

STEENROD OPERATIONS ON THE DE RHAM COHOMOLOGY OF ALGEBRAIC STACKS

FEDERICO SCAVIA 

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2
Canada

(scavia@math.ubc.ca)

(Received 20 December 2020; revised 25 March 2021; accepted 01 April 2021; first published online 4 May 2021)

Abstract Building upon work of Epstein, May and Drury, we define and investigate the mod p Steenrod operations on the de Rham cohomology of smooth algebraic stacks over a field of characteristic $p > 0$. We then compute the action of the operations on the de Rham cohomology of classifying stacks for finite groups, connected reductive groups for which p is not a torsion prime and (special) orthogonal groups when $p = 2$.

Keywords: Stacks, de Rham cohomology, Steenrod operations, cohomology operations, operads

2020 Mathematics Subject Classification: Primary 14F40
Secondary 55S10, 14A20, 14F20

1. Introduction

Let R be a commutative ring with identity, and let X be a smooth algebraic stack over R . We will mostly be interested in the case when $R = k$ is a field and $X = BG$ is the classifying stack of a linear algebraic k -group G . (Note that BG is always smooth over k , even if G is not.) We denote by $\Omega_{X/R}$ the de Rham complex of abelian sheaves on the big étale site of X :

$$0 \rightarrow \mathcal{O}_X \rightarrow \Omega_{X/R}^1 \rightarrow \Omega_{X/R}^2 \rightarrow \dots$$

By definition, the de Rham cohomology $H_{\text{dR}}^*(X/R)$ of X is the hypercohomology of $\Omega_{X/R}$; see [21, §1].

Let p be a prime number, let $R = \mathbb{F}_p$ be a field of p elements and let G be a finite discrete group. In [21, Lemma 10.2], B. Totaro produced a canonical isomorphism of graded rings

$$H_{\text{dR}}^*(BG/\mathbb{F}_p) \xrightarrow{\sim} H_{\text{sing}}^*(BG; \mathbb{F}_p) = H^*(G, \mathbb{F}_p), \tag{1.1}$$

where the ring on the right is group cohomology. (The identification of the singular cohomology of BG with the group cohomology of G is classical.)

Let G be a split connected reductive group over \mathbb{Z} . Assume that p is not a torsion prime for G ; that is, the p -torsion subgroup of $H_{\text{sing}}^*(BG(\mathbb{C}); \mathbb{Z})$ is trivial, where we regard $G(\mathbb{C})$ as a complex Lie group. Totaro showed in [21, Theorem 9.2] that $H_{\text{dR}}^*(BG/\mathbb{F}_p)$ is a polynomial ring on generators of degrees equal to two times the fundamental degrees of G . As a consequence, he obtained an isomorphism

$$H_{\text{dR}}^*(BG/\mathbb{F}_p) \cong H_{\text{sing}}^*(BG(\mathbb{C}); \mathbb{F}_p). \tag{1.2}$$

When p is a torsion prime for G , it is an interesting problem to compute the ring $H_{\text{dR}}^*(BG/\mathbb{F}_p)$ and to see whether (1.2) is still valid for G . For example, when $p = 2$, Totaro showed in [21, Theorem 11.1] that

$$\begin{aligned} H_{\text{dR}}^*(BO_{2r}/\mathbb{F}_2) &= \mathbb{F}_2[u_1, u_2, \dots, u_{2r}], \\ H_{\text{dR}}^*(BO_{2r+1}/\mathbb{F}_2) &= \mathbb{F}_2[v_1, c_1, u_2, u_3, \dots, u_{2r}]/(v_1^2), \\ H_{\text{dR}}^*(BSO_n/\mathbb{F}_2) &= \mathbb{F}_2[u_2, u_3, \dots, u_n], \end{aligned} \tag{1.3}$$

where $|u_i| = i$ for every i , $|v_1| = 1$ and $|c_1| = 2$. In particular, when $p = 2$, (1.2) holds for $G = O_{2r}, SO_n$, but not for $G = O_{2r+1}$. (In characteristic 2, O_{2r} is disconnected, whereas $O_{2r+1} = SO_{2r+1} \times \mu_2$ is connected.) Moreover, he proved in [21, Theorem 12.1] that (1.2) fails for $p = 2$ and $G = \text{Spin}_{11}$.

In [18], E. Primožic computed $H_{\text{dR}}^*(BG/\mathbb{F}_2)$ when G is the split group of type G_2 and when $G = \text{Spin}_n$ for $n \leq 11$. In the examples considered by him, with the exception of Spin_{11} , (1.2) always holds. Primožic then asked whether Steenrod operations on the de Rham cohomology of smooth stacks over \mathbb{F}_p may be defined and whether they agree with the topological Steenrod operations on $H_{\text{sing}}^*(BG(\mathbb{C}); \mathbb{F}_p)$, when $X = BG$ and (1.2) holds.

In the present work, we adapt a construction of R. Drury [7] to define the Steenrod p -power operations on $H_{\text{dR}}^*(X/k)$ for a smooth algebraic stack X over a field k of characteristic p . Drury’s work fits in the setting of J. P. May [12]. Presumably, one could also proceed by extending the work of D. Epstein [9] to hypercohomology functors.

We summarise the properties that we have been able to establish in Theorem 1.1. When k is perfect, we write $W_2(k)$ for the ring of Witt vectors of length 2 with coefficients in k .

Theorem 1.1. *Let p be a prime number, let k be a field of characteristic p and let X be a smooth algebraic stack over k . Then, for all $i \in \mathbb{Z}$, we have group homomorphisms*

$$\text{Sq}^i : H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^{*+i}(X/k)$$

when $p = 2$ and

$$P^i : H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^{*+2(p-1)i}(X/k), \quad \beta P^i : H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^{*+2(p-1)i+1}(X/k)$$

when $p > 2$. These homomorphisms are natural in X (i.e., they commute with pullbacks along 1-morphisms of k -stacks) and satisfy the following properties.

(i) For every $n \geq 0$ and every $x \in H_{\text{dR}}^n(X/k)$, we have

$$\text{Sq}^i(x) = \begin{cases} x^2 & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases} \quad P^i(x) = \begin{cases} x^p & \text{if } 2i = n, \\ 0 & \text{if } 2i > n, \end{cases}$$

when $p = 2$ and $p > 2$, respectively.

- (ii) The (internal) Cartan formulas of [12, (1) p. 165] hold.
- (iii) The Ádem relations of [12, Theorem 4.7] hold.
- (iv) Assume that $X = [Y/G]$, where Y is a smooth quasiprojective k -scheme and G is a linear algebraic k -group. Then, for all $i < 0$, we have

$$\mathrm{Sq}^i = 0 \quad (p = 2), \quad \mathrm{P}^i = 0 \quad (p > 2)$$

in $H_{\mathrm{dR}}^*(X/k)$. Moreover, P^0 and Sq^0 factor as

$$H_{\mathrm{dR}}^*(X/k) \rightarrow H^*(X, \mathcal{O}_X) \rightarrow H^*(X, \mathcal{O}_X) \rightarrow H_{\mathrm{dR}}^*(X/k),$$

where the first map is an edge homomorphism in the Hodge spectral sequence, the second map is induced by the Frobenius endomorphism of \mathcal{O}_X and the third map is an edge homomorphism in the conjugate spectral sequence.

(v) Assume that k is perfect and that $X = \tilde{X} \times_{W_2(k)} k$, where \tilde{X} is a smooth algebraic $W_2(k)$ -stack of finite type and with affine diagonal. Let $\iota : X \hookrightarrow \tilde{X}$ denote the canonical closed embedding. Then there is a Bockstein homomorphism

$$\beta : H_{\mathrm{dR}}^*(X/k) \rightarrow H_{\mathrm{dR}}^{*+1}(X/k),$$

defined as the connecting map of the hypercohomology long exact sequence associated to

$$0 \rightarrow \iota_* \Omega_{X/k} \rightarrow \Omega_{\tilde{X}/W_2(k)} \rightarrow \iota_* \Omega_{X/k} \rightarrow 0.$$

The homomorphism β is a derivation in the graded sense, and for all $i \in \mathbb{Z}$ we have

$$\beta \circ \mathrm{Sq}^i = i \cdot \mathrm{Sq}^{i+1} \quad (p = 2), \quad \beta \mathrm{P}^i = \beta \circ \mathrm{P}^i \quad (p > 2).$$

Note that a Bockstein homomorphism does not exist in Epstein’s generality (see [9, §7.1]), and it is not automatically defined in May’s setting. Our definition of β is inspired by the one given by Totaro during the proof of [21, Theorem 11.1] in Hodge cohomology. In order to prove that β satisfies the properties of (v), one must make a connection between the definition of β and the definition of the Sq^i and P^i . To achieve this, we first define a Bockstein $\check{\beta}$ that fits in May’s setting, and thus automatically satisfies the properties at the end of (v), and then show that $\beta = \check{\beta}$. Our alternative construction appears to be new even in the case of smooth projective k -varieties. (If X is a separated scheme over a ring R , the diagonal $X \rightarrow X \times_R X$ is a closed embedding, hence affine.)

In (iv), there is no loss of generality in assuming that $G = \mathrm{GL}_n$ for some $n \geq 0$.

In [12], the Cartan formulas and Ádem relations are written with homological indexing; see [12, §5] for the changes required to pass to cohomology. Properties (i), (ii) and (iii) have been proved by Drury for the hypercohomology of a commutative differential graded \mathbb{F}_p -algebra on a topological space. Drury’s proof immediately generalises to topoi with sufficiently many points, and in particular to the big étale topos of X . The most difficult part of the proof of Theorem 1.1 consists of establishing properties (iv) and (v).

Our second result is the following computation of the Steenrod operations for classifying stacks of linear algebraic groups. We denote by Sq and P the total Steenrod square and

the total Steenrod power, respectively. By definition, for all $x \in H_{\text{dR}}^*(X/k)$, we have

$$\text{Sq}(x) = \sum_{i \in \mathbb{Z}} \text{Sq}^i(x) \quad (p = 2), \quad \text{P}(x) = \sum_{i \in \mathbb{Z}} \text{P}^i(x) \quad (p > 2).$$

Note that by Theorem 1.1(i) the sums contain only finitely many nonzero terms.

Theorem 1.2. *Let p be a prime number, and let G be a linear algebraic group over \mathbb{F}_p .*

- (a) *If G is finite and discrete, then the isomorphism (1.1) is compatible with the Steenrod operations and the Bockstein homomorphisms.*
- (b) *If G is split reductive and p is not a torsion prime for G , then $\beta = 0$ and the Steenrod operations on $H_{\text{dR}}^*(BG/\mathbb{F}_p)$ are trivial; that is, for all $x \in H_{\text{dR}}^*(BG/\mathbb{F}_p)$ we have*

$$\text{Sq}(x) = x^2 \quad (p = 2), \quad \text{P}(x) = x^p \quad (p > 2).$$

- (c) *Assume that $p = 2$. The Steenrod operations on $H_{\text{dR}}^*(BO_n/\mathbb{F}_2)$ for $n \geq 1$, and on $H_{\text{dR}}^*(BSO_n/\mathbb{F}_2)$ for $n \geq 3$, are nontrivial. More precisely, we have*

$$\text{Sq}(u_{2a}) = u_{2a}^2, \quad \text{Sq}(u_{2a+1}) = u_{2a+1}^2 + u_{4a+1} + \sum_{t=0}^{2a-1} u_{2a-t} u_{2a+1+t}.$$

Here $u_i := 0$ for $i = 0$ and $i > 2r$, and in the case of SO_n we set $u_1 := 0$. Moreover,

$$\beta(u_{2a}) = u_{2a+1} + u_1 u_{2a}, \quad \beta(u_{2a+1}) = u_1 u_{2a+1}.$$

In the case of O_{2r+1} , we also have

$$\text{Sq}(c_1) = c_1^2, \quad \beta(c_1) = 0, \quad \text{Sq}(v_1) = 0, \quad \beta(v_1) = c_1.$$

In algebraic topology Steenrod operations are never trivial, because Sq^0 and P^0 are equal to the identity. Thus, (b) stands in stark contrast to its topological counterpart. In general, the triviality of the Steenrod operations on the de Rham cohomology is related to subtle arithmetic information; see Remark 6.6.

Because $SO_2 \cong \mathbb{G}_m$, the condition $n \geq 3$ in (c) is necessary. In (c), the only interesting nontrivial operation on u_{2a+1} is Sq^{2a} . The expression for $\text{Sq}^{2a}(u_{2a+1})$ has the same form as Wu’s formula for the topological Sq^{2a} applied to the Stiefel–Whitney class w_{2a+1} .

As the computations of Totaro and Primožic show, (1.2) may hold even if p is a torsion prime for G . Theorem 1.2 shows that, even though the fact that 2 is a torsion prime for SO_n is not detected by the failure of (1.2), it is detected by the existence of nontrivial Steenrod operations.

We now give a brief description of the content of each section. In Section 2, we recall the definition of May’s category $\mathcal{C}(p)$, the Steenrod operations associated to objects of $\mathcal{C}(p)$ and Drury’s construction of the Steenrod operations on the hypercohomology of a sheaf of commutative differential graded \mathbb{F}_p -algebras A . In Section 3, we first prove the naturality of the operations with respect to morphisms of topoi. Then, we assume that A is concentrated in degree 0, and we compare Drury’s setting to that of Epstein. As a consequence, we obtain in Proposition 3.5 that negative Steenrod operations on

the cohomology of A are zero and that P^0 and Sq^0 are induced by the Frobenius endomorphism of A . In Section 4, we return to the case of an arbitrary commutative differential graded algebra A . We give a way to compute Steenrod operations using Čech cohomology; see Proposition 4.9. In Section 5, we prove that we may approximate the de Rham cohomology of quotient stacks X as in Theorem 1.1(iv) by smooth schemes; that is, for all $d \geq 0$ we construct a morphism $Z_d \rightarrow X$ such that Z_d is a smooth k -scheme of finite type and the pullback $H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^*(Z_d/k)$ is injective in degrees $\leq d$. In Section 6, we define Steenrod operations on de Rham cohomology of stacks. We then prove Theorem 1.2(iv) by using the results of Section 5 to reduce to the case of smooth schemes and then conclude in that case by combining the crystalline Poincaré lemma with Section 3. In Proposition 6.10, we prove Theorem 1.1(v) using the results of Section 4. Finally, in Section 7 we combine the work of the previous sections with some explicit calculations to prove Theorem 1.2.

Notation. Let \mathcal{A} be an additive category. A cochain complex A in \mathcal{A} is an increasing sequence of objects and homomorphisms

$$\dots \rightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \rightarrow \dots$$

in \mathcal{A} , such that $d^i \circ d^{i-1} = 0$ for all integers i . If \mathcal{A} is abelian, we denote by $H^*(A)$ the cohomology of A : $H^i(A) := \text{Ker } d^i / \text{Im } d^{i-1}$. If $A, B \in \mathcal{A}$, we denote by $\text{Hom}(A, B)$ the group of homomorphisms from A to B in \mathcal{A} .

