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TRIPLES AND LOCALIZATIONS(1)

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1. **Introduction.** Let A be a ring (associative) with unity, and let ${}_A\mathcal{M}$ denote the category of unital left A-modules. If $\mathcal{G}$ is a strongly complete Serre class in ${}_A\mathcal{M}$, then (see [3], and also [6]) there is an exact functor $S: {}_A\mathcal{M} \to \mathcal{C}$, where $\mathcal{C}$ is the quotient category ${}_A\mathcal{M}/\mathcal{G}$, and $\mathcal{C}$ is an abelian category. Denoting the opposite ring of $\mathrm{Hom}_{\mathcal{C}}(SA, SA)$ by C, the localization functor $L: {}_A\mathcal{M} \to {}_C\mathcal{M}$ is defined by $L = \mathrm{Hom}_{\mathcal{C}}(SA, S\_\_)$. The functor S induces a ring homomorphism from A to C, and this in turn induces a forgetful functor $V: {}_C\mathcal{M} \to {}_A\mathcal{M}$. It is shown in [3] that S is left adjoint to $V\cdot\mathrm{Hom}_{\mathcal{C}}(SA, \_\_)$, and that the natural transformation from $S\cdot V\cdot\mathrm{Hom}_{\mathcal{C}}(SA, \_\_)$ to $I_{\mathcal{C}}$ associated with adjointness is in fact an equivalence. Furthermore, if $T': \mathcal{B} \to {}_A\mathcal{M}$ and $S': {}_A\mathcal{M} \to \mathcal{B}$ are functors, where (i) $\mathcal{B}$ is abelian, (ii) S' is exact and left adjoint to T', and (iii) the associated natural transformation from $S'T'$ to $I_{\mathcal{B}}$ is an equivalence, then $\mathcal{B}$ is equivalent to $\mathcal{C}$.

The purpose of this paper is to give an alternative account of this theory which is more "module-theoretic", i.e. free of reference to the category ${}_A\mathcal{M}/\mathcal{G}$. We shall see that a category defined by Eilenberg and Moore in [2] serves as a suitable replacement for ${}_A\mathcal{M}/\mathcal{G}$. This category has the advantage of being isomorphic (not just equivalent) to an easily defined full subcategory of ${}_A\mathcal{M}$.

The account of localization theory presented here is completely independent of the results of Gabriel [3] and of Walker and Walker [6], except when we consider the equivalence of the two approaches. However, familiarity with §3 and §4 of Goldman [4], and with §2 of Eilenberg and Moore [2] is assumed.

2. **Quotient modules.** A class $\mathcal{G}$ of modules is called a *strongly complete Serre class* [6] if it is closed under formation of submodules, homomorphic images, extensions, and arbitrary direct sums. (These classes are also called *hereditary torsion classes*—see [1].) Given a strongly complete Serre class, for any module X define σX to be the (unique) largest submodule X' of X for which X' is in $\mathcal{G}$. Then σ gives rise to an idempotent kernel functor (as defined in [4]). Conversely, if σ is an idempotent kernel functor, the class $\mathcal{G}$ of modules X satisfying $\sigma X = X$ is a strongly complete Serre class.

Suppose we are given a strongly complete Serre class $\mathcal{G}$. Denoting $X/\sigma X$ by $\bar{X}$ for any module X, let $Q_\sigma(X)$ be the largest submodule of an injective hull $E(\bar{X})$ of $\bar{X}$ which satisfies $Q_\sigma(X) \supseteq \bar{X}$ and $Q_\sigma(X)/\bar{X} = \sigma(E(\bar{X})/\bar{X})$. Then $Q_\sigma(X)$ is a faithfully σ-injective module ([4], §3). We have a map $\eta X: X \to Q_\sigma(X)$, defined as the

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composition of the canonical map from X to $\bar{X}$ with the embedding map of $\bar{X}$ into $Q_\sigma(X)$. Also, given a map $f: X \to Y$, there is a unique map $Q_\sigma(f)$ from $Q_\sigma(X)$ to $Q_\sigma(Y)$ satisfying $\eta Y \cdot f = Q_\sigma(f) \cdot \eta X$. In this way we obtain a covariant functor $Q_\sigma: {}_A\mathcal{M} \to {}_A\mathcal{M}$ and a natural transformations $\eta: I_{{}_A\mathcal{M}} \to Q_\sigma$. It is easily verified that the map $\eta Q_\sigma(X)$ from $Q_\sigma(X)$ to $Q_\sigma^2(X)$ is an isomorphism, and that this map coincides with the map $Q_\sigma(\eta X)$. Also [4, Theorem 3.9], the functor Q_σ is left exact. Furthermore, for any module X, $\sigma(X) = \ker \eta X$ and $\mathcal{G}$ is the class of modules for which $Q_\sigma(X) = 0$.

