CONSTRUCTING AN AUTOMORPHISM FROM AN ANTI-AUTOMORPHISM

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We consider the following problem: Let G be a group with distinct automorphisms β and σ and an anti-automorphism α such that

(1) $$x \in G \Rightarrow \sigma(x) = \beta(x) \quad \text{or} \quad \alpha(x).$$

What can be said about G?

If $\sigma = \alpha$, σ is both an automorphism and an anti-automorphism so that G is abelian. Hence we assume that $\sigma \neq \alpha$. In this case, we show that G is non-abelian, but has an abelian subgroup of index 2. Conversely, for such a group G there always exist distinct automorphisms β and σ and an anti-automorphism α such that (1) holds.

The case when β is the identity mapping and α is the mapping $x \to x^{-1}$ was the content of a problem (# 5471) in the Monthly. It was required to prove that G is solvable. Theorem 4 shows what structure G must have.

THEOREM 1. *Let* G *be a group,* α *an anti-automorphism of* G, *and* $\sigma \neq \alpha$ *a non-trivial automorphism of* G *and assume*

(2) $$x \in G \Rightarrow \sigma(x) = x \quad \text{or} \quad \sigma(x) = \alpha(x).$$

Then G *has a (normal) abelian subgroup* H *of index* 2. α *induces a non-trivial automorphism on* H. *If* $G = \langle H, g \rangle$, $g^{-1}hg = \alpha(h)$ *for* $h \in H$. *Furthermore,* $\alpha(g) = bg$, *where* $1 \neq b \in H$ *and* $\alpha(b) = b^{-1}$.

Proof. Let $H = \{x \in G \mid \sigma(x) = x\}$. Then H is a proper subgroup of G. If $h \in H$, $g \notin H$, $\sigma(hg) = \alpha(hg) = \alpha(g)\alpha(h) = \sigma(h)\sigma(g) = h\alpha(g)$. Hence we have

(3) $$\alpha(h) = \alpha(g)^{-1} h\, \alpha(g), \quad \text{for} \quad h \in H,\ g \notin H.$$

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From (3) we see that for x and y in H, $\alpha(xy) = \alpha(x)\alpha(y)$, but $\alpha(yx) = \alpha(x)\alpha(y)$. Hence $xy = yx$ and H is abelian.

If there exists an element g_o in $G \setminus H$ such that $\alpha(g_o) \in H$, we have: $\alpha(h) = \alpha(g_o)^{-1} h \alpha(g_o) = h$, for all $h \in H$. This implies that $\sigma(g) = \alpha(g)$ for all $g \in G$, or that $\sigma = \alpha$, contrary to hypothesis. Hence $\alpha^{-1}(H) \leq H$. Applying α^{-1} to (3) we get

(4) $$\alpha^{-1}(h) = g^{-1} hg \quad \text{for } h \in H,\ g \notin H.$$

Since $\alpha^{-1}(h) \in H$, (4) shows that H is normal in G.

Now let $x, y \in G \setminus H$. Then for $h \in H$, $(xy)^{-1} h(xy) = y^{-1} \alpha^{-1}(h) y = \alpha^{-2}(h)$ since $\alpha^{-1}(h) \in H$. On the other hand, if $xy \notin H$, $(xy)^{-1} h(xy) = \alpha^{-1}(h)$ so that $\alpha^{-1}(h) = \alpha^{-2}(h) \Rightarrow \alpha(h) = h$ for $h \in H$. But this implies that $\alpha = \sigma$, contrary to hypothesis.

Hence if $x, y \in G \setminus H$, $xy \in H$ and $\alpha^{-2}(h) = h$ for $h \in H$. Thus G/H has order 2, and α induces an automorphism of order 2 on H.

If we let $G = \langle H, g \rangle$, then $g^2 = a \in H$; and by (4), $g^{-1}hg = \alpha(h)$ for $h \in H$. Clearly $\alpha(a) = a$. Since $\alpha(g) \notin H$, we have $\alpha(g) = bg$, where $b \in H$. If $b = 1$, $\sigma(h) = h$ and $\sigma(hg) = \sigma(h)\sigma(g) = h\alpha(g) = hg$ for $h \in H$, i.e. σ is the trivial automorphism of G. Hence $b \neq 1$.

Now $a = \alpha(a) = \alpha(g^2) = \alpha(g)^2 = (bg)^2 = bg^2 g^{-1} bg = ba\alpha(b)$. Thus $\alpha(b) = b^{-1}$.

THEOREM 2. <u>Let</u> G <u>be a non-abelian group with an abelian subgroup</u> H <u>of index</u> 2. <u>Then there exists an anti-automorphism</u> α <u>of</u> G <u>and an automorphism</u> σ <u>of</u> G <u>such that</u>

(5) $$\begin{cases} \sigma(h) = h & \text{for } h \in H \\ \sigma(x) = \alpha(x) \neq x & \text{for } x \notin H. \end{cases}$$

<u>Proof</u>. Let $G = \langle H, g \rangle$, and let $g^2 = a \in H$. Let $\alpha(h) = g^{-1} hg$ for $h \in H$. We note next that there exists an element $b \neq 1$ such that $\alpha(b) = b^{-1}$. In fact, if we choose $h \in H$ with $g^{-1} hg \neq h$ (h exists since G is non-abelian) and let $b = h^{-1}\alpha(h)$, then $b \neq 1$ and $\alpha(b) = \alpha(h)^{-1} \alpha^2(h) = \alpha(h)^{-1} h = b^{-1}$ since H is abelian. Define $\alpha(hg) = bhg$ for $h \in H$. Then $\alpha(hg) = bg^{-1} hg = \alpha(g)\alpha(h)$. Define σ by equations (5).