If A is a simplicial or cosimplicial object of \mathcal{A} , we denote by $K(A)$ and $K_N(A)$ the unnormalised and normalised cochain complexes associated to A , respectively; see [20, Tags 0194, 019D, 019H]. When A is a simplicial object, $K(A)$ and $K_N(A)$ are nonpositively graded, and when A is a cosimplicial object, they are nonnegatively graded.

A double complex A in \mathcal{A} is a cochain complex in the category of cochain complexes of \mathcal{A} . (In particular, the squares in a double complex are commutative, not anticommutative.) If A is a double complex, we denote by $\text{Tot}(A)$ the associated total complex

$$\text{Tot}(A)^n := \bigoplus_{i \in \mathbb{Z}} A^{i, n-i}, \quad d^n(a) := d^v(a) + (-1)^{n-i} d^h(a),$$

where $|a| = (i, n - i)$, $d^v : A^{i, n-i} \rightarrow A^{i, n-i+1}$ is the vertical differential and $d^h : A^{i, n-i} \rightarrow A^{i+1, n-i}$ is the horizontal differential.

Let R be a commutative ring with identity, let G be a finite group and let \mathcal{A} be the category of R -modules with left G -action; that is, the category of $R[G]$ -modules. If M and N are $R[G]$ -modules, we view the group of endomorphisms $\text{Hom}_R(M, N)$ and $M \otimes_R N$ as $R[G]$ -modules in the usual way, and we let $\text{Hom}_{R[G]}(M, N)$ and $M \otimes_{R[G]} N$ be the module of G -equivariant homomorphisms and the tensor product of M and N as $R[G]$ -modules, respectively. If A and B are cochain complexes of $R[G]$ -modules, we define the cochain complex of $R[G]$ -modules $\underline{\text{Hom}}_R(A, B)$ by

$$\underline{\text{Hom}}_R^n(A, B) := \prod_{i \in \mathbb{Z}} \text{Hom}_R(A^i, B^{n+i}), \quad d^n(f) := \prod_{i \in \mathbb{Z}} (d_B^{i+n} \circ f^i + (-1)^{n+1} f^{i+1} \circ d_A^i).$$

We define the complex of R -modules $\underline{\text{Hom}}_{R[G]}(A, B)$ in a similar way. Furthermore, we define their tensor product $A \otimes_R B$ by

$$(A \otimes_R B)^n := \bigoplus_{i \in \mathbb{Z}} A^i \otimes_R B^{n-i}, \quad d^n(f) := \bigoplus_{i \in \mathbb{Z}} (d_A^i \otimes \text{id}_B^{n-i} + (-1)^n (\text{id}_A^i \otimes d_B^{n-i})).$$

We define $A \otimes_{R[G]} B$ in a similar way. The signs are chosen so that, for all cochain complexes A, B, C of $R[G]$ -modules, the canonical adjunction

$$\Phi : \text{Hom}_{R[G]}(A, \text{Hom}_R(B, C)) \xrightarrow{\sim} \text{Hom}_{R[G]}(A \otimes_R B, C)$$

is given by the sign-free formula

$$\Phi^n(f^{k+l})(a \otimes b) = (f^k(a))^l(b), \quad |f| = n, |a| = k, |b| = l.$$

We will sometimes combine the adjunction formula with the switch isomorphism

$$\tau : A \otimes_R B \xrightarrow{\sim} B \otimes_R A, \quad \tau(a \otimes b) = (-1)^{kl}(b \otimes a), \quad |a| = k, |b| = l.$$

Let \mathcal{T} be a topos, and let Set be the set topos (also called point topos). We have a morphism of topoi $(e_*, e^{-1}) : \mathcal{T} \rightarrow \text{Set}$, where $e_* = \Gamma(\mathcal{T}, -) := \text{Hom}(e, -)$, where e is a terminal object of \mathcal{T} and e^{-1} is the constant sheaf functor. When writing a morphism of topoi, we do not explicitly mention the adjunction between pullback and pushforward.

Let M and N be left $R[G]$ -modules on \mathcal{T} (here we view \mathcal{T} as a topos ringed by the constant ring object associated to $R[G]$). We denote by $\mathcal{H}om_R(M, N)$ and $M \otimes_R N$ the sheaf of homomorphisms and the sheaf tensor product, respectively; see [3, IV, Proposition 12.1, Proposition 12.7] for the definition. It immediately follows from the definition that $\Gamma(\mathcal{T}, \mathcal{H}om_R(M, N)) = \text{Hom}_R(M, N)$. If A and B are complexes of R -modules on \mathcal{T} , we define the $R[G]$ -modules $\underline{\mathcal{H}om}_R(A, B)$ and $A \otimes_R B$ on \mathcal{T} as in the case of $R[G]$ -modules (on Set). It easily follows from the definition that $\Gamma(\mathcal{T}, \underline{\mathcal{H}om}_R(A, B)) = \underline{\text{Hom}}_R(A, B)$. We will be mostly interested in the case when A is constant, in which case our definition agrees with [7, Definition 3.2.1].

By definition, a homotopy associative differential graded R -algebra is a cochain complex (C, d) of R -modules, where $C^i = 0$ for $i < 0$, together with a homomorphism of complexes $m : C \otimes C \rightarrow C$ such that

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b),$$

for all homogeneous a and b , where $x \cdot y := m(x \otimes y)$, and the diagram

$$\begin{array}{ccc} C \otimes C \otimes C & \xrightarrow{\text{id} \otimes m} & C \otimes C \\ \downarrow m \otimes \text{id} & & \downarrow m \\ C \otimes C & \xrightarrow{m} & C \end{array}$$

is homotopy commutative. A (two-sided) homotopy identity for C is an element $u \in C^0$ such that the homomorphisms $C \rightarrow C$ of left and right multiplication by u are homotopic to the identity.

2. Steenrod operations on sheaf cohomology

We start by recalling the part of the setup of May [12] that is relevant to us. In [12, §2], the definitions are given with applications to homology in mind, and in [12, §5] how to change the indexing when dealing with cohomology is explained. We give the definitions directly in the cohomological setting.

Let R be a commutative ring with identity, let p be a prime number, let Σ_p be the symmetric group on p letters and let π be the cyclic subgroup of Σ_p generated by some $\alpha \in \Sigma_p$ of order p . Let $\epsilon : W(p, R) \rightarrow R$ be the $R[\pi]$ -free resolution of R defined as follows. For every $i \geq 0$, we let $W(p, R)_i = W(p, R)^{-i}$ be $R[\pi]$ -free on one generator e_i . Define $N, T \in R[\pi]$ by the formulas

$$N := 1 + \alpha + \dots + \alpha^{p-1}, \quad T := \alpha - 1.$$

We define a differential d and the augmentation by the formulas

$$d(e_{2i+1}) := Te_{2i}, \quad d(e_{2i+2}) := Ne_{2i+1}, \quad \epsilon(\alpha^j e_0) := 1$$

for all $i, j \geq 0$.

We also let V be a $R[\Sigma_p]$ -free resolution of R such that $V^i = 0$ for all $i > 0$. In particular, V is a $R[\pi]$ -free resolution of R ; hence, by [20, Tag 0649] there exists a homomorphism $j : W(p, R) \rightarrow V$ of $R[\pi]$ -complexes that commutes with the augmentations maps.

2.1. The category $\mathcal{C}(p)$

Following [12, Definitions 2.1], we define a category $\mathcal{C}(p, R)$ as follows. The objects of $\mathcal{C}(p, R)$ are pairs (K, θ) , where K is a homotopy associative differential graded R -algebra, π acts trivially on K and on $K^{\otimes p}$ by cyclic permutations of the factors and $\theta : W(p, R) \otimes K^{\otimes p} \rightarrow K$ is a homomorphism of $R[\pi]$ -complexes such that

- (i) the restriction of θ to $(Re_0) \otimes K^{\otimes p} \cong K^{\otimes p}$ is R -homotopic to the iterated product $K^{\otimes p} \rightarrow K$ (associated in some order) and
- (ii) there exists a homomorphism $\phi : V \otimes K^{\otimes p} \rightarrow K$ of $R[\Sigma_p]$ -complexes such that θ is $R[\pi]$ -homotopic to the composition

$$W(p, R) \otimes K^{\otimes p} \xrightarrow{j \otimes 1} V \otimes K^{\otimes p} \xrightarrow{\phi} K.$$

Note that (ii) does not depend on the choice of V or j . A morphism $f : (K_1, \theta_1) \rightarrow (K_2, \theta_2)$ in $\mathcal{C}(p, R)$ is defined as a homomorphism of R -complexes $g : K_1 \rightarrow K_2$ such that the diagram

$$\begin{array}{ccc} W(p, R) \otimes K_1^{\otimes p} & \xrightarrow{\theta_1} & K_1 \\ \downarrow \text{id} \otimes g^{\otimes p} & & \downarrow g \\ W(p, R) \otimes K_2^{\otimes p} & \xrightarrow{\theta_2} & K_2 \end{array}$$

is $R[\pi]$ -homotopy commutative. We write

$$\mathcal{C}(p) := \mathcal{C}(p, \mathbb{Z}/p\mathbb{Z}), \quad W := W(p, \mathbb{Z}/p\mathbb{Z}).$$

Let $i : R^{\otimes p} \xrightarrow{\sim} R$ be the isomorphism $a_1 \otimes \cdots \otimes a_p \mapsto a_1 \cdots a_p$. Note that $(R, \epsilon \otimes i)$ is an object of $\mathcal{C}(p, R)$. We say that $(K, \theta) \in \mathcal{C}(p, R)$ is *unital* if K has a two-sided homotopy identity u such that the homomorphism $R \rightarrow K$ given by $1 \mapsto u$ is a morphism in $\mathcal{C}(p, R)$.

If $R \rightarrow S$ is a ring homomorphism and (K, θ) is an object of $\mathcal{C}(p, R)$, the pair $(K \otimes_R S, \theta \otimes_R \text{id}_S)$ defines an object of $\mathcal{C}(p, S)$. We say that $(K, \theta) \in \mathcal{C}(p)$ is *reduced* if there exists an object $(\tilde{K}, \tilde{\theta}) \in \mathcal{C}(p, \mathbb{Z}/p^2\mathbb{Z})$ whose reduction modulo p is isomorphic to (K, θ) and such that \tilde{K} is flat over $\mathbb{Z}/p^2\mathbb{Z}$. If (K, θ) is reduced, tensorisation by \tilde{K} of the short exact sequence $0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$ gives rise to a short exact sequence of cochain complexes

$$0 \rightarrow K \rightarrow \tilde{K} \rightarrow K \rightarrow 0,$$

where $\tilde{K} \rightarrow K$ is the natural projection. The associated connecting map

$$\beta : H^*(K) \rightarrow H^{*+1}(K)$$

is called a Bockstein homomorphism or, more precisely, the Bockstein homomorphism associated to $(\tilde{K}, \tilde{\theta})$. (Simple examples show that β really depends on the choice of $(\tilde{K}, \tilde{\theta})$.)

2.2. The Steenrod operations

Given a prime p and an object (K, θ) of $\mathcal{C}(p)$, May constructs mod p Steenrod operations on $H^*(K)$ as follows; see [12, Definitions 2.2]. The homomorphism θ induces a map

$$\theta_* : H^*(W \otimes_{\mathbb{F}_p[\tau]} K^{\otimes p}) \rightarrow H^*(K).$$

Let $q, i \geq 0$ be integers, and let $x \in H^q(K)$. Define

$$D_i(x) := \theta_*(e_i \otimes x^{\otimes p}) \in H^{pq-i}(K).$$

It is checked in [12, p. 161] (using homological indexing) that this construction does not depend on the choice of representative of x . If $i < 0$, we set D_i equal to zero. When $p > 2$, for every integer n we let

$$\nu(n) := (-1)^{n(n-1)(p-1)/4} ((p-1)/2)!.$$

The mod p Steenrod operations on $H^*(K)$ are defined by the following formulas:

- If $p = 2$, $\text{Sq}^s(x) := D_{q-s}(x) \in H^{q+s}(K)$;
- If $p > 2$, $\text{P}^s(x) := (-1)^s \nu(-q) D_{(q-2s)(p-1)}(x) \in H^{q+2s(p-1)}(K)$; and $\beta \text{P}^s(x) := (-1)^s \nu(-q) D_{(q-2s)(p-1)-1}(x) \in H^{q+2s(p-1)+1}(K)$.

One then extends the definitions on arbitrary elements of $H^*(K)$ by linearity. Note that this defines Steenrod operations Sq^s and P^s for every integer s . However, for every $x \in H^*(K)$ at most finitely many operations are nonzero on x . For every $x \in H^*(K)$,

we define

$$\text{Sq}(x) := \sum_s \text{Sq}^s(x), \quad \text{P}(x) := \sum_s \text{P}^s(x).$$

There is not a construction of a Bockstein homomorphism β on $H^*(K)$ taking as input an arbitrary (K, θ) . In particular, the expression βP^s appearing in the definition is a single symbol and not a composition.

Lemma 2.1. *Let (K, θ) be a reduced object of $\mathcal{C}(p)$, and let $\beta : H^*(K) \rightarrow H^{*+1}(K)$ be a Bockstein homomorphism. Then β satisfies the following properties:*

(i) *It is a derivation in the graded sense: for every two homogeneous elements a, b of $H^*(K)$, we have*

$$\beta(a \cdot b) = \beta(a) \cdot b + (-1)^{|a|} a \cdot \beta(b).$$

(ii) *When $p = 2$, we have $\beta \circ \text{Sq}^s = s \cdot \text{Sq}^{s+1}$ for all $s \in \mathbb{Z}$. When $p > 2$, we have $\beta P^s = \beta \circ P^s$ for all $s \in \mathbb{Z}$.*

Proof. (i) This follows from the definition of β as a connecting homomorphism, together with the fact that the differential of \tilde{K} is a derivation in the graded sense.

(ii) This is proved in [12, Proposition 2.3(v), Corollary 2.5]. □

Let (K', θ') be another object of $\mathcal{C}(p)$, and let $f : K \rightarrow K'$ be a homomorphism of complexes. If f is a morphism in $\mathcal{C}(p)$, then the induced homomorphism $f_* : H^*(K) \rightarrow H^*(K')$ respects Steenrod operations.

Remark 2.2. If K is associative and $\theta = \epsilon \otimes m_p$, where $m_p : K^{\otimes p} \rightarrow K$ is the p -fold iterated product, then (K, θ) is an object of $\mathcal{C}(p)$. Moreover, it immediately follows from the definition that the p th-power Steenrod operations on $H^*(K)$ are trivial; that is, for all $x \in H^*(K)$ we have $\text{Sq}(x) = x^2$ (if $p = 2$) or $\text{P}(x) = x^p$ (if $p > 2$).