We shall define (G, η) to be a *localization functor* on ${}_A\mathcal{M}$ if $G: {}_A\mathcal{M} \to {}_A\mathcal{M}$ is a covariant left exact functor, and η is a natural transformation from $I_{{}_A\mathcal{M}}$ to G satisfying the condition that the natural transformations $G\eta$ and ηG from G to G^2 are the same, and are in fact natural equivalences. With this definition, the results from [4] cited above can be summarized as

THEOREM 2.1. *If σ is an idempotent kernel functor, then there is a localization functor (G, η) for which $\sigma(X) = \ker \eta X$, and for which $\mathcal{G}$ is the class of modules X for which $GX = 0$.*

We now show that every localization functor can be realized in this fashion.

In any category with kernels, define a morphism $f: X \to Y$ to be an *essential morphism* if, given a morphism $g: Y \to Z$, g is a monomorphism if $\ker f = \ker gf$. For the category ${}_A\mathcal{M}$, this is equivalent to: $\operatorname{im} f$ is an essential submodule of Y.

LEMMA 2.2. *If (G, η) is a localization functor, $X \xrightarrow{\eta X} GX$ is an essential morphism for every X.*

Proof. Suppose that $g: GX \to Y$ satisfies $\ker \eta X = \ker g \cdot \eta X$, and let this kernel be $i: K \to X$. Since G is left exact,

$$Gi = \ker G(\eta X) = \ker Gg \cdot G\eta X.$$

However, $G\eta$ is an equivalence, so it follows that $\ker Gg = 0$, and Gg is a monomorphism. Suppose that $j: Z \to GX$ is $\ker g$. We then have the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z & \xrightarrow{\ j\ } & GX & \xrightarrow{\ g\ } & Y \\
 & & \downarrow{\scriptstyle \eta Z} & & \downarrow{\scriptstyle \eta GX} & & \downarrow{\scriptstyle \eta Y} \\
0 & \longrightarrow & GZ & \xrightarrow{\ Gj\ } & G^2X & \xrightarrow{\ Gg\ } & GY
\end{array}$$

Since $Gj = 0$ and ηG is an equivalence, $j = 0$ and hence g is a monomorphism.

COROLLARY 2.3. *If (G, η) is a localization functor, then $GX = 0$ if and only if $\eta X = 0$.*

Proof. This follows trivially from the fact that $\operatorname{im} \eta X$ is an essential submodule of GX.

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THEOREM 2.4. *Let (G, η) be a localization functor. Then*

(1) *the class $\mathcal{G}$ of modules X for which $GX=0$ is a strongly complete Serre class;*

(2) *if σ is the corresponding idempotent kernel functor, then $\sigma(X)=\ker \eta X$ for each module X;*

(3) *for each X, $GX \cong Q_\sigma(X)$ in a natural way.*

Proof. (1) We must show that $\mathcal{G}$ is closed under (a) homomorphic images, (b) submodules, (c) extensions, and (d) arbitrary direct sums. Now (b) and (c) follow immediately from the left exactness of G. To establish (a), suppose $X \xrightarrow{f} Y \to 0$ is exact, and that X is in $\mathcal{G}$. Then $\eta Y \cdot f = Gf \cdot \eta X = 0$, and so (since f is epic) $\eta Y = 0$. By the above corollary, $GY=0$ as desired. Finally, let $Y = \bigoplus_{\alpha \in \Lambda} M_\alpha$, where each M_α is in $\mathcal{G}$, and let $q_\alpha : M_\alpha \to Y$ be the canonical embedding. For each α, $\eta Y \cdot q_\alpha = Gq_\alpha \cdot \eta M_\alpha = 0$, and so, by the universal property of direct sums, $\eta Y = 0$ and Y is in $\mathcal{G}$.

