THE OPTIMAL REINSURANCE TREATY

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1. Some years ago I discussed optimal reinsurance treaties, without trying to give a precise definition of this term [1]. I suggested that a reinsurance contract could be called "most efficient" if it, for a given net premium, maximized the reduction of the variance in the claim distribution of the ceding company. I proved under fairly restricted conditions that the Stop Loss contract was most efficient in this respect.

I do not consider this a particularly interesting result. I pointed out at the time that there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party, may be quite unacceptable to the other.

2. In spite of my own reservations, it seems that this result —which I did not think deserved to be called a theorem—has caused some interest. Kahn [4] has proved that the result is valid under far more general conditions, and recently Ohlin [5] has proved that the result holds for a much more general class of measures of dispersion.

In view of these generalizations it might be useful to state once more, why I think the original result has relatively little interest. In doing so, it is by no means my purpose to reduce the value of the mathematical generalizations of Kahn and Ohlin. Such work has a value in itself, whether the results are immediately useful or not. I merely want to point out that there are other lines of research, which appear more promising, if our purpose is to develop a realistic theory of insurance.

3. To illustrate my point, let us consider an insurance contract, which can lead to the claim payments:

0 with probability 0.99
1 with probability 0.01

Let us next consider a portfolio of 10.000 such contracts, and

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assume that claim payments under different contracts are stochastically independent.

Expected claim payment under the contracts in the portfolio, i.e. the net premium, is then

$$P = 100$$

The variance of the claim payment is

$$V = 99$$

This is a portfolio which most insurance companies would be glad to have—provided there was a reasonable loading—and it is unlikely that the question of reinsurance should be brought up. It is possible that the portfolio may lead to a total payment of 10.000, but the probability of such a catastrophe is 10^{-20000}, and is almost certain to be ignored.

4. Let us now assume—in spite of the argument above—that our company considers reinsuring a quota of 5 % of all contracts in the portfolio. The net premium of this reinsurance cover is obviously

$$P_q = 0.05\ P = 5$$

The variance of claim payments in the portfolio retained by the company is

$$V_q = (0.95)^2\ V = 89$$

Let us next assume that the company considers a Stop Loss contract at 100, i.e. if claim payments should exceed 100, the excess will be paid by the reinsurer.

It seems fairly safe to represent the claim distribution by a Normal distribution with mean = 100 and standard deviation = 10.

With this approximation, the net premium for the reinsurance cover is:

$$P_s = \frac{1}{10\sqrt{2\pi}} \int_{100}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} dx - \frac{100}{10\sqrt{2\pi}} \int_{100}^{\infty} e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} dx$$

or

$$P_s = 4$$

The variance of the portfolio retained by the company is found to be:

$$V_s = \frac{1}{10\sqrt{2\pi}} \int_0^{100} x^2 e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} dx + \frac{100^2}{10\sqrt{2\pi}} \int_{100}^{\infty} e^{-\frac{1}{2}\left(\frac{x-100}{10}\right)^2} dx - (100 - P_s)^2$$

or

$$V_s = 34$$

5. Our example seems to demonstrate the striking superiority of Stop Loss reinsurance. It is cheaper, and it gives a far greater reduction of variance than the conventional form of reinsurance. In addition the Stop Loss contract gives our company an absolute guarantee—against ruin—provided of course, that the company holds reserves exceeding 100, after the reinsurance premium has been paid.

This seems almost too good to be true, and we should ask ourselves if it is likely that an insurance company ever will have an option of the kind indicated by our example. Do we really expect a reinsurer to offer a Stop Loss contract and a conventional quota treaty with the same loading on the net premium? If the reinsurer is worried about the variance in the portfolio he accepts, he will prefer to sell the quota contract, and we should expect him to demand a higher compensation for the Stop Loss contract.

Experience seems to confirm this. The net premium will usually play only a minor part in negociations over non-proportional reinsurance treaties.

These considerations should remind us that there are two parties to a reinsurance contract, and that these parties have conflicting interests. The *optimal contract* must then appear as a reasonable compromise between these interests. To me the most promising line of research seems to be the study of contracts, which in different ways can be said to be optimal from the point of view of both parties. I discussed this problem first in a paper [2] on reciprocal treaties.

6. The variance has a long tradition as a "measure of risk", but it has also been clear for a long time, that it is not always an

adequate measure. To illustrate this, let us consider a reinsurance contract as reinsurers, and assume:

E = expected profit from the contract
V = the variance of profits.

We can then lay down a rule to the effect that for given $E > 0$ we accept the contract if V is not too large. If $V = 0$, the contract is obviously acceptable, since it then offers a certain profit of E.

If V is too large, we reject the contract, i.e. we prefer the certainty of a zero profit to accepting the contract.

Let now the pair (E, V) represent an unacceptable contract, and consider the following contract, which obviously is better than the certainty of a zero profit:

Profit 0 with probability $1—p$
Profit x with probability p

Expected profit is px, and the variance is $x^2p(1 - p)$. If we now take

$$p = \frac{E^2}{E^2 + V}$$

$$x = \frac{E^2 + V}{E}$$

our rule will tell us to reject this contract, on which we cannot loose. This is an obvious contradiction.

7. The counter example, which we used above, is evidently artificial. One may well object that such reinsurance contracts do not exist, and hence that there is no need for a rule as to whether they should be accepted or not. Nevertheless, the example should serve as a warning that we cannot always evaluate reinsurance contracts by computing only mean and variance of the claim distribution.

We need a more general rule, and this we find in the utility theory, which I have discussed in another paper [3].

8. It may be thought that the preceding paragraphs dismiss expected values (i.e. net premiums) and variances in a rather high-handed manner. It may, therefore, be useful to recall why

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these two concepts have played such important parts in actuarial theory:

(i) Net premiums are obviously important when we can appeal to the *Law of Large Numbers*, so that deviations from expected values can be disregarded.

(ii) The variance is a fairly adequate measure of risk, if we only consider probability distributions, which are approximately *symmetrical*. This will be the case if all distributions are approximately Normal, which they may be if the Law of Large Numbers and the Central Limit Theorem apply.

These two conditions are not usually fulfilled in reinsurance. The number of contracts, held by a reinsurer, will be small compared to that of a direct underwriter, and the distributions which occur in non-proportional reinsurance, are usually extremely skew. This should indicate that game theory, and the utility theory associated with it, are the proper tools.

References

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