2.3. Operations on sheaf cohomology

We now review the construction of the Steenrod operations on sheaf cohomology given by Drury in [7]. Let p be a prime number, let \mathcal{T} be a topos with sufficiently many points¹ and let (A, d) be a commutative differential graded \mathbb{F}_p -algebra on \mathcal{T} . We denote by $\mathbb{H}^*(\mathcal{T}, A)$ the hypercohomology of A ; that is, $\mathbb{H}^*(\mathcal{T}, A) := H^*(\Gamma(\mathcal{T}, I))$, where $A \rightarrow I$ is an injective resolution of \mathbb{F}_p -vector spaces on \mathcal{T} . We denote by $m : A \otimes A \rightarrow A$ the multiplication map.

Let $\nu : A \rightarrow I$ be an injective resolution of A in the category of \mathbb{F}_p -vector spaces on \mathcal{T} . By [20, Tag 013K], we may choose the resolution to be *standard*: ν is a monomorphism in every degree, and $I^i = 0$ for $i < 0$.

Because we are working over a field, for every $r \geq 2$ the r -fold tensor product $\nu^{\otimes r} : A^{\otimes r} \rightarrow I^{\otimes r}$ is a monomorphism in every degree and a quasi-isomorphism; see [7, Lemma 3.2.16]. Here we use the fact that \mathcal{T} has sufficiently many points, because then exactness

¹Drury works in the context of sheaves on a topological space. As we will see, his definitions and arguments easily adapt to topoi with sufficiently many points.

may be checked on stalks. By [20, Tag 013P], there exists $\tilde{m} : I \otimes I \rightarrow I$ making the following diagram commute:

$$\begin{CD} A \otimes A @>{m}>> A \\ @VV{v \otimes v}V @VV{v}V \\ I \otimes I @>{\tilde{m}}>> I \end{CD}$$

By [20, Tag 013S], \tilde{m} is unique up to homotopy.

We define $K := \Gamma(\mathcal{T}, I)$, with differential induced by that of I . We define a homomorphism $M : K \otimes K \rightarrow K$ in degree $n \geq 0$ as the composition

$$\oplus_q \Gamma(\mathcal{T}, I^q) \otimes \Gamma(\mathcal{T}, I^{n-q}) \rightarrow \Gamma(\mathcal{T}, \oplus_q (I^q \otimes I^{n-q})) \xrightarrow{\Gamma(\tilde{m})} \Gamma(\mathcal{T}, I^n).$$

For every $a, b \geq 0$, we have a cup product given by the composition

$$\cup : H^a(K) \otimes H^b(K) \rightarrow H^{a+b}(K \otimes K) \xrightarrow{H^{a+b}(M)} H^{a+b}(K).$$

By [7, Lemma 5.1.3], the product $M : K \otimes K \rightarrow K$ makes K into a homotopy associative differential graded \mathbb{F}_p -algebra, and the induced cup product $\cup : H^*(K) \otimes H^*(K) \rightarrow H^*(K)$ is graded commutative and associative.

The natural map

$$A = \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}\mathbb{F}_p, A) \rightarrow \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I)$$

is an $\mathbb{F}_p[\pi]$ -resolution by $\mathbb{F}_p[\pi]$ -injective objects; see [9, Corollary 4.3.4] for the case when A is concentrated in degree 0 and [7, Lemma 3.2.21] for the general case. It follows that there exists a commutative diagram of π -equivariant maps

$$\begin{CD} A^{\otimes p} @>{m_p}>> A \\ @VV{v^{\otimes p}}V @VV{v}V \\ I^{\otimes p} @>{\tilde{m}_p}>> \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I). \end{CD} \tag{2.1}$$

The map \tilde{m}_p is unique up to a π -equivariant homotopy. Passing to global sections, we obtain an $\mathbb{F}_p[\pi]$ -homomorphism

$$\Gamma(\mathcal{T}, I)^{\otimes p} \rightarrow \Gamma(\mathcal{T}, I^{\otimes p}) \xrightarrow{\Gamma(\tilde{m}_p)} \Gamma(\mathcal{T}, \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I)) = \underline{\mathcal{H}om}_{\mathbb{F}_p}(W, \Gamma(\mathcal{T}, I)),$$

where the last equality is the projection formula [3, IV, Proposition 10.3]. By the tensor-hom adjunction, this is equivalent to a π -equivariant homomorphism

$$\theta : W \otimes \Gamma(\mathcal{T}, I)^{\otimes p} \rightarrow \Gamma(\mathcal{T}, I).$$

By [7, Lemmas 5.3.1 and 5.3.2], the pair (K, θ) satisfies the properties (i) and (ii) of Section 2.1 and so is an object in May’s category $\mathcal{C}(p)$. We thus obtain Steenrod operations on $\mathbb{H}^*(\mathcal{T}, A)$. It is straightforward to verify that the operations do not depend on the choice of $v : A \rightarrow I$, \tilde{m} and \tilde{m}_p .

Proposition 2.3. *Let \mathcal{T} be a topos with sufficiently many points, and let A be commutative differential graded \mathbb{F}_p -algebra on \mathcal{T} .*

(i) *For every $n \geq 0$ and every $x \in \mathbb{H}^*(\mathcal{T}, A)$, we have*

$$\text{Sq}^i(x) = \begin{cases} x^2 & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases} \quad \text{P}^i(x) = \begin{cases} x^p & \text{if } 2i = n, \\ 0 & \text{if } 2i > n, \end{cases}$$

when $p = 2$ and $p > 2$, respectively.

(ii) *The (internal) Cartan formulas of [12, (1) p. 165] hold.*

(iii) *The Ádem relations of [12, Theorem 4.7] hold.*

Proof. Property (i) holds for the Steenrod operations on $H^*(K)$, for every object $(K, \theta) \in \mathcal{C}(p)$, as is easily seen by unwinding the definitions of Sq^i and P^i .

Let now $(K, \theta) \in \mathcal{C}(p)$ be an object associated to A by the construction above. By [7, Lemma 5.3.3, Lemma 5.3.4], (K, θ) is a Cartan object and an Ádem object. We refer the reader to [12, Definition 2.1, Definition 4.1] for the definition of Cartan and Ádem objects. It now follows from [12, (1) p. 165, Theorem 4.7] that (ii) and (iii) are also satisfied. □

Remark 2.4. Let $u \in I^0$ be the image of the multiplicative identity of A . Then u is a homotopy identity for $\Gamma(\mathcal{T}, I)$, and the homomorphism $\mathbb{F}_p \rightarrow I$ given by $1 \mapsto u$ induces a morphism $(\mathbb{F}_p, \epsilon \otimes i) \rightarrow (\Gamma(\mathcal{T}, I), \theta)$ in $\mathcal{C}(p)$. Thus, $(\Gamma(\mathcal{T}, I), \theta)$ is unital. By [12, Proposition 3.1(iii)], if we let $1 \in H^*(K)$ denote the multiplicative unit, we have $\text{Sq}(1) = 1$ and $\text{P}(1) = 1$.

3. General properties

In this section, we establish properties of the Steenrod operations on $\mathbb{H}^*(\mathcal{T}, A)$ that hold in wide generality. In later sections, we will restrict our attention to the case when \mathcal{T} is the big étale site of a smooth algebraic stack X over a field of positive characteristic and A is the de Rham complex of X .

3.1. Functoriality

Let $f = (f_*, f^{-1}) : \mathcal{T}' \rightarrow \mathcal{T}$ be a morphism of topoi. Let (A', d') and (A, d) be commutative differential graded \mathbb{F}_p -algebras on \mathcal{T}' and on \mathcal{T} , respectively. Then $f^{-1}A$ is a commutative differential graded \mathbb{F}_p -algebra over \mathcal{T}' ; see [3, IV, 3.1.2]. Let $f^{-1}A \rightarrow A'$ be a homomorphism of commutative differential graded \mathbb{F}_p -algebras on \mathcal{T}' . There is an induced homomorphism

$$\mathbb{H}^*(\mathcal{T}, A) \rightarrow \mathbb{H}^*(\mathcal{T}', A'), \tag{3.1}$$

which is constructed as follows. Let $A \rightarrow I$ and $A' \rightarrow I'$ be standard injective resolutions of \mathbb{F}_p -vector spaces on \mathcal{T} . Because f^{-1} is exact, $f^{-1}A \rightarrow f^{-1}I$ is a quasi-isomorphism. By

[20, Tag 013P], there exists a morphism $f^{-1}I \rightarrow I'$ making the square

$$\begin{array}{ccc} f^{-1}A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ f^{-1}I & \longrightarrow & I' \end{array}$$

commute. By adjunction, we get a map $I \rightarrow f_*(I')$, and taking global sections yields (3.1). It is easy to check that (3.1) does not depend on the choice of resolutions I, I' and of the map $f^{-1}I \rightarrow I'$.

Lemma 3.1. *The homomorphism (3.1) is compatible with Steenrod operations.*

Proof. There is a canonical Σ_r -equivariant isomorphism $f^{-1}(A^{\otimes r}) \xrightarrow{\sim} (f^{-1}A)^{\otimes r}$ for every $r \geq 2$; see [3, Proposition 13.4(c)]. It follows that for every $r \geq 2$ we have a commutative Σ_r -equivariant diagram

$$\begin{array}{ccc} f^{-1}A^{\otimes r} & \longrightarrow & f^{-1}A \\ \downarrow & & \downarrow \\ (A')^{\otimes r} & \longrightarrow & A', \end{array}$$

where the horizontal maps are the r -fold multiplication maps. Denote by $(e'_*, (e')^{-1}) : \mathcal{T}' \rightarrow \mathcal{S}et$ the global sections morphism for \mathcal{T}' . We obtain a diagram of π -equivariant homomorphisms

$$\begin{array}{ccc} f^{-1}I^{\otimes p} & \xrightarrow{\quad\quad\quad} & f^{-1}\underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I) \\ \downarrow & \swarrow & \downarrow \\ & f^{-1}A^{\otimes p} \longrightarrow f^{-1}A & \\ & \downarrow & \downarrow \\ & (A')^{\otimes p} \longrightarrow A' & \\ \downarrow & \swarrow & \downarrow \\ (I')^{\otimes p} & \xrightarrow{\quad\quad\quad} & \underline{\mathcal{H}om}_{\mathbb{F}_p}((e')^{-1}W, I') \end{array}$$

where each of the five inner squares is commutative. To construct the left and right squares, apply [20, Tag 013P]. The bottom square is (2.1) for A' , and the top square is the pullback of (2.1) for A . Thus, because f^{-1} is exact, the four diagonal arrows are π -equivariant quasi-isomorphisms. We deduce that the outer square is commutative in the derived category of π -equivariant \mathbb{F}_p -vector spaces on \mathcal{T} . Because every term of $\underline{\mathcal{H}om}_{\mathbb{F}_p}((e')^{-1}W, I')$ is injective, by [20, Tag 05TG] the outer square is π -homotopy commutative. Using the adjunction between f^{-1} and f_* , we obtain a π -homotopy

commutative diagram

$$\begin{array}{ccc}
 I^{\otimes p} & \longrightarrow & \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I) \\
 \downarrow & & \downarrow \\
 f_*(I')^{\otimes p} & \longrightarrow & f_*\underline{\mathcal{H}om}_{\mathbb{F}_p}((e')^{-1}W, I').
 \end{array}$$

We thus get a diagram

$$\begin{array}{ccccc}
 \Gamma(\mathcal{T}, I)^{\otimes p} & \longrightarrow & \Gamma(\mathcal{T}, I^{\otimes p}) & \longrightarrow & \underline{\mathcal{H}om}_{\mathbb{F}_p}(W, \Gamma(\mathcal{T}, I)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Gamma(\mathcal{T}', I')^{\otimes p} & \longrightarrow & \Gamma(\mathcal{T}', (I')^{\otimes p}) & \longrightarrow & \underline{\mathcal{H}om}_{\mathbb{F}_p}(W, \Gamma(\mathcal{T}', I')),
 \end{array}$$

where the square on the left is π -equivariantly commutative and the square on the right is obtained by taking global sections in the square above and using the projection formula [3, IV, Proposition 10.3], and so it is π -homotopy commutative. Using the tensor-hom adjunction on the outer rectangle in the previous diagram, we deduce that the square

$$\begin{array}{ccc}
 W \otimes \Gamma(\mathcal{T}, I)^{\otimes p} & \xrightarrow{\theta} & \Gamma(\mathcal{T}, I) \\
 \downarrow & & \downarrow \\
 W \otimes \Gamma(\mathcal{T}', I')^{\otimes p} & \xrightarrow{\theta'} & \Gamma(\mathcal{T}', I')
 \end{array}$$

is π -homotopy commutative. This means that (3.1) gives a morphism in $\mathcal{C}(p)$, and so it is compatible with Steenrod operations. □

Remark 3.2. Consider the special case when $A' = f^{-1}A$. We have a commutative square

$$\begin{array}{ccc}
 \mathbb{H}^*(\mathcal{T}, A) & \longrightarrow & \mathbb{H}^*(\mathcal{T}', f^{-1}A) \\
 \downarrow & & \downarrow \wr \\
 \mathbb{H}^*(\mathcal{T}, f_*f^{-1}A) & \longrightarrow & \mathbb{H}^*(\mathcal{T}, Rf_*f^{-1}A),
 \end{array}$$

where the top horizontal map is (3.1), the vertical map on the left is induced by the unit $A \rightarrow f_*f^{-1}A$ and the bottom horizontal map is induced by the natural map $f_*f^{-1}A \rightarrow Rf_*f^{-1}A$. The vertical isomorphism on the right is obtained using the fact that f_* respects injectives. The proof is a simple exercise in homological algebra; we leave it to the reader. This diagram will be used during the proof of Proposition 6.1.

3.2. The case when A is concentrated in degree 0

Let \mathcal{T} be a topos, and let $q = (q_*, q^{-1})$ be a point of \mathcal{T} ; that is, a morphism from the set topos $\mathcal{S}et$ to \mathcal{T} ; see [3, IV, Définition 6.1]. If E is a set, q_*E is called a skyscraper sheaf; see [20, Tag 00Y9].

Lemma 3.3. *Let k be a field, and let $F = \prod_{j \in J} q_{j*} V_j$, where J is a set and, for every $j \in J$, V_j is a k -vector space and q_j is a point of \mathcal{T} . Then F is an injective object of the category of sheaves of k -vector spaces on \mathcal{T} .*

Note that, in Lemma 3.3, J is allowed to be infinite, and the V_j may be infinite-dimensional over k .