(2) Let $i : K \to X$ be $\ker \eta X$. Since G is left exact, $\ker G\eta X$ is $Gi : GK \to GX$, and so, since $G\eta$ is an equivalence, $GK=0$ and K is in $\mathcal{G}$. Thus $\operatorname{im} i \subseteq \sigma(X)$. Now let $j : \sigma(X) \to X$ be the inclusion mapping. Since $\sigma(X)$ is in $\mathcal{G}$, $\eta\sigma(X)=0$, so $0 = Gj \cdot \eta\sigma(X) = \eta X \cdot j$. Therefore $\sigma(X) \subseteq \ker \eta X = \operatorname{im} i$, and the desired equality results.

(3) Let $Y = \operatorname{im} \eta X$. In view of Lemma 2.2, we have

$$Y \subseteq GX \subseteq E(Y)$$

where $E(Y)$ is an injective hull of Y. Recalling Goldman's construction of $Q_\sigma(X)$, the desired isomorphism will follow if we can show $GX/Y = \sigma(E(Y)/Y)$, and this in turn will follow if we can prove (i) GX/Y is in $\mathcal{G}$, and (ii) $\sigma(E(Y)/GX)=0$. Let $Z = GX/Y$, and let $p : GX \to Z$ be the canonical epimorphism. Then we have the commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\eta X} & GX & \xrightarrow{p} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta X} & & \downarrow{\scriptstyle \eta GX} & & \downarrow{\scriptstyle \eta Z} & & \\
GX & \xrightarrow{G\eta X} & G^2X & \xrightarrow{Gp} & GZ & &
\end{array}$$

whose first row is exact. Then $0 = G(p \cdot \eta X) = Gp \cdot G\eta X$ and, since $G\eta$ is an equivalence, $Gp=0$. Thus $\eta Z \cdot p = Gp \cdot \eta GX = 0$ and so $\eta Z = 0$ and Z is in $\mathcal{G}$. This establishes (i). To prove (ii), let $W = E(Y)/GX$. The following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & GX & \xrightarrow{i} & E(Y) & \longrightarrow & W & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \eta GX} & & \downarrow{\scriptstyle \eta E(Y)} & & \downarrow{\scriptstyle \eta W} & & \\
0 & \longrightarrow & G^2X & \xrightarrow{Gi} & GE(Y) & \longrightarrow & GW & &
\end{array}$$

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The first column is an isomorphism, so $\eta E(Y)\cdot i$ is a monomorphism. Since i is an essential monomorphism, $\eta E(Y)$ is a monomorphism. Since $E(Y)$ is injective and $\eta E(Y)$ is an essential morphism, $\eta E(Y)$ is in fact an isomorphism. A standard diagram chase reveals ηW to be a monomorphism. Hence $\sigma(W)=\ker \eta W=0$.

3. **The ring of quotients.** Suppose σ is an idempotent kernel functor, and (G, η) is a corresponding localization functor. In [4], it was shown that GA has a unique ring structure for which ηA is a ring homomorphism. It was also shown in [4] that each GX can be given a (unique) GA-module structure extending the A-module structure of GX. We shall show now how GA has a ring structure.

We write endomorphisms on the opposite side to scalars, and thus avoid talking about opposite rings. Since A is an A–A bimodule, GA has a right A-module structure given by $g\cdot a=(g)G\rho_a$, where $\rho_a : A \to A$ maps a' to $a'a$ for all a' in A.

THEOREM 3.1. *The left A-module GA is isomorphic to* $\mathrm{Hom}_A({}_AGA_A, {}_AGA)$.

Proof. For g in GA, the map $\rho_g : A \to GA$ where $(a)\rho_g=ag$ induces a map $\overline{\rho_g}$ from $\mathrm{im}\,\eta A$ to GA. Since $GA/\mathrm{im}\,\eta A$ is in $\mathscr{G}$, and GA is faithfully σ-injective, $\overline{\rho_g}$ lifts to a unique map θ_g from GA to GA. We define $\delta : GA \to \mathrm{Hom}_A({}_AGA, {}_AGA)$ by $(g)\delta=\theta_g$. Let 1 denote $(1)\eta_A$, and let $\epsilon : \mathrm{Hom}_A({}_AGA, {}_AGA) \to GA$ be defined by $(f)\epsilon=(1)f$. It is easily verified that ϵ is an A-homomorphism, and that $(g)\delta\epsilon=g$ for all g in GA. Furthermore, if f is in $\mathrm{Hom}_A({}_AGA, {}_AGA)$, the restrictions of f and $(f)\epsilon\delta$ to $\mathrm{im}\,\eta A$ coincide. Since GA is *faithfully* σ-injective, $f=(f)\epsilon\delta$, so δ and ϵ are inverses of one another.

