. Since the Weil restriction is compatible with extensions, and since the Weil restriction of $\mathbb{G}_a$ along a finite flat map of degree m is $\mathbb{G}_a^m$, we can assume that $L = k$ and $k' = k \oplus N$ with N a nilpotent ideal. We can find a sequence of local Artin algebras k_i such that $k_0 = k$, $k_n = k'$ and $k_{i+1} = k_i \oplus N_i$ with $N_i^2 = 0$. Since an extension of smooth, connected, unipotent commutative group schemes is of the same type, we can assume that $k' = k \oplus N$ with $N^2 = 0$. For a given scheme $T \to k$,

$$\begin{aligned}\Gamma(T \times_k k', \mathcal{O}_{T\times_k k'})^\times &\cong (\Gamma(T, \mathcal{O}_T) \otimes_k k')^\times \\ &\cong (\Gamma(T, \mathcal{O}_T) \oplus \Gamma(T, \mathcal{O}_T) \otimes_k N)^\times \cong \Gamma(T, \mathcal{O}_T)^\times \oplus \Gamma(T, \mathcal{O}_T) \otimes_k N,\end{aligned}$$

hence the cokernel of $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k'}$ is represented by the vector group N.

If $\dim X > 0$, we consider the conductor square (3). By hypothesis, $Z' \to Z$ is a bijection of schemes of lower dimension which induces isomorphism on residue fields, and Z is still reduced. From the injectivity of $\mathbb{G}_{m,Z} \to \mathbb{G}_{m,Z'}$ and the short exact sequence

$$0 \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,X'} \oplus i_*\mathbb{G}_{m,Z} \to i_*\mathbb{G}_{m,Z'} \to 0$$

we conclude that the map $\mathbb{G}_{m,X} \to \mathbb{G}_{m,X'}$ is injective with cokernel equal to the cokernel of $i_*\mathbb{G}_{m,Z} \to i_*\mathbb{G}_{m,Z'}$. The proposition follows because

$$R^j p_*(\mathbb{G}_{m,X'}/\mathbb{G}_{m,X}) = R^j p_*(i_*\mathbb{G}_{m,Z'}/i_*\mathbb{G}_{m,Z}) = R^j(pi)_*(\mathbb{G}_{m,Z'}/\mathbb{G}_{m,Z}). \qquad \square$$

Proof of Theorem 3. (a) This follows from the exact sequence of étale sheaves

$$0 \to p_*\mathbb{G}_{m,X} \to p_*\mathbb{G}_{m,X^+} \to p_*(\mathbb{G}_{m,X^+}/\mathbb{G}_{m,X}) \to \operatorname{Pic}_X \to \operatorname{Pic}_{X^+} \to R^1 p_*(\mathbb{G}_{m,X^+}/\mathbb{G}_{m,X})$$

and Proposition 9.

(b) Recall that $q : X_T \to T$, and consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X\times T}) \arrow[r] \arrow[d] & \operatorname{Pic}(X \times T) \arrow[r] \arrow[d, "r"'] & \operatorname{Pic}_{X/k}(T) \arrow[r] \arrow[d, "s"'] & H^2_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X\times T}) \arrow[d] \\ 0 \arrow[r] & H^1_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X^+\times T}) \arrow[r] & \operatorname{Pic}(X^+ \times T) \arrow[r] & \operatorname{Pic}_{X^+/k}(T) \arrow[r] & H^2_{\mathrm{et}}(T, q_*\mathbb{G}_{m,X^+\times T}) \end{tikzcd}$$

https://doi.org/10.1112/S0010437X08003710 Published online by Cambridge University Press

Since $q_*\mathbb{G}_{a,X\times T} = H^0(X,\mathcal{O}_X)\otimes\mathcal{O}_T = H^0(X^+,\mathcal{O}_{X^+})\otimes\mathcal{O}_T = q_*\mathbb{G}_{a,X^+\times T}$ is an isomorphism as in Proposition 9(a), the outer maps are isomorphisms, and it suffices to calculate $\ker r$. Let $Y = X\times T$ and $Y' = X^+\times T$, and consider the tautological map

$$f : \{\mathcal{O}_Y\text{-line bundles } \mathcal{L}\subseteq\mathcal{O}_{Y'} \text{ which are invertible in } \mathcal{O}_{Y'}\}\to \operatorname{Pic}(Y).$$

It suffices to show the following statements:

(a) the image of f is contained in $\ker(\operatorname{Pic}(Y)\to\operatorname{Pic}(Y'))$;

(b) f surjects onto $\ker(\operatorname{Pic}(Y)\to\operatorname{Pic}(Y'))$;

(c) f is injective.