Proof. If $\{F_j\}_{j \in J}$ is a collection of sheaves of injective k -vector spaces on \mathcal{T} , then $\prod_j F_j$ is also injective. Therefore, it suffices to show that $q_{j*} V_j$ is injective for every $j \in J$. In the category of k -vector spaces, every object is injective. Note that q_{j*} is a right adjoint even when viewed as a functor between the associated ringed topoi of k -vector spaces; see [3, IV, Proposition 13.4]. It follows that each q_{j*} preserves injective objects, and the conclusion follows. \square

Assume now that \mathcal{T} has sufficiently many points, and let $\{q_i\}_{i \in I}$ be a conservative family of points of \mathcal{T} ; see [20, Tag 00YK]. Let k be a field. There is an endofunctor of the category of k -vector spaces on \mathcal{T} given on objects V by

$$S(V) := \prod_{i \in I} q_{i*} q_i^{-1} V$$

and defined in an obvious way on morphisms. Iterating S , for every k -vector space V on \mathcal{T} we obtain a cosimplicial k -vector space $S^*(V)$; see [9, 8.1.4]. By definition, the degree i component $S^i(V)$ of $S^*(V)$ is given by iteratively applying S on V $i + 1$ times. Applying the associated (nonnormalised) cochain complex functor $K(-)$, we get a quasi-isomorphism $V \rightarrow K(S^* V)$. By Lemma 3.3, every term of $K(S^* V)$ is injective, and so $V \rightarrow K(S^* V)$ is an injective resolution of k -vector spaces on \mathcal{T} . It may be helpful for the reader to note that this is a special case of [9, Theorem 8.1.5].

Let now $k = \mathbb{F}_p$, and let A be an \mathbb{F}_p -algebra on \mathcal{T} . In [9, (8.3.2)], a π -equivariant map

$$K(S^* A)^{\otimes p} \rightarrow \underline{\mathcal{H}om}(e^{-1} W, K(S^* A)) \tag{3.2}$$

is constructed. We will need the details of the construction of (3.2) during the proof of Lemma 3.4. We have a sequence of π -equivariant cochain maps

$$\begin{aligned} W(p, \mathbb{Z}) &\rightarrow \underline{\mathcal{H}om}(\nabla k, t) \rightarrow \underline{\mathcal{H}om}(K(S^* A)^{\otimes p}, K((S^* A)^{\otimes p})) \\ &\rightarrow \underline{\mathcal{H}om}(K(S^* A)^{\otimes p}, K(S^* A)), \end{aligned} \tag{3.3}$$

defined as follows:

- The first map (from left to right) is the composition of two homomorphisms $\Psi : W(p, \mathbb{Z}) \rightarrow \underline{\mathcal{H}om}(\nabla k, t)$ and $\Theta : \underline{\mathcal{H}om}(\nabla k, t) \rightarrow \underline{\mathcal{H}om}(\nabla k, t)$, both defined in [9, p. 217].²
- The second map is induced from the composition $E_A \hat{M}$ of the functors \hat{M} and E_A , as defined in [9, p. 193]. That this is a homomorphism of complexes is checked in [9, p. 194].

²In [9, p. 217], $W(p, \mathbb{Z})$ is denoted by \mathcal{W} . The complex $\underline{\mathcal{H}om}(\nabla k, t)$ is defined in [8, 3.1], where it is denoted by $\mathcal{H}om(\nabla k, t)$.

- The multiplication map $A^{\otimes p} \rightarrow A$ induces a cosimplicial map $S^*(A^{\otimes p}) \rightarrow S^*A$, and hence a cochain map $K((S^*A)^{\otimes p}) \rightarrow K(S^*A)$. Consider the cosimplicial map $(S^*A)^{\otimes p} \rightarrow S^*(A^{\otimes p})$ given by [9, Lemma 8.2.3] and the induced cochain map $K(S^*A)^{\otimes p} \rightarrow K(S^*(A^{\otimes p}))$. The third map is induced by the composition $K(S^*A)^{\otimes p} \rightarrow K(S^*(A^{\otimes p})) \rightarrow K(S^*A)$.

Because A is p -torsion, (3.3) factors through the projection $W(p, \mathbb{Z}) \rightarrow W$. Using the adjunction between e_* and e^{-1} and the tensor-hom adjunction, we get the following canonical isomorphisms:

$$\begin{aligned} \text{Hom}(W, \underline{\text{Hom}}(K(S^*A)^{\otimes p}, K(S^*A))) &= \text{Hom}(e^{-1}W, \underline{\mathcal{H}om}(K(S^*A)^{\otimes p}, K(S^*A))) \\ &= \text{Hom}(e^{-1}W \otimes K(S^*A)^{\otimes p}, K(S^*A)) \quad (3.4) \\ &= \text{Hom}(K(S^*A)^{\otimes p}, \underline{\mathcal{H}om}(e^{-1}W, K(S^*A))). \end{aligned}$$

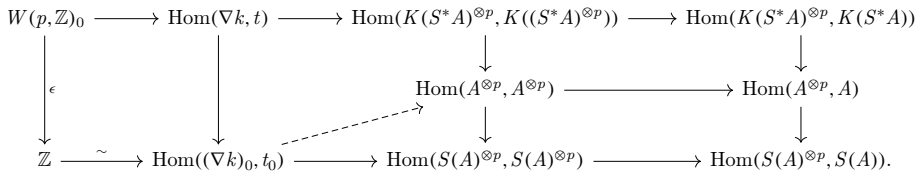
We define (3.2) to be the image of (3.3) under this chain of isomorphisms.

On the other hand, applying (2.1) with $I = K(S^*A)$, we have another homomorphism

$$K(S^*A)^{\otimes p} \rightarrow \underline{\mathcal{H}om}(e^{-1}W, K(S^*A)). \quad (3.5)$$

Lemma 3.4. *The homomorphisms (3.2) and (3.5) are homotopic.*

Proof. We construct the following commutative diagram of π -equivariant maps:



Here, the top row is obtained by applying the functor $Z^0(-)$ of 0-cocycles to (3.3).

If C_1 and C_2 are cochain complexes in an additive category \mathcal{A} , there is a natural map

$$\text{Hom}(C_1, C_2) \rightarrow \text{Hom}((C_1)^0, (C_2)^0).$$

Moreover, if \mathcal{A} is abelian, this map factors as

$$\text{Hom}(C_1, C_2) \rightarrow \text{Hom}(Z^0(C_1), Z^0(C_2)) \rightarrow \text{Hom}((C_1)^0, (C_2)^0).$$

All of these homomorphisms are contravariantly functorial in C_1 and covariantly functorial in C_2 . This is how the square on the left and the two squares on the right are constructed. The pentagon of solid arrows in the center is induced by the functor $E_A \hat{M}$. This completes the construction of the solid arrows in the previous diagram.

We have $(\nabla k)_0 = t_0 = M(0, \dots, 0)$ (see [8, End of p. 213] for the notation), and the bottom map $\mathbb{Z} \rightarrow \text{Hom}((\nabla k)_0, t_0)$ sends 1 to the identity. Hence, the composition $\mathbb{Z} \rightarrow \text{Hom}(S(A)^{\otimes p}, S(A)^{\otimes p})$ sends 1 to the identity. Because $\text{id} \in \text{Hom}(A^{\otimes p}, A^{\otimes p})$ is π -invariant, we obtain the dashed arrow.

The composition $\eta : \mathbb{Z} \rightarrow \text{Hom}(A^{\otimes p}, A)$ of the bottom row sends 1 to the p -fold multiplication map $m_p : A^{\otimes p} \rightarrow A$. Because A is p -torsion, the composition of the top

row factors through W_0 and $\mathbb{Z} \rightarrow \text{Hom}(A^{\otimes p}, A)$ factors through \mathbb{F}_p . This proves that the square of π -equivariant maps

$$\begin{array}{ccc} W_0 & \longrightarrow & \text{Hom}(K(S^*A)^{\otimes p}, K(S^*A)) \\ \downarrow \epsilon & & \downarrow \\ \mathbb{F}_p & \xrightarrow{\eta} & \text{Hom}(A^{\otimes p}, A) \end{array}$$

is commutative. We have adjunction isomorphisms

$$\begin{aligned} \text{Hom}(\mathbb{F}_p, \underline{\text{Hom}}(A^{\otimes p}, A)) &= \text{Hom}(e^{-1}\mathbb{F}_p, \underline{\mathcal{H}om}(A^{\otimes p}, A)) \\ &= \text{Hom}(e^{-1}\mathbb{F}_p \otimes A^{\otimes p}, A) \\ &= \text{Hom}(A^{\otimes p}, \underline{\mathcal{H}om}(e^{-1}\mathbb{F}_p, A)), \end{aligned}$$

which are compatible with those of (3.4). Applying the adjunctions to the above square, we get a commutative square of π -equivariant homomorphisms

$$\begin{array}{ccc} K(S^*A)^{\otimes p} & \longrightarrow & \underline{\mathcal{H}om}(e^{-1}W, K(S^*A)) \\ \uparrow & & \uparrow \\ A^{\otimes p} & \xrightarrow{m_p} & A, \end{array}$$

where the top horizontal map is (3.2), the left vertical map is a monomorphism in every degree and a quasi-isomorphism and the right vertical map is a π -injective resolution. By definition, the map (3.5) fits in a square (2.1) of the same form. The conclusion now follows from [20, Tag 013S]. □

Proposition 3.5. *Let A be an \mathbb{F}_p -algebra on \mathcal{T} .*

- (a) *All negative Steenrod operations on $\mathbb{H}^*(\mathcal{T}, A)$ are zero.*
- (b) *The operations Sq^0 and P^0 are induced by the Frobenius endomorphism $A \rightarrow A$.*

Proof. (a) It suffices to show that $D_i(x) = 0$ for every $x \in H^q(K(S^*A))$ and every $i > q(p-1)$ ($i \geq (q+2)(p-1)$ would be sufficient). Let $x \in H^q(K(S^*A))$.

By definition, $\Theta\Psi(e_i) \in \prod_j \text{Hom}((\nabla k)_j, t_{i+j})$; thus,

$$\Theta\Psi(e_i)^{q'} \in \text{Hom}((\nabla k)_{q'}, t_{i+q'}) = \text{Hom}(M(q', \dots, q'), \oplus_{\sum i_h=i+q'} M(i_1, \dots, i_p)).$$

Now, $E_A \hat{M}$ and the multiplication $A^{\otimes p} \rightarrow A$ induce a homomorphism

$$f^{i, q'} : \text{Hom}(M(q', \dots, q'), \oplus_{\sum i_h=i+q'} M(i_1, \dots, i_p)) \rightarrow \text{Hom}((K(S^*A)^{\otimes p})^{i+q'}, S^{q'}A).$$

By Lemma 3.4,

$$\theta_*(e_i \otimes x^{\otimes p}) = (f^{i, q'}(\Theta\Psi(e_i)^{q'}))(x^{\otimes p}),$$

where $i + q' = pq$; that is, $q' = pq - i$. It thus suffices to prove that $\Theta\Psi(e_i)^{q'} = 0$, where $q' = pq - i$. By [8, Theorem 5.1.2], this holds when $i > q'(p-1)$. This is equivalent to $i > (pq - i)(p-1)$; that is, $i > q(p-1)$, which is true by assumption.

(b) As in Section 2.1, in the case $p \neq 2$, for every integer n we let

$$\nu(n) := (-1)^{n(n-1)(p-1)/4}((p-1)/2)!.$$

Note that $((p-1)/2)!^2 \equiv (-1)^{(p+1)/2} \pmod{p}$, and so $\nu(n)\nu(-n) \equiv 1 \pmod{p}$. By definition, $P^0(x) = \nu(-q)D_{q(p-1)}(x)$ when $p > 2$, and $Sq^0(x) = D_q(x)$ when $p = 2$.

Assume that $p > 2$. Let

$$f(q)^q := \Theta\Psi(e_{q(p-1)})^q : M(q, \dots, q) \rightarrow \bigoplus_{\sum i_h = pq} M(i_1, \dots, i_p).$$

By [8, Theorem 5.1.2], all of the components of $f(q)^q$ are zero, except $M(q, \dots, q) \rightarrow M(q, \dots, q)$, which is given by multiplication by an integer congruent to $\nu(q)$ modulo p . It follows that

$$D_{q(p-1)}(x) = \theta_*(e_{q(p-1)} \otimes x^{\otimes p}) = \nu(q)S^{q(p-1)}(m^{\otimes p})(x^{\otimes p}) = \nu(q)S^{q(p-1)}(\text{Fr})(x)$$

for every $x \in H^q(K)$. Here $\text{Fr} : A \rightarrow A$ denotes the Frobenius endomorphism of A . In the last step, we have used the fact that $\text{Fr} = m^{\otimes p} \circ i$, where $i : A \rightarrow A^{\otimes p}$ is defined on sections by $a \mapsto a^{\otimes p}$. Therefore,

$$P^0(x) = \nu(-q)D_{q(p-1)}(x) = \nu(-q)\nu(q)S^{q(p-1)}(\text{Fr})(x) = S^{q(p-1)}(\text{Fr})(x),$$

as desired. A similar argument (with no sign issues) works when $p = 2$. □

4. Steenrod operations on Čech cohomology

4.1. The Eilenberg–Zilber operad

Let R be a commutative ring with identity. For every $n \geq 0$, let Δ^n be the standard n -dimensional simplicial set and denote by $R[\Delta^n]$ the free simplicial R -module on Δ^n . We let Λ and Λ_N be the cosimplicial cochain complexes of R -modules such that

$$\Lambda^n := K(R[\Delta^n]), \quad \Lambda_N^n := K_N(R[\Delta^n])$$

for all $n \geq 0$. Here $K(-)$ and $K_N(-)$ are the unnormalised and normalised cochain complex functors, respectively. For all $n \geq 0$, Λ^n and Λ_N^n are nonpositively graded, Λ_N^n is bounded and Λ^n is unbounded in the negative direction.

Let \mathcal{T} be a topos. If M and N are two cosimplicial cochain complexes of R -modules on \mathcal{T} , we define a cochain complex $\underline{\text{Hom}}_{\Delta}(M, N)$ of R -modules on \mathcal{T} as the equaliser of

$$\prod_{r \geq 0} \underline{\text{Hom}}(M^r, N^r) \rightrightarrows \prod_{[r] \rightarrow [s]} \underline{\text{Hom}}(M^r, N^s),$$

where the second product is over all arrows in the simplicial category Δ , and the two maps are induced by pre-composition and post-composition. This definition is a special case of [14, (2.1)]. When $\mathcal{T} = \text{Set}$, so that M and N are cosimplicial cochain complexes of R -modules, we denote $\underline{\text{Hom}}_{\Delta}(M, N)$ by $\underline{\text{Hom}}_{\Delta}(M, N)$.

Recall that we write $(e_*, e^{-1}) : \mathcal{T} \rightarrow \text{Set}$ for the morphism of topoi such that $e_* = \Gamma(\mathcal{T}, -)$ and e^{-1} is the constant sheaf functor.