The maps δ and ϵ are in fact bimodule homomorphisms. If we use Theorem 3.1 to define a ring structure on GA, ηA is seen to be a ring homomorphism. Denoting GA by C, the ring C is the double centralizer of the module ${}_AC$.

Let L' be the functor from ${}_A\mathscr{M}$ to ${}_C\mathscr{M}$ defined as

$$ {}_A\mathscr{M} \xrightarrow{G} {}_A\mathscr{M} \xrightarrow{\mathrm{Hom}_A({}_AC_C,\, -)} {}_C\mathscr{M} $$

and let $V=\mathrm{Hom}_C({}_CC_A, \_\_)$ be the forgetful functor from ${}_A\mathscr{M}$ to ${}_C\mathscr{M}$.

THEOREM 3.2. *If (G, η) is a localization functor, and L' and V are as defined above, then G is naturally equivalent to VL'.*

Proof. There is a natural transformation from VL' to G given by

$$ \begin{aligned} VL'X &= \mathrm{Hom}_C({}_CC_A, \mathrm{Hom}_A({}_AC_C, GX)) \\ &\cong \mathrm{Hom}_A({}_AC_A, GX) \xrightarrow{\mathrm{Hom}(\eta A,1)} \mathrm{Hom}_A({}_AA_A, GX) \cong GX. \end{aligned} $$

To show this is an equivalence, it suffices to show $\mathrm{Hom}_A(\eta A, 1_{GX})$ is an isomorphism. Let f be in $\mathrm{Hom}_A(A, GX)$. Since $f(\sigma A)\subseteq\sigma(GX)=0$, and $\sigma(A)=\ker \eta A$, f factors uniquely through *im* ηA. Since GX is faithfully σ-injective, the map from *im* ηA to GX lifts uniquely to a map f' from $C=GA$ to GX. Thus, given f in

$\mathrm{Hom}_A(A, GX)$, there is a unique f' in $\mathrm{Hom}_A(C, GX)$ such that $f = f' \cdot \eta A$. Thus $\mathrm{Hom}_A(\eta A, l_{GX})$ is an isomorphism.

REMARK. The ring C described here is isomorphic (over A) to the ring of quotients defined in [3] and in [6]. If we identify these rings, then the functor VL (see Introduction), which gives the module of quotients as an A-module, is equivalent to Q_σ. (See [6, the remark following Proposition 2.3.]) Since, for any A-module X, LX and $L'X$ are (essentially) C-modules with the same A-module structure, Goldman's remark [4] about the uniqueness of the C-module structure on $Q_\sigma(X)$ shows L and L' are equivalent functors. This comment (or rather a rigorous version thereof) shows that, if L is a localization functor in the sense of [6], VL gives a localization functor in the present sense, and if (G, η) is a localization functor in the present sense, $\mathrm{Hom}_A({}_AC_C, G\_\_)$ is equivalent to a localization functor in the sense of [6].

4. **Triples.** We begin this section with a review of §2 of [2]. For an arbitrary category $\mathscr{A}$, a triple $\bar{T} = (G, \eta, \mu)$ consists of a covariant functor $G : \mathscr{A} \to \mathscr{A}$, and natural transformations $\eta : I_{\mathscr{A}} \to G$ and $\mu : G^2 \to G$ for which the following diagrams commute:

$$(1)\qquad \begin{array}{ccc} G & \xrightarrow{G\eta} & G^2 \\ {\scriptstyle \eta G}\downarrow & \searrow^{I_G} & \downarrow{\scriptstyle \mu} \\ G^2 & \xrightarrow[\mu]{} & G \end{array} \qquad\qquad (2)\qquad \begin{array}{ccc} G^3 & \xrightarrow{G\mu} & G^2 \\ {\scriptstyle \mu G}\downarrow & & \downarrow{\scriptstyle \mu} \\ G^2 & \xrightarrow[\mu]{} & G \end{array}$$