(a) We claim that the map $\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'}\to\mathcal{O}_{Y'}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'}\xrightarrow{\mu}\mathcal{O}_{Y'}$ is an isomorphism. We can check this on an affine covering, and in this case it is proved in [RS93, Lemma 2.2(4)].

(b) Let $\mathcal{L}\in\operatorname{Pic}(Y)$ with $\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'}\cong\mathcal{O}_{Y'}$. Since $\mathcal{L}$ is flat, we obtain an injection $\mathcal{L} = \mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y\to\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'}\cong\mathcal{O}_{Y'}$. We claim that the inverse of $\mathcal{L}$ in $\mathcal{O}_{Y'}$ is the sheaf associated to the presheaf $U\mapsto\{x\in\mathcal{O}_{Y'}(U)\mid x\mathcal{L}(U)\subseteq\mathcal{O}_Y(U)\}\subseteq\mathcal{O}_{Y'}(U)$. This can be checked on an affine covering, and then it is [RS93, Lemma 2.2(2)].

(c) Let $\mathcal{L}$ and $\mathcal{L}'$ be subsheaves of $\mathcal{O}_{Y'}$ which are invertible in $\mathcal{O}_{Y'}$ and isomorphic as abstract invertible sheaves. Multiplying with the inverse of $\mathcal{L}'$ inside $\mathcal{O}_{Y'}$, it suffices to show that if $\mathcal{L}$ is a subsheaf of $\mathcal{O}_{Y'}$, and $f:\mathcal{O}_Y\to\mathcal{L}$ an isomorphism, then $\mathcal{L}=\mathcal{O}_Y\subseteq\mathcal{O}_{Y'}$. However, $f(1)$ is a global unit of $\mathcal{O}_{Y'}(Y)$ and, by Proposition 9(a), $\mathcal{O}_Y(Y)^\times = \mathcal{O}_{Y'}(Y)^\times$. Hence, $\mathcal{L} = f(1)^{-1}\mathcal{L} = \mathcal{O}_Y$. □

Acknowledgements

This paper was written while the author was visiting T. Saito at the University of Tokyo, whom we thank for his hospitality. We are indebted to G. Faltings for pointing out a mistake in a previous version, and the referee, whose comments helped to improve the exposition and to give more concise proofs.

References

BLR90 S. Bosch, W. Lütkebohmert and M. Raynaud, *Neron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21 (Springer, Berlin, 1990).

Che60 C. Chevalley, *Une démonstration d'un théorème sur les groupes algébriques*, J. Math. Pures Appl. (9) **39** (1960), 307–317.

Gei06 T. Geisser, *Arithmetic cohomology over finite fields and special values of ζ-functions*, Duke Math. J. **133** (2006), 27–57.

Gei09 T. Geisser, *Duality via cycle complexes*, Ann. of Math (2), to appear, arXiv:math.AG/0608456.

Gro62 A. Grothendieck, *Technique de descente et théoremes d'existence en géometrie algébrique. VI. Les schémas de Picard: propriétés générales,* in *Seminaire Bourbaki, 1961/62*, Exposé no. 236 (Société Mathématique de France, Paris, 1962).

Mil80 J.S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33 (Princeton University Press, Princeton, NJ, 1980).

Mur64 J.P. Murre, *On contravariant functors from the category of pre-schemes over a field into the category of abelian groups (with an application to the Picard functor)*, Publ. Math. Inst. Hautes Études Sci. **23** (1964), 5–43.

Oor62 F. Oort, *Sur le schéma de Picard*, Bull. Soc. Math. France **90** (1962), 1–14.

Oor66 F. Oort, *Commutative group schemes*, Lecture Notes in Mathematics, vol. 15 (Springer, Berlin, 1966).

Ray70 M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. Inst. Hautes Études Sci. **38** (1970), 27–76.

RS93 L. Roberts and B. Singh, *Subintegrality, invertible modules and the Picard group*, Compositio Math. **85** (1993), 249–279.

SGA3 M. Raynaud, *Groups algébriques unipotents. Extensions entre groupes unipotents et groupes de type multiplicative*, in *Séminarie de Géometrie Algébrique du Bois Marie 1963/64 (SGA3)*, Lecture Notes in Mathematics, vol. 152 (Springer, Berlin, 1970).

Tra70 C. Traverso, *Seminormality and Picard group*, Ann. Sc. Norm. Super. Pisacl. Sci. (3) **24** (1970), 585–595.

Wei91 C. Weibel, *Pic is a contracted functor*, Invent. Math. **103** (1991), 351–377.

Thomas Geisser geisser@usc.edu
Department of Mathematics, University of Southern California, 3620 S Vermont Av. KAP 108, Los Angeles, CA 90089-2532, USA

https://doi.org/10.1112/S0010437X08003710 Published online by Cambridge University Press