Lemma 4.1. *Let C be a cosimplicial nonnegatively graded cochain complex of R -modules on \mathcal{T} . Then we have a commutative diagram*

$$\begin{CD} \text{Tot}(K_N(C)) @>>> \text{Tot}(K(C)) \\ @VV\cong V @VV\cong V \\ \underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda_N, C) @>>> \underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda, C), \end{CD}$$

where the vertical arrows are isomorphisms and the horizontal arrows are cochain homotopy equivalences.

Proof. The cochain complex $\underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda, C)$ is given in degree n by the equaliser of

$$\prod_{r \geq 0} \prod_{q \in \mathbb{Z}} \mathcal{H}om(e^{-1}K(\mathbb{Z}[\Delta^r])^q, (C^r)^{q+n}) \rightrightarrows \prod_{[r] \rightarrow [s]} \prod_{q \in \mathbb{Z}} \mathcal{H}om(e^{-1}K(\mathbb{Z}[\Delta^r])^q, (C^s)^{q+n}).$$

If M is an R -module on \mathcal{T} , then $\mathcal{H}om(e^{-1}R, M) \cong M$. It follows that

$$\mathcal{H}om(e^{-1}K(\mathbb{Z}[\Delta^r])^q, (C^s)^{q+n}) \cong ((C^s)^{q+n})^{\oplus \Delta_{-q}^r}$$

is a direct sum of copies of $(C^s)^{q+n}$, parametrised by the set Δ_{-q}^r of $(-q)$ -dimensional simplices of Δ^r . Thus, $\underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda, C)$ in degree n is the equaliser of

$$\prod_{r \geq 0} \prod_{q \in \mathbb{Z}} ((C^r)^{q+n})^{\oplus \Delta_{-q}^r} \rightrightarrows \prod_{[r] \rightarrow [s]} \prod_{q \in \mathbb{Z}} ((C^s)^{q+n})^{\oplus \Delta_{-q}^r}.$$

We now construct the map $\text{Tot}(K(C)) \rightarrow \underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda, C)$ in degree n . For every integer $0 \leq i \leq n$, let $\sigma_i \in \Delta_i^i$ be the fundamental class. If r is another integer and $\sigma \in \Delta_i^r$, we define $(C^i)^{n-i} \rightarrow ((C^r)^{n-i})^{\oplus \Delta_i^r}$ on the component relative to σ as the homomorphism induced by the map $[i] \rightarrow [r]$ given by $\sigma_i \mapsto \sigma$. where the first map is induced by the inclusion $\{\sigma_i\} \hookrightarrow \Delta_i^i$, and the other two maps are induced by the map $[i] \rightarrow [r]$ arising from $\sigma_i \mapsto \sigma$. We set $(C^i)^{n-i} \rightarrow ((C^r)^{q+n})^{\oplus \Delta_{-q}^r}$ equal to zero when $q \neq -i$. These maps assemble to a homomorphism from $\text{Tot}(K(C))$ to the product on the left, which respects the equaliser condition. We thus get a homomorphism

$$\text{Tot } K(C) \rightarrow \underline{\mathcal{H}om}_\Delta(e^{-1}\Lambda, C).$$

It is an isomorphism whose inverse is induced by projection onto the factors corresponding to the fundamental classes of the Δ_i^i . The proof for $K_N(C)$ is entirely analogous, with the difference that one only works with nondegenerate simplices. The commutativity of the diagram is then obvious.

If A is a cosimplicial object of an abelian category, the map $K_N(A) \rightarrow K(A)$ is a homotopy equivalence. Applying this to the category of cochain complexes of R -modules on \mathcal{T} , we see that $K_N(C) \rightarrow K(C)$ is a homotopy equivalence of double complexes (that is, complexes of complexes). Passing to total complexes, we obtain the required cochain homotopy equivalence $\text{Tot}(K_N(C)) \rightarrow \text{Tot}(K(C))$. □

We define $\mathcal{Z} := \text{End}(\Lambda)$, the endomorphism operad of Λ in the category of cochain complexes of R -modules; see [14, Construction 2.3]. Similarly, we define $\mathcal{Z}_N := \text{End}(\Lambda_N)$.

By definition, for every $j \geq 0$ we have cochain complexes

$$\mathcal{Z}(j) = \text{Hom}_\Delta(\Lambda, \Lambda^{\otimes j}), \quad \mathcal{Z}_N(j) = \text{Hom}_\Delta(\Lambda_N, \Lambda_N^{\otimes j}),$$

which are concentrated in nonpositive degrees.

Lemma 4.2. *For every $j \geq 0$, we have a homotopy equivalence $\mathcal{Z}_N(j) \rightarrow \mathcal{Z}(j)$.*

The maps of Lemma 4.2 assemble to a quasi-isomorphism of operads $\mathcal{Z}_N \rightarrow \mathcal{Z}$, but we will not need this.

Proof. By Lemma 4.1, it is enough to show that the composition

$$\text{Tot}(K_N(\Lambda_N^{\otimes j})) \rightarrow \text{Tot}(K(\Lambda_N^{\otimes j})) \rightarrow \text{Tot}(K(\Lambda^{\otimes j})) \tag{4.1}$$

is a homotopy equivalence.

If A is a cosimplicial object in an abelian category \mathcal{A} , by the Dold–Kan correspondence we have a decomposition $K(A) = K_N(A) \oplus D(A)$ in the category of cochain complexes of \mathcal{A} ; see [20, Tag 019I] or [10, Theorem III.2.5]. The complex $D(A)$ is homotopically equivalent to zero, and so the inclusion $K_N(A) \rightarrow K(A)$ is a cochain homotopy equivalence, natural in A . The cochain homotopy between the identity on $K(A)$ and the composition $K(A) \rightarrow K_N(A) \rightarrow K(A)$ is also natural in A ; see the paragraph preceding [10, Theorem 2.4].

We apply this to the case when \mathcal{A} is the category of cochain complexes of R -modules and then pass to total complexes. If a bicomplex is homotopy equivalent to zero (as a complex in the category of complexes), then its totalisation is also homotopy equivalent to zero. Thus, if C is a cosimplicial R -module, we have a decomposition $\text{Tot } K(C) = \text{Tot } K_N(C) \oplus \text{Tot } D(C)$, and $\text{Tot } D(C)$ is homotopically equivalent to zero. Letting $C = \Lambda_N^{\otimes j}$, we deduce that the first map in (4.1) is a homotopy equivalence.

We have a decomposition $\Lambda = \Lambda_N \oplus \Lambda'$. (For every $n \geq 0$, $(\Lambda')^n$ is the subcomplex of $K(R[\Delta^n])$ generated by degenerate simplices.) Thus, $\Lambda^{\otimes j} = \Lambda_N^{\otimes j} \oplus U$, where U is a direct sum of terms of the form $U_1 \otimes \cdots \otimes U_j$, where $U_h \in \{\Lambda_N, \Lambda'\}$ and at least one of the U_h equals Λ' . Passing to total complexes, we see that in order to prove that the second map is a homotopy equivalence, it suffices to show that each $K(U_1 \otimes \cdots \otimes U_j)$ is homotopy equivalent to zero (as a complex of complexes), because then $\text{Tot}(K(U_1 \otimes \cdots \otimes U_j))$ will also be homotopically trivial. The classical Alexander–Whitney map gives a homotopy equivalence between $K(U_1 \otimes \cdots \otimes U_j)$ and $K(U_1) \otimes \cdots \otimes K(U_j)$, so it suffices to show that the latter are homotopically trivial.

Because the homotopies in the Dold–Kan correspondence are functorial, $K(\Lambda')$ is homotopically trivial (as a complex in the category of complexes). Let M, N and N' be double complexes, and let $f : N \rightarrow N'$ be a homotopy equivalence in the category of complexes of complexes: $f^n : N^n \rightarrow (N')^n$ is a homotopy equivalence for every n , and the homotopies commute with vertical differentials. Then $\text{id} \otimes f : M \otimes N \rightarrow M \otimes N'$ is also a homotopy equivalence (here \otimes is the tensor product in the category of complexes of complexes). Because $K(\Lambda')$ is homotopically trivial and at least one U_h is equal to Λ' , we conclude that each $K(U_1) \otimes \cdots \otimes K(U_j)$ is homotopically equivalent to zero (as a complex of complexes), as desired. We conclude that the $K(U_1) \otimes \cdots \otimes K(U_j)$ are homotopically trivial, and so the second map of (4.1) is also a homotopy equivalence. \square

Note that the operad defined in [14, Definition 3.1] is denoted there by \mathcal{Z} , but it coincides with our \mathcal{Z}_N . By a result of Mandell [14, Proposition 3.2], for all $j \geq 0$ there is an augmentation map $\mathcal{Z}_N(j) \rightarrow R$ which is a quasi-isomorphism. By [14, Proposition 3.3], there are an E_∞ -operad \mathcal{E} and a quasi-isomorphism of operads $\alpha_N : \mathcal{E} \rightarrow \mathcal{Z}_N$. In particular, $\mathcal{E}(j)$ is an $R[\Sigma_j]$ -free resolution of R for every $j \geq 0$, concentrated in nonpositive degrees, and we have a Σ_j -equivariant quasi-isomorphism of cochain complexes $\alpha_N(j) : \mathcal{E}(j) \rightarrow \mathcal{Z}_N(j)$.

Pre-composing with $\alpha_N(j)$, we obtain a Σ_j -equivariant quasi-isomorphism

$$\alpha(j) : \mathcal{E}(j) \rightarrow \mathcal{Z}(j).$$

Because $W(p, R)$ and $\mathcal{E}(p)$ are both $R[\pi]$ -free resolutions of R , by [20, Tag 0649] there exists a π -homotopy equivalence $W(p, R) \rightarrow \mathcal{E}(p)$ commuting with the augmentations. The composition

$$W(p, R) \rightarrow \mathcal{E}(p) \xrightarrow{\alpha(p)} \mathcal{Z}(p) \tag{4.2}$$

is a π -equivariant quasi-isomorphism. Because e^{-1} is exact (it commutes with colimits and finite limits), $e^{-1}\alpha(p)$ and the pullback of (4.2) are also quasi-isomorphisms.

4.2. The relative Čech complex

In [14], May defined an E_∞ -algebra structure on Čech cochains, thus obtaining Steenrod operations on the Čech cohomology of sheaves. This alternative definition is a crucial ingredient in the proofs of Theorem 1.1(v) and Theorem 1.2(a). Note that May works in the category of sheaves on a topological space but, as we explain below, his arguments may be easily adapted to the setting of sheaves on a site; this is explicitly mentioned in [14, §4, p. 9 and Remark 5.10].

Let \mathcal{T} be a topos, let R be a commutative ring with identity and let F be a cochain complex of R -modules on \mathcal{T} . We fix an equivalence of \mathcal{T} with the category of sheaves on a site \mathcal{S} admitting a terminal object; by Giraud’s criterion [3, IV, Théorème 1.2(i’)] such \mathcal{S} always exists (it is usually straightforward to exhibit a concrete \mathcal{S} in practice). We view F as a complex of sheaves of R -modules on \mathcal{S} via this equivalence.

Let e be a terminal object of \mathcal{S} , and let U be the Čech nerve (that is, the 0-coskeleton) of a cover of e in \mathcal{S} . For every $r \geq 0$ let $\eta_r : U_r \rightarrow e$ denote the unique map to e , and let $(\eta_{r*}, \eta_r^{-1}) : \mathcal{T}/U_r \rightarrow \mathcal{T}$ be the associated morphism of topoi. We define $\check{C}_0(U, F)$ as the cosimplicial complex of sheaves on \mathcal{S} such that

$$\check{C}_0(U, F)^{r,s} := \eta_{r*}(\eta_r^{-1}(F^s))$$

and whose differentials are induced from those of F and the degeneracy maps of U . Similarly, we denote by $\check{C}(U, F) := K(\check{C}_0(U, F))$ the bicomplex of sheaves on \mathcal{S} such that

$$\check{C}(U, F)^{r,s} := \eta_{r*}(\eta_r^{-1}(F^s))$$

and whose differentials are induced from those of F and the alternating sums of the face maps of U . We set

$$\check{C}_0(U, F) := \Gamma(\mathcal{T}, \check{C}_0(U, F)), \quad \check{C}(U, F) := \Gamma(\mathcal{T}, \check{C}(U, F)),$$

the global section functor being applied level-wise. Thus, for all $r, s \geq 0$,

$$\check{C}(U, F)^{r,s} = \Gamma(\mathcal{T}, \eta_{r*}(\eta_r^{-1}(F^s))) = \Gamma(\mathcal{T}/U_r, \eta_r^{-1}(F^s)) = \Gamma(U_r, F^s).$$

When \mathcal{S} is the big fppf site of a scheme, for every $s \geq 0$ the cochain complex $\check{C}(U, F)^{*,s}$ is the relative Čech complex of F^s of [20, Tag 06X7]. The unit maps $F^s \rightarrow \eta_{0*}(\eta_0^{-1}(F^s))$ induce a homomorphism of bicomplexes $F \rightarrow \check{C}(U, F)$ that is natural in F ; here F is regarded as a bicomplex concentrated in the zeroth row. Passing to total complexes, we obtain a homomorphism of complexes of sheaves

$$\iota_F : F \rightarrow \text{Tot} \check{C}(U, F).$$

The map ι_F is a quasi-isomorphism of complexes of sheaves; we refer the reader to [16, Lemma 2.4.18] for a proof in the case when F is concentrated in degree 0 (i.e., a sheaf). The proof for arbitrary F is entirely analogous and goes as follows. Because $\eta_0 : U_0 \rightarrow e$ is a cover, one may check the property after pulling back to \mathcal{S}/U_0 . The pullback of $\check{C}(U, F)$ is the Čech complex of η_0^*F with respect to the Čech nerve of $\text{pr}_2 : U_0 \times U_0 \rightarrow U_0$. In other words, we may assume that $U_0 \rightarrow e$ has a section. In this case, $F \rightarrow \check{C}(U, F)$ is a homotopy equivalence (of complexes of complexes), given by the same formulas as [16, Lemma 2.4.18]. Passing to total complexes, we deduce that ι_F is a homotopy equivalence, and hence a quasi-isomorphism.

If F' is another cochain complex of R -modules on \mathcal{T} , we have a natural homomorphism

$$\cup : \text{Tot} \check{C}(U, F) \otimes \text{Tot} \check{C}(U, F') \rightarrow \text{Tot} \check{C}(U, F \otimes F'). \tag{4.3}$$

The map (4.3) is compatible with $\iota_F, \iota_{F'}$ and $\iota_{F \otimes F'}$ in an obvious way.