If $S : \mathscr{A} \to \mathscr{B}$ and $T : \mathscr{B} \to \mathscr{A}$ are functors for which S is left adjoint to T, and $\alpha : ST \to I_{\mathscr{B}}$ and $\beta : I_{\mathscr{A}} \to TS$ are the associated natural transformations, then $(TS, \beta, T\alpha S)$ is a triple, and we say S and T *generate* the triple. Also, any triple (G, η, μ) in $\mathscr{A}$ has a generator. To see this, define $\mathscr{A}^G$ to be the category whose objects are pairs (X, ϕ), where X is an object in $\mathscr{A}$, and $\phi : GX \to X$ is a morphism in $\mathscr{A}$ for which $\phi \cdot \eta X = 1_X$ and $\phi \cdot \mu X = \phi \cdot G\phi$. A morphism $[f] : (X, \phi) \to (X', \phi')$ in $\mathscr{A}^G$ is given by a morphism $f : X \to X'$ in $\mathscr{A}$ for which $f\phi = \phi' \cdot Gf$, and composition of morphisms in $\mathscr{A}^G$ is given by $[f] \cdot [g] = [fg]$. We define a functor S^G from $\mathscr{A}$ to $\mathscr{A}^G$ by $S^G X = (GX, \mu X)$, $S^G f = [Gf]$, and a functor T^G from $\mathscr{A}^G$ to $\mathscr{A}$ by $T^G(X, \phi) = X$, $T^G[f] = f$.

It is shown in [2] that S^G is left adjoint to T^G, and the natural transformations associated with adjointness, $\alpha^G : S^G T^G \to I_{\mathscr{A}^G}$ and $\mathscr{B}^G : I_{\mathscr{A}} \to T^G S^G$ are given by $\alpha^G(X, \phi) = [\phi]$ and $\beta^G X = \mathscr{N} X$. The triple generated by S^G and T^G is

$$(T^G S^G, \beta^G, T^G \alpha^G S^G) = (G, \eta, \mu).$$

Furthermore, it is shown that, if $S : \mathscr{A} \to \mathscr{B}$ and $T : \mathscr{B} \to \mathscr{A}$ is another generator of the triple (G, η, μ), there is a unique functor $\Gamma : \mathscr{B} \to \mathscr{A}^G$ satisfying (i) $\Gamma S = S^G$, (ii) $\Gamma \cdot \alpha = \alpha^G \cdot \Gamma$, and (iii) $T^G \Gamma = T$.

THEOREM 4.1. *If (G, η) is a localization functor, there is a (unique) equivalence $\mu: G^2 \to G$ for which (G, η, μ) is a triple. Conversely, if (G, η, μ) is a triple on ${}_A\mathcal{M}$ for which G is left exact and μ is an equivalence, (G, η) is a localization functor.*

Proof. If (G, η) is a localization functor, choose $\mu=(G\eta)^{-1}=(\eta G)^{-1}$. The verification of the conditions for (G, η, μ) to be a triple is easy. Conversely, if (G, η, μ) is a triple, and μ is an equivalence, condition (1) for triples guarantees $G\eta=\eta G=\mu^{-1}$. Thus, if G is, in addition, left exact, (G, η) is a localization functor.

A triple (G, η, μ) for which G is left exact and μ is an equivalence will be called *localizing*. Combining Theorems 2.1, 2.4, and 4.1 we have

THEOREM 4.2. *Every localizing triple (G, η, μ) on ${}_A\mathcal{M}$ has an associated idempotent kernel functor σ given by $\sigma(X)=\ker \eta X$, and every idempotent kernel functor on ${}_A\mathcal{M}$ arises in this way from a unique (up to equivalence) localizing triple.*

The next theorem shows that, for a localizing triple (G, η, μ), the category $({}_A\mathcal{M})^G$ is a suitable replacement for Gabriel's quotient category.

THEOREM 4.3. *Let $\mathcal{G}$ be a strongly complete Serre class, and let (G, η, μ) be an associated localizing triple. Let $S^G: {}_A\mathcal{M} \to ({}_A\mathcal{M})^G$ and $T^G: ({}_A\mathcal{M})^G \to {}_A\mathcal{M}$ be the Eilenberg–Moore universal generator for the triple, and let α^G and β^G be the associated natural transformations. Then*

(1) *$({}_A\mathcal{M})^G$ is isomorphic (not just equivalent) to the full subcategory of ${}_A\mathcal{M}$ whose objects are all modules X for which ηX is an isomorphism,*

(2) *S^G is an exact functor, α^G is an equivalence, and $({}_A\mathcal{M})^G$ is an abelian category,*