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We have to verify that α is an anti-automorphism and σ an automorphism of G, i.e. for $x_1, x_2 \in G$ we have to show that $\alpha(x_1x_2) = \alpha(x_2)\,\alpha(x_1)$ and $\sigma(x_1x_2) = \sigma(x_1)\,\sigma(x_2)$. There are four cases to distinguish:

(i) $x_1, x_2 \in H$.

(ii) $x_1 \in H$, $x_2 \notin H$.

(iii) $x_1 \notin H$, $x_2 \in H$.

(iv) $x_1 \notin H$, $x_2 \notin H$.

It is a simple matter to compute that the required equations hold in each of these cases. We prove case (iv) as an example: Let $x_1 = h_1g, x_2 = h_2g$ where $h_1, h_2 \in H$. Then $x_1x_2 = h_1gh_2g = h_1a\,\alpha(h_2)$; $\alpha(x_1x_2) = \alpha(h_1)\,a\,h_2$;

$$\alpha(x_2)\,\alpha(x_1) = bh_2gbh_1g = bh_2\,a\,g^{-1}(bh_1)g = bh_2\,a\,\alpha(b)\alpha(h_1)$$

$$= bh_2ab^{-1}\,\alpha(h_1) = \alpha(h_1)\,a\,h_2\ .$$

Hence $\alpha(x_1x_2) = \alpha(x_2)\,\alpha(x_1)$. $\sigma(x_1x_2) = x_1x_2$ since $x_1x_2 \in H$. $\sigma(x_1)\,\sigma(x_2) = \alpha(x_1)\,\alpha(x_2) = \alpha(x_2x_1) = \alpha(h_2)\,a\,h_1 = x_1x_2$. Hènce $\sigma(x_1x_2) = \sigma(x_1)\,\sigma(x_2)$.

Note: If H is an abelian group of order $\neq 1$ and $\neq 2$, then H has a non-trivial automorphism of order 2, and hence there exists a non-abelian extension G of H such that G/H has order 2.

THEOREM 3. Let G be a group with distinct automorphisms β and σ and an anti-automorphism $\alpha \neq \sigma$ such that (1) $x \in G \Rightarrow \sigma(x) = \beta(x)$ or $\alpha(x)$. Then G is non-abelian and has an abelian subgroup H of index 2.

Proof. Let $\rho = \beta^{-1}\sigma$. Then ρ is an automorphism of G and $\rho(x) = x$ or $\rho(x) = \beta^{-1}\,\alpha(x)$. $\beta^{-1}\alpha$ is an anti-automorphism of G and $\rho \neq \beta^{-1}\alpha$, since $\sigma \neq \alpha$. The theorem follows by applying Theorem 1.

THEOREM 4. Let G be a group and assume that G has a non-trivial automorphism σ such that (6) $x \in G \Rightarrow \sigma(x) = x$ or $\sigma(x) = x^{-1}$. Then either: (a) $\sigma(x) = x^{-1}$ for all x in G, G is abelian and $G^2 \neq 1$, or: (b) $G = \langle H, g\rangle$, where H is an abelian group which contains an element a of order 2, and $H^2 \neq 1$.

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$g^2 = a$ and $g^{-1} hg = h^{-1}$ for all $h \in H$. Then:

$$(7) \qquad \begin{cases} \sigma(x) = x & \text{for } x \in H \\ \sigma(x) = x^{-1} & \text{for } x \notin H \end{cases}$$

Conversely, if G is defined by (b) the mapping given by (7) is an automorphism of G.

Proof. If $\sigma(x) = x^{-1}$ for all x in G, then G is abelian; $G^2 \neq 1$ since σ is not trivial. Let α be defined by $\alpha(x) = x^{-1}$ and assume that $\alpha \neq \sigma$. By Theorem 1, $G = \langle H, g \rangle$, where H is abelian, $g^{-1} hg = h^{-1}$, $g^2 = a \in H$. Since α is non-trivial on H, $H^2 \neq 1$. Also $a^{-1} = \alpha(a) = g^{-1} a g = a$ so that $a^2 = 1$. If $g^2 = 1$, $\sigma(hg) = \sigma(h) \sigma(g) = h g^{-1} = hg$ for all $h \in H$ and this implies that σ is trivial, contrary to hypothesis. Hence a has order 2. (7) holds from the definition of H in Theorem 1.

Conversely, suppose that $G = \langle H, g \rangle$, where H is abelian, $g^{-1} hg = h^{-1}$, $g^2 = a \in H$ has order 2, and $H^2 \neq 1$. Then by Theorem 2 there exists an anti-automorphism α of G and an automorphism σ of G such that (5) holds. To show that (7) holds it is only necessary to show that $\alpha(h) = h^{-1}$. But in the proof of Theorem 2, we defined α so that $\alpha(h) = g^{-1} hg$. Hence $\alpha(h) = h^{-1}$ for $h \in H$, and the theorem is proved.

Remark. If instead of studying the problem stated in the introduction, we require that α and β both be automorphisms and σ an automorphism such that (1) holds, it is easy to see that $\sigma = \alpha$. For let $A = \{g \in G \mid \sigma(g) = \alpha(g)\}$ and $B = \{g \in G \mid \sigma(g) = \beta(g)\}$. Then A and B are subgroups of G and $G = A \cup B$. This implies that $G = A$ or $G = B$. But $G \neq B$ and hence $G = A$, $\sigma = \alpha$.

If on the other hand, we require both α and β to be anti-automorphisms, the answer seems to be much more difficult. I was not able to determine when this could happen.

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