Let $j \geq 0$ be an integer, and let C be a cosimplicial complex of R -modules on \mathcal{T} . The Alexander–Whitney map is a homomorphism

$$e^{-1} \mathcal{Z}(j) \otimes (\text{Tot} K(C))^{\otimes j} \rightarrow \text{Tot} K(C^{\otimes j}), \tag{4.4}$$

where on the right we are taking tensor products of cosimplicial objects. Under the identifications provided by Lemma 4.1, (4.4) is given by the composition

$$\begin{aligned} \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda, e^{-1} \Lambda^{\otimes j}) \otimes \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda, C)^{\otimes j} \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda, e^{-1} \Lambda^{\otimes j}) \otimes \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda^{\otimes j}, C^{\otimes j}) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda^{\otimes j}, C^{\otimes j}) \otimes \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda, e^{-1} \Lambda^{\otimes j}) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1} \Lambda, C^{\otimes j}), \end{aligned}$$

where the first map is induced by the j -fold tensor product, the second is the (graded) switch isomorphism and the third is induced by composition. This is a special case of [14, Construction 2.6] (to see this, note that e^{-1} commutes with finite limits, and so $e^{-1} \text{End}(\Lambda)(j) = \text{End}(e^{-1} \Lambda)(j)$). Of course, if \mathcal{T} is the set topos, so that C is just a cosimplicial cochain complex of R -modules, (4.4) takes the form

$$\mathcal{Z}(j) \otimes (\text{Tot} K(C))^{\otimes j} \rightarrow \text{Tot} K(C^{\otimes j}). \tag{4.5}$$

4.3. May’s construction

Let A be a commutative differential graded R -algebra on \mathcal{T} . We have a Σ_p -equivariant homomorphism³

$$\mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \text{Tot } \check{C}(U, A). \tag{4.6}$$

Under the identifications of Lemma 4.1, (4.6) is defined as the composition

$$\begin{aligned} \underline{\text{Hom}}_{\Delta}(\Lambda, \Lambda^{\otimes p}) \otimes \underline{\text{Hom}}_{\Delta}(\Lambda, \check{C}_0(U, A))^{\otimes p} \\ \rightarrow \underline{\text{Hom}}_{\Delta}(\Lambda, \check{C}_0(U, A)^{\otimes p}) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(\Lambda, \check{C}_0(U, A^{\otimes p})) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(\Lambda, \check{C}_0(U, A)), \end{aligned}$$

where the first homomorphism is the Alexander–Whitney map, the second homomorphism is induced by the natural map $\check{C}_0(U, A)^{\otimes p} \rightarrow \check{C}_0(U, A^{\otimes p})$ and the third homomorphism is induced by the multiplication map $A^{\otimes p} \rightarrow A$.

Remark 4.3. It is clear that (4.4) is functorial in C and that (4.6) is functorial in A .

Let \mathcal{T}' be another topos, let e' be a terminal object in some site with associated topos equivalent to \mathcal{T}' and let $\eta' : U' \rightarrow e'$ be the Čech resolution of a cover of e' . Let $\phi : \check{C}_0(U, R) \rightarrow \check{C}_0(U', R)$ be a map of cosimplicial (cochain) complexes, such that for every $n \geq 0$ the map $\Gamma(\mathcal{T}, \eta_{n*} \eta_n^{-1} R) \rightarrow \Gamma(\mathcal{T}', \eta'_{n*} (\eta'_n)^{-1} R)$ is a homomorphism of R -algebras. (In fact, because R is concentrated in degree zero, ϕ is just a homomorphism of cosimplicial R -algebras.) Because (4.5) is functorial in C , it is compatible with ϕ . Moreover, thanks to the additional assumptions on ϕ , the second and third maps in the definition of (4.6) are also compatible with ϕ . In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, R)^{\otimes p} & \longrightarrow & \text{Tot } \check{C}(U, R) \\ \downarrow & & \downarrow \\ \mathcal{Z}(p) \otimes \text{Tot } \check{C}(U', R)^{\otimes p} & \longrightarrow & \text{Tot } \check{C}(U', R), \end{array}$$

where the horizontal arrows are the Alexander–Whitney maps and the vertical maps are induced by ϕ . We will use this remark in the proof of Theorem 1.2(a).

We now show that (4.6) is induced by a morphism of sheaves.

Lemma 4.4. *There exists a morphism of complexes of sheaves of R -modules*

$$e^{-1} \mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \text{Tot } \check{C}(U, A)$$

³In [14, Theorem 5.5] a similar map is constructed, with \mathcal{Z} replaced by \mathcal{Z}_N and $\check{C}(U, A)$ replaced by the totalisation of the restricted Čech complex. The homomorphism (4.6) is the reason why we use \mathcal{Z} instead of \mathcal{Z}_N . The operad \mathcal{Z}_N only acts on the restricted Čech cochains (see [14, (4.2)]), which are only useful when the covers are monomorphisms (e.g., open embeddings); see, for example, [15, III, Remark 2.2(d)].

such that the composition

$$\mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \Gamma(\mathcal{T}, e^{-1}\mathcal{Z}(p)) \otimes \Gamma(\mathcal{T}, \text{Tot } \check{C}(U, A)^{\otimes p}) \rightarrow \text{Tot } \check{C}(U, A)$$

obtained by passage to global sections coincides with (4.6).

Proof. Because equalisers are limits and e^{-1} commutes with limits, we have

$$e^{-1}\underline{\text{Hom}}_{\Delta}(M, N) \cong \underline{\text{Hom}}_{\Delta}(e^{-1}M, e^{-1}N)$$

for any two cosimplicial cochain complexes M and N . Recall also that e^{-1} commutes with tensor products. We define the following composition:

$$\begin{aligned} \underline{\text{Hom}}_{\Delta}(e^{-1}\Lambda, e^{-1}\Lambda^{\otimes p}) \otimes \underline{\text{Hom}}_{\Delta}(e^{-1}\Lambda, \check{C}_0(U, A))^{\otimes p} \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1}\Lambda, \check{C}_0(U, A)^{\otimes p}) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1}\Lambda, \check{C}_0(U, A^{\otimes p})) \\ \rightarrow \underline{\text{Hom}}_{\Delta}(e^{-1}\Lambda, \check{C}_0(U, A)). \end{aligned}$$

Here the first homomorphism is the Alexander–Whitney map of [14, Construction 2.6], this time with the category of cochain complexes of R -modules on \mathcal{T} as target category. The proof of [14, Proposition 4.4] immediately adapts to give a map $\check{C}_0(U, A)^{\otimes p} \rightarrow \check{C}_0(U, A^{\otimes p})$, which in turn induces the second homomorphism in the composition above. The third homomorphism is induced by the multiplication $A^{\otimes p} \rightarrow A$. It is clear that passing to global sections yields the composition defining (4.6). \square

Pre-composing with the pullback of (4.2), we obtain a π -equivariant map

$$e^{-1}W(p, R) \otimes \text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \text{Tot } \check{C}(U, A). \tag{4.7}$$

Taking global sections and pre-composing with the adjunction unit and the natural map $\text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \Gamma(\mathcal{T}, \text{Tot } \check{C}(U, A))^{\otimes p}$, we finally get

$$\check{\theta} : W(p, R) \otimes \text{Tot } \check{C}(U, A)^{\otimes p} \rightarrow \text{Tot } \check{C}(U, A). \tag{4.8}$$

The map $\check{\theta}$ is uniquely defined up to π -homotopy equivalence. It is compatible with base change along homomorphisms of rings $R \rightarrow S$ in an obvious way.

Lemma 4.5. *Let A be an R -algebra on \mathcal{T} . Then the pair $(\text{Tot } \check{C}(U, A), \check{\theta})$ constructed above is an object of $\mathcal{C}(p, R)$.*

Proof. We have to check properties (i) and (ii) of Section 2.1.

(i) By Lemma 4.1, we see that $\mathcal{Z}(p)^0$ is the cochain complex

$$R \xrightarrow{0} R \xrightarrow{\text{id}} R \xrightarrow{0} R \rightarrow \dots,$$

where the first copy of R is in degree 0. The Alexander–Whitney map is the identity in the 0th row. It is now easy to check that, in degree 0, (4.6) is the map

$$\Gamma(\mathcal{T}, \eta_{0*}\eta_0^{-1}A)^{\otimes p} \rightarrow \Gamma(\mathcal{T}, \eta_{0*}\eta_0^{-1}A)$$

obtained from the multiplication $A^{\otimes p} \rightarrow A$ by applying the adjunction unit and then taking global sections. Composing with $\alpha(p)$ and $W(p, R) \rightarrow \mathcal{E}(p)$, we deduce (i).

(ii) We may take $V = \mathcal{E}(p)$ and j equal to the map $W(p, R) \rightarrow \mathcal{E}(p)$ appearing in (4.2). □

Lemma 4.6. *We have a π -homotopy commutative square*

$$\begin{array}{ccc} \check{C}(U, A)^{\otimes p} & \longrightarrow & \underline{\mathcal{H}om}(e^{-1}W(p, R), \check{C}(U, A)) \\ \iota_A^{\otimes p} \uparrow & & \uparrow \\ A^{\otimes p} & \xrightarrow{m_p} & A \end{array}$$

where the top horizontal map is adjoint to (4.7).

Proof. If we regard A as a cosimplicial cochain complex that is zero in positive levels (so all maps in the cosimplicial direction are zero), it is easy to check that the Alexander–Whitney map (4.4) for $F = A$ is just $e^{-1}(\epsilon) \otimes \text{id}$, where $\epsilon : \mathcal{Z}(p) \rightarrow R$ is the augmentation map.

By the functoriality and equivariance of the Alexander–Whitney map, we then have a π -equivariant commutative diagram

$$\begin{array}{ccccc} e^{-1}\mathcal{Z}(p) \otimes \text{Tot} \check{C}(U, A)^{\otimes p} & \longrightarrow & \text{Tot} \check{C}(U, A)^{\otimes p} & \longrightarrow & \text{Tot} \check{C}(U, A) \\ \text{id} \otimes \iota_A^{\otimes p} \uparrow & & \iota_A^{\otimes p} \uparrow & & \iota_A \uparrow \\ e^{-1}\mathcal{Z}(p) \otimes A^{\otimes p} & \xrightarrow{e^{-1}(\epsilon) \otimes \text{id}} & A^{\otimes p} & \xrightarrow{m_p} & A. \end{array}$$

By the tensor-hom adjunction, we obtain a π -equivariant commutative diagram

$$\begin{array}{ccc} \text{Tot} \check{C}(U, A)^{\otimes p} & \longrightarrow & \underline{\mathcal{H}om}(e^{-1}\mathcal{Z}(p), \text{Tot} \check{C}(U, A)) \\ \iota_A \uparrow & & \iota_A^* \uparrow \\ A^{\otimes p} & \xrightarrow{m_p} & A \longrightarrow \underline{\mathcal{H}om}(e^{-1}\mathcal{Z}(p), A) \end{array}$$

We obtain the conclusion by composing with the map

$$\underline{\mathcal{H}om}(e^{-1}\mathcal{Z}(p), \text{Tot} \check{C}(U, A)) \rightarrow \underline{\mathcal{H}om}(e^{-1}W(p, R), \text{Tot} \check{C}(U, A))$$

induced by (4.2). □

4.4. Relation with derived functor cohomology

Let $\nu : A \rightarrow I$ be a standard injective resolution by R -modules. We define a homomorphism

$$H^*(\text{Tot} \check{C}(U, A)) \rightarrow \mathbb{H}^*(\mathcal{T}, A) \tag{4.9}$$

as follows. The resolution ν induces a homomorphism

$$\nu^* : \text{Tot} \check{C}(U, A) \rightarrow \text{Tot} \check{C}(U, I).$$

The map ι_I induces a homomorphism $\Gamma(\iota_I) : \Gamma(\mathcal{T}, I) \rightarrow \text{Tot} \check{C}(U, I)$.

Lemma 4.7. *The map $\Gamma(\iota_I)$ is a quasi-isomorphism.*

Proof. By [20, Tag 03AW], we have $\check{H}^i(U, I^q) = 0$ for all $i > 0$ and $q \geq 0$. Here $\check{H}^i(U, -)$ denotes Čech cohomology with respect to U . The conclusion now follows from the application of [20, Tag 0133] to the double complex $\check{C}(U, I)$. \square

Thus, we have a morphism $\Gamma(\iota_I)^{-1} \circ \nu^*$ in the derived category, and we define (4.9) as the induced homomorphism in cohomology.

The next lemma shows that (4.9) comes from a homomorphism of sheaves.

Lemma 4.8. *There is a homomorphism $h : \text{Tot} \check{C}(U, A) \rightarrow I$ such that $h \circ \iota_A = \nu$ and the map induced in cohomology by $\Gamma(\mathcal{T}, h)$ is (4.9).*

Proof. By naturality, we have a commutative diagram

$$\begin{CD} A @>\iota_A>> \text{Tot} \check{C}(U, A) \\ @VV\nu V @VV\xi V \\ I @>\iota_I>> \text{Tot} \check{C}(U, I). \end{CD} \tag{4.10}$$

Because ν , ι_A and ι_I are quasi-isomorphisms, so is ξ . It follows that every arrow in (4.10) is an isomorphism in the derived category. Note that $\Gamma(\xi)$ is the map ν^* defined in the previous paragraph. Because I is injective in every degree, there exists a cochain map

$$\rho : \text{Tot} \check{C}(U, I) \rightarrow I$$

such that $\rho \circ \iota_I$ is homotopic to the identity on I (apply [20, Tag 05TG] to the map ι_I^{-1} in the derived category). Define $h := \rho \circ \xi$. Then by [20, Tag 05TG] the top triangle of (4.10) is homotopy commutative. Thus, we have a diagram

$$\begin{CD} A @>\iota_A>> \text{Tot} \check{C}(U, A) \\ @VV\nu V @VV\xi V \\ I @<\rho<< \text{Tot} \check{C}(U, I) \end{CD} \tag{4.11}$$

(Note: In the original image, a diagonal arrow labeled 'h' points from Tot C(U, A) to I, and a diagonal arrow labeled 'rho' points from Tot C(U, I) to I. The diagram is a square with a triangle on top.)

where the bottom triangle is commutative and the top triangle is commutative in the derived category and so is homotopy commutative by [20, Tag 013S].

We pass to global sections level-wise in (4.11):

$$\begin{CD} \Gamma(\mathcal{T}, A) @>\Gamma(\iota_A)>> \text{Tot} \check{C}(U, A) \\ @VV\Gamma(\nu)V @VV\Gamma(\xi)V \\ \Gamma(\mathcal{T}, I) @<\Gamma(\rho)<< \text{Tot} \check{C}(U, I). \end{CD} \tag{4.12}$$

(Note: In the original image, a diagonal arrow labeled 'Gamma(h)' points from Tot C(U, A) to Gamma(T, I), and a diagonal arrow labeled 'Gamma(rho)' points from Tot C(U, I) to Gamma(T, I).)