(3) *if $\mathcal{B}$ is any category, and $S: {}_A\mathcal{M} \to \mathcal{B}$ a left adjoint to $T: \mathcal{B} \to {}_A\mathcal{M}$ such that S and T generate (G, η, μ), and if the associated natural transformation $\alpha: ST \to I_{\mathcal{B}}$ is an equivalence, then the functor $\Gamma: \mathcal{B} \to ({}_A\mathcal{M})^G$ is an equivalence.*

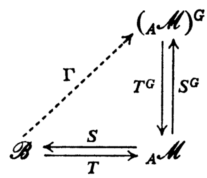
Proof. (1) Let (G, η, μ) be a localizing triple. Then $\mu=(G\eta)^{-1}=(\eta G)^{-1}$. If (X, ϕ) is an object in $({}_A\mathcal{M})^G$, $\phi\eta X=1_X$, so ηX is a monomorphism. Since ηX is an essential morphism (Lemma 2.2), ηX must in fact be an isomorphism. This being the case, ϕ must be ηX^{-1}, so every object of $({}_A\mathcal{M})^G$ is of the form $(X, \eta X^{-1})$. Conversely, if ηX is an isomorphism, $(X, \eta X^{-1})$ will be an object of $({}_A\mathcal{M})^G$ provided $\eta X^{-1}\cdot\mu X=\eta X^{-1}\cdot G(\eta X^{-1})$. But since $\mu=(G\eta)^{-1}=(\eta G)^{-1}$, this is always satisfied when ηX is an isomorphism. Thus the objects of $({}_A\mathcal{M})^G$ are all pairs $(X, \eta X^{-1})$ where ηX is an isomorphism. The functor T^G is faithful (2), and, in this case, is also full. Therefore T^G induces an isomorphism between $({}_A\mathcal{M})^G$ and the full subcategory $\mathcal{D}$ whose objects are A-modules X for which ηX is an isomorphism.

(2) Since T^G has a left adjoint S^G, $\mathcal{D}$ is a full coreflective subcategory of ${}_A\mathcal{M}$. In order to show $\mathcal{D}$ (or $({}_A\mathcal{M})^G$) is abelian, it is sufficient [5, Proposition V.5.3] to show that S^G is kernel preserving. Let $f: K \to X$ be a kernel of $g: X \to Y$ in ${}_A\mathcal{M}$. Then $S^Gg\cdot S^Gf=S^G(gf)=S^G0=[0]$. If $[w]:(Z, \phi) \to S^GX$ satisfies $S^Gg\cdot[w]=0$, then

$Gg \cdot w = 0$. Since G is left exact, Gf is the kernel of Gg, so there is a unique $h : Z \to GK$ such that $Gf \cdot h = w$. Since T^G is full, $[h]$ is the desired morphism in $({}_A\mathscr{M})^G$ satisfying $[w] = [Gf] \cdot [h]$. Thus $[Gf] = S^G f$ is a kernel of S^G, as desired.

The functor S^G is a left adjoint, and so preserves cokernels. Since it also preserves kernels, and since ${}_A\mathscr{M}$ and $({}_A\mathscr{M})^G$ are abelian categories, S^G is an exact functor. Finally, α^G is an equivalence, since $\alpha^G(X, \eta X^{-1}) = [\eta X^{-1}]$, which is invertible.

(3) Suppose the hypotheses of [3] hold. Then we have



where $T^G\Gamma = T$, $\Gamma S = S^G$, and $\Gamma \cdot \alpha = \alpha^G \cdot \Gamma$. Then $\alpha : (ST^G)\Gamma = ST \to I_{\mathscr{B}}$ and $\alpha^G : \Gamma(ST^G) = S^G T^G \to I_{({}_A\mathscr{M})^G}$ are equivalences.

COROLLARY 4.4. *Let $\mathscr{B}$ be an abelian category, and let $S : {}_A\mathscr{M} \to \mathscr{B}$ be an exact covariant functor which is left adjoint to $T : \mathscr{B} \to {}_A\mathscr{M}$. If, in addition, the associated natural transformation α from ST to $I_{\mathscr{B}}$ is an equivalence, then $\mathscr{B}$ is equivalent to $({}_A\mathscr{M})^{TS}$ for the localizing triple $(TS, \beta, T\alpha S)$ generated by S and T.*

Proof. S is exact, and T, being a right adjoint, is left exact. Therefore, TS is left exact. Since α is an equivalence, $(TS, \beta, T\alpha S)$ is a localizing triple. The conclusion follows from (3) of the theorem.

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