Note that taking global sections level-wise respects homotopy equivalences; therefore, the bottom triangle in (4.12) is commutative and the top triangle is homotopy commutative. By Lemma 4.7, the map $\Gamma(\iota_I)$ is a quasi-isomorphism and so is an isomorphism in the derived category. Because $\Gamma(\rho) \circ \Gamma(\iota_I)$ is homotopic to the identity on $\Gamma(\mathcal{T}, I)$, we deduce that $\Gamma(\rho)$ is a left inverse of $\Gamma(\iota_I)$ in the derived category, and because $\Gamma(\iota_I)$ is an isomorphism in the derived category, we obtain that $\Gamma(\rho) = \Gamma(\iota_I)^{-1}$ in the derived

category. It follows that $\Gamma(h)$ coincides with $\Gamma(\iota_I)^{-1} \circ \Gamma(\xi)$ in the derived category of cochain complexes. The latter is exactly the map of [20, Tag 08BN]. Because $\Gamma(h)$ and $\Gamma(\iota_I)^{-1} \circ \Gamma(\xi)$ coincide in the derived category, they induce the same homomorphism in cohomology, as desired. □

Denote by $\text{PSh}(\mathcal{S})$ the category of presheaves on \mathcal{S} . Following [3, Exposé V, 2.4.2], for all $i \geq 0$ we denote by $\mathcal{H}^i(S, -)$ the right derived functors of the forgetful functor $\mathcal{T} \rightarrow \text{PSh}(\mathcal{S})$. If S is an object of \mathcal{S} , we denote by $H^i(S, -)$ the right derived functors of $\Gamma(S, -) : \mathcal{T} \rightarrow \text{Set}$. For all sheaves F on \mathcal{S} , we have $H^i(S, F) = H^i(\mathcal{T}/S, F)$.

Proposition 4.9. *Assume that $H^i(U_n, A^q) = 0$ for every $i > 0$ and every $n, q \geq 0$. Then (4.9) is an isomorphism. If, moreover, $R = \mathbb{F}_p$, (4.9) is compatible with Steenrod operations.*

Proof. Because we do not have a reference for it, we derive the Cartan–Leray spectral sequence for A ; the original source [3, Exposé V, Théorème 3.2] only applies to the case when A is concentrated in degree 0. Let $A \rightarrow J$ be a Cartan–Eilenberg resolution of R -modules; see [20, Tag 015H] for the definition. We may then let $I = \text{Tot } J$ and $v : A \rightarrow I$ be the standard injective resolution induced by totalisation. Applying [20, 08BI] twice, we see that

$$\text{Tot}(\check{C}(U, I)) = \text{Tot}(\check{C}(U, J)) = \text{Tot}(\check{C}(U, \text{Tot}(K))),$$

where the $\text{Tot}(-)$ in the middle is the totalisation of a triple complex, and by definition K is the double complex with terms

$$K^{r,s} := \bigoplus_{a+b=r} \Gamma(U_a, J^{b,s})$$

and maps induced by those of J and U . On the other hand, by Lemma 4.7 we have a quasi-isomorphism

$$\Gamma(\iota_I) : R\Gamma(\mathcal{T}, A) = \Gamma(\mathcal{T}, I) \rightarrow \text{Tot}(\check{C}(U, I)).$$

Thus, the spectral sequence associated to K (see [20, Tags 0130, 0132]) reads:

$$E_2^{r,s} := H^r(\text{Tot}(\check{C}(U, \mathcal{H}^s(A)))) \Rightarrow H^{r+s}(\mathcal{T}, A). \tag{4.13}$$

Here $\mathcal{H}^s(A)$ is defined as the presheaf sending $V \mapsto H^s(V, A)$. It is easy to see that the edge maps on the bottom horizontal row of (4.13) coincide with (4.9). Under the assumptions of the lemma, the E_2 -page of (4.13) is concentrated in the bottom row; hence, (4.9) is an isomorphism.

Assume now that $R = \mathbb{F}_p$. We have the following diagram, where the two smaller triangles and rectangles are π -homotopy commutative:

$$\begin{array}{ccc}
 \text{Tot}\check{C}(U, A)^{\otimes p} & \xrightarrow{\quad} & \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, \text{Tot}\check{C}(U, A)) \\
 \downarrow h^{\otimes p} & \swarrow \iota_A^{\otimes p} & \downarrow h^* \\
 & A^{\otimes p} \longrightarrow A & \\
 & \swarrow & \searrow \\
 I^{\otimes p} & \xrightarrow{\quad} & \underline{\mathcal{H}om}_{\mathbb{F}_p}(e^{-1}W, I).
 \end{array}$$

Because ι_A is a quasi-isomorphism of complexes over a field, $\iota_A^{\otimes p}$ is also a quasi-isomorphism. A diagram chase now shows that the outer square is commutative in the derived category. Because every term of $\underline{\mathcal{H}om}(W, I)$ is $\mathbb{F}_p[\pi]$ -injective, it follows from [20, Tag 05TG] that the outer square is π -homotopy commutative. By the tensor-hom adjunction, we obtain a π -homotopy commutative square

$$\begin{array}{ccc}
 e^{-1}W \otimes \text{Tot}\check{C}(U, A)^{\otimes p} & \xrightarrow{\check{\theta}} & \text{Tot}\check{C}(U, A) \\
 \downarrow \text{id} \otimes h^{\otimes p} & & \downarrow h \\
 e^{-1}W \otimes I^{\otimes p} & \xrightarrow{\theta} & I.
 \end{array}$$

Passing to global sections in the last square and pre-composing with the adjunction unit, we see that $\Gamma(h)$ induces an morphism

$$(\text{Tot}\check{C}(U, A), \check{\theta}) \rightarrow (\Gamma(\mathcal{T}, I), \theta)$$

in $\mathcal{C}(p)$. Because (4.9) is induced by $\Gamma(h)$, it is compatible with Steenrod operations, as desired. □

Remark 4.10. (i) The homomorphism (4.9) is defined in [20, Tag 08BN], at least when \mathcal{T} is the ringed topos of a ringed space (X, \mathcal{O}_X) . The construction easily adapts to the case of arbitrary topoi; this is how we constructed (4.9).

(ii) The main ingredient in our proof of Proposition 4.9 is Lemma 4.8, which shows that (4.9) comes from a map of sheaves. This is crucial, because in order to compare the Steenrod operations of May with those of Drury in Proposition 4.9 one cannot pass to global sections too soon. More precisely, we cannot prove directly that the outer square of global section is π -homotopy commutative, without first showing that the outer square of sheaves is π -homotopy commutative. This is because $\Gamma(\mathcal{T}, \nu) : \Gamma(\mathcal{T}, A) \rightarrow \Gamma(\mathcal{T}, I)$ is not necessarily a quasi-isomorphism.

5. De Rham cohomology of stacks and approximation arguments

Let p be a prime number, let k be a field of characteristic p and let X be a smooth algebraic stack over k . We denote by $\Omega_{X/k}$ the de Rham complex of X , viewed as a complex of big étale sheaves on X . We write $H_{\text{dR}}^*(X/k)$ for the de Rham cohomology of X ; that is, the hypercohomology of $\Omega_{X/k}$. We consider the following property of X .

Property 5.1. *For every $d \geq 0$, there exist a smooth k -scheme of finite type Z_d and a morphism $Z_d \rightarrow X$ such that the induced map $H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^*(Z_d/k)$ is injective in degrees $\leq d$.*

We denote by $D(k)$ the derived category of k -vector spaces, and for all $h \in \mathbb{Z}$ we let $D(k)^{\geq h}$ be the subcategory of $D(k)$ consisting of complexes with cohomology equal to zero in all degrees $< h$. Let $f : X \rightarrow Y$ be a morphism of smooth k -stacks, and let $d \geq 0$ be an integer. Following [1, Definition 5.1], we say that f is a d -Hodge equivalence if for all $j \geq 0$ the cone of $Rf^* : R\Gamma(Y, \Omega_{Y/k}^j) \rightarrow R\Gamma(X, \Omega_{X/k}^j)$ belongs to $D(k)^{\geq d-j}$. If f is a d -Hodge equivalence, then by [1, Remark 5.2] the induced map $f^* : H_{\text{dR}}^i(Y/k) \rightarrow H_{\text{dR}}^i(X/k)$ is an isomorphism for all $i < d$.

Lemma 5.2. *Let X be a smooth k -scheme of finite type, let $Z \subseteq X$ be a closed subscheme of codimension $d + 1$ and let $U := X \setminus Z$. Then the inclusion $U \hookrightarrow X$ is a d -Hodge equivalence.*

Proof. If $i \geq 0$ is an integer and F is a sheaf on X , we denote by $H^i(U, F)$ the value at F of the i th derived functor of $\Gamma(U, -)$, viewed as a functor from abelian sheaves on X to abelian groups. Using [20, Tag 01E1] and the identification $\Omega_{U/k}^j = \Omega_{X/k}^j|_U$, we see that

$$H^*(U, \Omega_{X/k}^j) = H^*(U, \Omega_{X/k}^j|_U) = H^*(U, \Omega_{U/k}^j).$$

Thus, it suffices to show that the natural map $H^i(X, \Omega_{X/k}^j) \rightarrow H^i(U, \Omega_{X/k}^j)$ is an isomorphism for all $0 \leq i \leq d$.

Because X is a smooth k -scheme, $\Omega_{X/k}^j$ is a locally free sheaf on X . Recall that if R is a local Noetherian ring and M and N are finitely generated R -modules, then $\text{depth}(M \oplus N) = \min\{\text{depth } M, \text{depth } N\}$. (To see this, use the Ext characterisation of depth.) Thus, for all $x \in X$ and all $0 \leq j \leq \dim \mathcal{O}_{X,x}$, we have

$$\text{depth } \Omega_{X/k,x}^j = \text{depth } \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} = \text{codim}_X \overline{\{x\}}.$$

Here we are computing depths of $\mathcal{O}_{X,x}$ -modules. The last equality is the fact that a regular local ring is Cohen–Macaulay. Therefore, for all $z \in Z$ we have

$$\text{depth } \Omega_{X/k,z}^j \geq \text{codim}_X Z = d + 1.$$

By [11, Exposé III, Proposition 3.3], this implies that $H^i(X, \Omega_{X/k}^j) \rightarrow H^i(U, \Omega_{X/k}^j)$ is an isomorphism for all $0 \leq i \leq d$, as desired. □

Proposition 5.3. *Let Y be a smooth quasi-projective k -scheme, let G be a linear algebraic k -group and let $X := [Y/G]$. Then X satisfies Property 5.1.*

Proof. Let $d \geq 0$ be an integer. There exists a representation V of G , a closed G -invariant subscheme $Z \subseteq V$ of codimension $\geq d + 2$, such that the complement $U := V \setminus Z$ is the total space of a G -torsor $U \rightarrow U/G$, where U/G is a smooth quasi-projective k -scheme. By Lemma 5.2, the inclusion $U \hookrightarrow V$ is a $(d + 1)$ -Hodge equivalence. By [1, Proposition 5.10(1)], the inclusion $U \times_k Y \hookrightarrow V \times_k Y$ is also a $(d + 1)$ -Hodge equivalence, and it is G -equivariant if we let G act diagonally. By [1, Proposition 5.10(2)], the induced open

embedding $(U \times_k Y)/G \hookrightarrow [(V \times_k Y)/G]$ is a $(d + 1)$ -Hodge equivalence. By [1, Remark 5.2], this implies that for all $i \leq d$ the natural maps

$$H_{\text{dR}}^i([(V \times_k Y)/G]/k) \rightarrow H_{\text{dR}}^i(((U \times_k Y)/G)/k)$$

are isomorphisms.

Using the zero section of the vector bundle $[(V \times_k Y)/G] \rightarrow [Y/G]$, we see that $H_{\text{dR}}^*(X/k)$ is a direct summand of $H_{\text{dR}}^*([(V \times_k Y)/G]/k)$. We conclude that for all $i \leq d$ the pullback homomorphisms

$$H_{\text{dR}}^i(X/k) \rightarrow H_{\text{dR}}^i(((U \times_k Y)/G)/k)$$

are injective. Because Y is quasiprojective, $(U \times_k Y)/G$ is a scheme, and hence

$$Z_d := (U \times_k Y)/G \rightarrow X$$

satisfies Property 5.1. □

Remark 5.4. (i) Let $G \hookrightarrow \text{GL}_n$ be a faithful representation of G over k , for some $n \geq 2$. Then $X = [Y/G] \cong [(Y \times^G \text{GL}_n)/\text{GL}_n]$. Here $Y \times^G \text{GL}_n$ is the fppf-quotient of $Y \times_k \text{GL}_n$ by the diagonal action of G . Because Y is smooth and quasiprojective, the fppf-quotient is represented by a smooth quasiprojective k -scheme. Thus, in the course of proving Proposition 5.3, we could have assumed that $G = \text{GL}_n$ for some $n \geq 1$.

When $G = \text{GL}_n$ for some $n \geq 1$, we can be more explicit about the representations that we use. Namely, let $r \geq 0$ be an integer, let $V := M_{n, n+r}$, the k -vector space of $n \times (n + r)$ -matrices, on which GL_n acts by multiplication on the left, let Z be the locus of matrices of rank $< n$, and $U := V \setminus Z$. If r is sufficiently large, Z has codimension $\geq d + 1$ in V . We have a GL_n -torsor $U \rightarrow U/\text{GL}_n$, where $U/\text{GL}_n = \text{Gr}(n, n + r)$ is a Grassmannian.

(ii) In the setting of Lemma 5.2, assume that k is of characteristic zero. Then $H_{\text{dR}}^i(X/k) \rightarrow H_{\text{dR}}^i(U/k)$ is injective for all $0 \leq i \leq 2d + 1$. The proof goes by induction, cutting X by a smooth divisor Y and using the logarithmic de Rham complex $\Omega_{X/k}(\log Y)$; see [6, Lemma p. 11].

(iii) When $X = BG$ for a finite or reductive k -group G , Property 5.1 has been established by B. Antieau et al. in [1, Theorem 1.2] by a completely different argument. The Z_d exhibited by them are smooth projective k -varieties.

6. Steenrod operations on de Rham cohomology

Let p be a prime number, let k be a field of characteristic p and let X be a smooth algebraic stack over k . We apply the construction of Section 2 to the case where \mathcal{T} is the big étale topos of X and $A = \Omega_{X/k}$ is the de Rham complex, viewed as a commutative differential graded \mathbb{F}_p -algebra on \mathcal{T} . We thus obtain Steenrod operations on $H_{\text{dR}}^*(X/k)$.

The purpose of this section is to show, under additional assumptions on X , that negative Steenrod operations on $H_{\text{dR}}^*(X/k)$ are zero and to compute Sq^0 and P^0 .

6.1. Compatibility with cohomology on the crystalline topoi

Let X be a smooth scheme over k . Let X_{Zar} , X_{ZAR} , $X_{\text{ét}}$, $X_{\text{ÉT}}$ denote the small and big Zariski sites of X , and the small and big étale sites of X , respectively. We may view $\Omega_{X/k}$ as a complex of sheaves on each of these four sites. We have an obvious commutative diagram of morphisms of sites:

$$\begin{CD} X_{\text{ÉT}} @>>> X_{\text{ét}} \\ @VVV @VVV \\ X_{\text{ZAR}} @>>> X_{\text{Zar}}. \end{CD}$$

Because X is of finite type over k , $\Omega_{X/k}$ is a complex of quasicoherent sheaves on X ; hence, each map in the previous diagram induces an isomorphism between the hypercohomology of $\Omega_{X/k}$ computed in each of those sites. Applying Lemma 3.1 to the morphisms of topoi associated to each of the arrows appearing, we see that the Steenrod operations on $H_{\text{dR}}^*(X/k)$ do not depend on the choice of site.

We write $(X/k)_{\text{cris}}$ for the crystalline site of X over k , where we regard k as a divided power ring, with the unique divided power structure with respect to the ideal (0) . Let $(u_*, u^{-1}) : (X/k)_{\text{cris}} \rightarrow X_{\text{Zar}}$ be the morphism of topoi defined in [4, Proposition 5.18].⁴ By [4, 5.19], the unit $\text{id} \rightarrow u_* u^{-1}$ is an isomorphism of functors, and in particular we have a canonical isomorphism

$$\Omega_{X/k} \xrightarrow{\sim} u_* u^{-1} \Omega_{X/k}. \tag{6.1}$$

Let $\mathcal{O}_{X/k}$ be the structure sheaf of $(X/k)_{\text{cris}}$; see [4, Example 5.2(1)]. By the Poincaré lemma in crystalline cohomology [4, Theorem 6.12], there is a canonical quasi-isomorphism $\mathcal{O}_{X/k} \rightarrow L(\Omega_{X/k})$ of complexes of abelian sheaves on $(X/k)_{\text{cris}}$, where $\mathcal{O}_{X/k}$ is viewed as a complex concentrated in degree zero. Here L is the linearisation functor of [4, Construction 6.9 and below]. Applying [4, Proposition 6.10] to $S = \text{Spec } k$ and $Y = X$, we see that there is a natural isomorphism $L(\Omega_{X/k}) \xrightarrow{\sim} u^{-1} \Omega_{X/k}$. Composing these two maps, we obtain a canonical quasi-isomorphism

$$\psi : \mathcal{O}_{X/k} \xrightarrow{\sim} u^{-1} \Omega_{X/k}$$

on $(X/k)_{\text{cris}}$. Moreover, ψ is a homomorphism of sheaves of differential graded k -algebras ($\mathcal{O}_{X/k}$ is concentrated in degree 0).

We have $\Gamma((X/k)_{\text{cris}}, -) = \Gamma(X_{\text{Zar}}, -) \circ u_*$, and so

$$R\Gamma((X/k)_{\text{cris}}, -) = R\Gamma(X_{\text{Zar}}, -) \circ Ru_*. \tag{6.2}$$

Because $u^{-1} \Omega_{X/k}$ is acyclic with respect to u_* , the natural map

$$u_* u^{-1} \Omega_{X/k} \rightarrow Ru_* u^{-1} \Omega_{X/k} \tag{6.3}$$

is an isomorphism in the derived category; see [4, Corollary 5.27] or the proof of [4, Theorem 7.1]. The composition of $Ru_*(\psi)$, the inverse of (6.3) and the inverse of (6.1)

⁴In [4], u^{-1} is denoted by u^* .

yield a canonical isomorphism

$$Ru_*(\mathcal{O}_{X/k}) \xrightarrow{\sim} \Omega_{X/k} \tag{6.4}$$

in the derived category of sheaves of abelian groups on X_{Zar} . It may be helpful for the reader to note that this is exactly the isomorphism of [4, (7.1.2)]. Using (6.2), we obtain a canonical isomorphism

$$H^*((X/k)_{\text{cris}}, \mathcal{O}_{X/k}) \xrightarrow{\sim} \mathbb{H}^*(X_{\text{Zar}}, \Omega_{X/k}) = H_{\text{dR}}^*(X/k). \tag{6.5}$$

Because $\mathcal{O}_{X/k}$ is a sheaf of k -algebras on $(X/k)_{\text{cris}}$, on the left side of (6.5) we also have Steenrod operations.

Proposition 6.1. *Let X be a smooth scheme over k . For every $i \geq 0$, the isomorphism (6.5) is compatible with Steenrod operations.*

Proof. We apply Lemma 3.1 to $\mathcal{T} = \mathcal{T}' = (X/k)_{\text{cris}}$, $(f_*, f^{-1}) = (\text{id}, \text{id})$, $A = f^{-1}A = \mathcal{O}_{X/k}$, $A' = u^{-1}\Omega_{X/k}$ and ψ as the map $f^{-1}A' \rightarrow A$. We obtain that the isomorphism

$$\psi^* : H^*((X/k)_{\text{cris}}, \mathcal{O}_{X/k}) \xrightarrow{\sim} \mathbb{H}^*((X/k)_{\text{cris}}, u^{-1}\Omega_{X/k})$$

is compatible with Steenrod operations.

We now apply Lemma 3.1 to $\mathcal{T}' = (X/k)_{\text{cris}}$, $\mathcal{T} = X_{\text{Zar}}$, $(f_*, f^{-1}) = (u_*, u^{-1})$, $A = \Omega_{X/k}$, $A' = u^{-1}\Omega_{X/k}$ and the identity as homomorphism $f^{-1}A \rightarrow A'$. We obtain that the induced homomorphism

$$H_{\text{dR}}^*(X/k) \rightarrow \mathbb{H}^*((X/k)_{\text{cris}}, u^{-1}\Omega_{X/k})$$

is compatible with Steenrod operations.

By Remark 3.2, we see that this is in fact the isomorphism induced by the inverse of (6.3). Thus, (6.5) is compatible with Steenrod operations, as desired. \square

Remark 6.2. Let $\text{Fr} : \mathcal{O}_{X/k} \rightarrow \mathcal{O}_{X/k}$ be the Frobenius endomorphism, and let $\Phi : \Omega_{X/k} \rightarrow \Omega_{X/k}$ be the endomorphism given by the Frobenius $\mathcal{O}_X \rightarrow \mathcal{O}_X$ in degree zero, and zero in all other degrees. Note that Φ is a map of complexes because $d(f^p) = pf^{p-1}d(f) = 0$ for every section f of \mathcal{O}_X . We have a commutative square

$$\begin{CD} \mathcal{O}_{X/k} @>\psi>> u^{-1}\Omega_{X/k} \\ @V\text{Fr}VV @VVu^{-1}\Phi V \\ \mathcal{O}_{X/k} @>\psi>> u^{-1}\Omega_{X/k}. \end{CD}$$

It follows that we have a commutative diagram

$$\begin{CD} H^*((X/k)_{\text{cris}}, \mathcal{O}_{X/k}) @>\psi^*>> \mathbb{H}^*((X/k)_{\text{cris}}, u^{-1}\Omega_{X/k}) @<< H_{\text{dR}}^*(X/k) \\ @V\text{Fr}^*VV @VV(u^{-1}\Phi)^*V @VV\Phi^*V \\ H^*((X/k)_{\text{cris}}, \mathcal{O}_{X/k}) @>\psi^*>> \mathbb{H}^*((X/k)_{\text{cris}}, u^{-1}\Omega_{X/k}) @<< H_{\text{dR}}^*(X/k), \end{CD}$$

where the horizontal arrows on the right come from Lemma 3.1 as in the proof of Proposition 6.1. As we observed in the course of proving Proposition 6.1, the horizontal

maps on the right are isomorphisms, and the composition of their inverse with ψ^* is (6.5). It follows that (6.5) transports the Frobenius endomorphism to Φ^* . We will use this remark during the proof of Proposition 6.5.

6.2. Vanishing of negative operations

Let X be a smooth algebraic stack of finite type over k .

Proposition 6.3. *If X satisfies Property 5.1, the negative Steenrod operations on $H_{\text{dR}}^*(X/k)$ are zero.*

Proof. By Lemma 3.1 and Proposition 5.3, we may assume that X is a smooth scheme of finite type over k . By Proposition 6.1, it suffices to show that the negative Steenrod operations on $H^*((X/k)_{\text{cris}}, \mathcal{O}_{X/k})$ are zero. Because $\mathcal{O}_{X/k}$ is concentrated in degree 0, this follows from Proposition 3.5. □

6.3. Determination of Sq^0 and P^0

Let \mathcal{A} be an abelian category with sufficiently many injectives, let \mathcal{B} be an abelian category and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact additive functor. Let (A, d) be a cochain complex with $A^i \in \mathcal{A}$ for all i and $A^i = 0$ for $i < 0$. We have the two hypercohomology spectral sequences

$$'E_1^{rs} := R^s F(A^r) \Rightarrow H^{r+s}(RF(A))$$

and

$$''E_2^{rs} := R^r F(H^s(A)) \Rightarrow H^{r+s}(RF(A)).$$

Let $A \rightarrow I^{**}$ be a Cartan–Eilenberg resolution of A ; see [20, Tag 015H]. The two spectral sequences above are the spectral sequences associated to the double complex $F(I^{**})$; see [20, Tag 015J]. Associated to these spectral sequences, we have edge homomorphisms

$$'e^i : H^i(RF(A)) \rightarrow R^i F(A^0), \quad ''e^i : R^i F(H^0(A)) \rightarrow H^i(RF(A)).$$

Now let $\phi \in \text{Hom}(A^0, A^0)$ and assume that $d^1\phi = 0$. We also denote by ϕ the induced map $A^0 \rightarrow \text{Ker}(d^1) = H^0(A)$. We have a cochain map $\Phi : A \rightarrow A$, given by $\Phi^0 := \phi$ and $\Phi^i := 0$ for all $i > 0$.

Lemma 6.4. *The diagram*

$$\begin{array}{ccc} H^*(RF(A)) & \xrightarrow{\Phi^*} & H^*(RF(A)) \\ \downarrow 'e & & ''e \uparrow \\ R^*F(A^0) & \xrightarrow{\phi^*} & R^*F(H^0(A)) \end{array}$$

is commutative.

Proof. By assumption, the map $\Phi : A \rightarrow A$ factors as

$$A \rightarrow A^0[0] \xrightarrow{\phi} H^0(A)[0] \rightarrow A.$$

We deduce that Φ^* factors as

$$H^*(RA(F)) \xrightarrow{\psi'} H^*(RF(A^0[0])) \xrightarrow{\phi^*} H^*(RF(H^0(A)[0])) \xrightarrow{\psi''} H^*(RA(F)).$$

Here ψ' and ψ'' come from the functoriality of the E_∞ -pages of the first and second spectral sequences, respectively. Consider now the following commutative diagram:

$$\begin{array}{ccccccc} H^*(RA(F)) & \xrightarrow{\psi'} & H^*(RF(A^0[0])) & \xrightarrow{\phi^*} & H^*(RF(H^0(A)[0])) & \xrightarrow{\psi''} & H^*(RA(F)) \\ \parallel & & \downarrow \wr & & \downarrow \wr & & \parallel \\ H^*(RA(F)) & \xrightarrow{e} & R^*F(A^0) & \xrightarrow{\phi^*} & R^*F(H^0(A)) & \xrightarrow{e} & H^*(RF(A)). \end{array}$$

The commutativity of the square on the left (respectively right) follows from the naturality of the edge maps in the first (respectively second) hypercohomology spectral sequence. The square in the middle is commutative, because the E_2 -pages of the first and second spectral sequences for the hypercohomology of a complex concentrated in degree 0 coincide. \square

We immediately obtain the following result.

Proposition 6.5. *Assume that X satisfies Property 5.1, and let Sq^0 (if $p = 2$) and P^0 (if $p > 2$) be the zeroth Steenrod operation on $H_{dR}^*(X/k)$.*

(a) *The homomorphisms P^0 and Sq^0 factor as*

$$H_{dR}^*(X/k) \rightarrow H^*(X, \mathcal{O}_X) \rightarrow H^*(X, \mathcal{O}_X) \rightarrow H_{dR}^*(X/k),$$

where the first map is an edge homomorphism in the Hodge spectral sequence, the second map is induced by the Frobenius endomorphism of \mathcal{O}_X and the third map is an edge homomorphism in the conjugate spectral sequence.

(b) *If $p > 2$, the composition*

$$H^0(X, \Omega_{X/k}^1) \hookrightarrow H_{dR}^1(X/k) \xrightarrow{P^0} H_{dR}^1(X/k)$$

is equal to zero. Similarly, if $p = 2$, the composition

$$H^0(X, \Omega_{X/k}^1) \hookrightarrow H_{dR}^1(X/k) \xrightarrow{Sq^0} H_{dR}^1(X/k)$$

is equal to zero.

Proof. By Property 5.1 and Lemma 3.1, we may assume that X is a smooth scheme of finite type over k . By Proposition 6.5, Sq^0 and P^0 are induced by the Frobenius endomorphism of $\mathcal{O}_{X/k}$. We deduce from Remark 6.2 that Sq^0 and P^0 are induced by the homomorphism $\Phi : \Omega_{X/k} \rightarrow \Omega_{X/k}$ given by the Frobenius $\phi : \mathcal{O}_X \rightarrow \mathcal{O}_X$ in degree zero, and zero everywhere else.

(a) The Hodge and conjugate spectral sequences for X are a special case of the first and second hypercohomology spectral sequences, letting \mathcal{A} be the category of sheaves of k -vector spaces over X , A be the de Rham complex of X and Φ and ϕ as in the previous paragraph. Now (a) follows from Lemma 6.4.

(b) This follows from (a) and the exact sequence of low degree terms in the Hodge spectral sequence

$$0 \rightarrow H^0(X, \Omega_{X/k}^1) \rightarrow H_{\text{dR}}^1(X/k) \rightarrow H^1(X, \mathcal{O}_X). \quad \square$$

Remark 6.6. The Steenrod operations P^0 and Sq^0 are not equal to the identity; see Proposition 3.5(b) or Proposition 6.5(b). From Theorem 1.2(a) (to be proved in the next section) we also see that they are not identically zero. As another example, let E be an elliptic curve over \mathbb{F}_p . Then the Steenrod operations in degree 0 are trivial on $H_{\text{dR}}^*(E/\mathbb{F}_p)$ if and only if E is ordinary. More generally, the behavior of Sq^0 and P^0 is related to the Frobenius and Hodge filtration on crystalline cohomology; see, for example, [4, Chapter 8]. We will not make use of this remark in the sequel.

Remark 6.7. (i) It would be interesting to know whether Property 5.1 holds for an arbitrary smooth stack of finite type over k . By Proposition 6.5, this would imply the validity Theorem 1.1(iv) for all such stacks.

(ii) One could remove the requirement that Y be quasiprojective by showing that the crystalline Poincaré lemma for algebraic spaces of M. Olsson [17, Corollary 2.5.4] is compatible with Steenrod operations. We chose not to do so in this article, in order to remain within classical crystalline cohomology. Note that the crystalline Poincaré lemma for algebraic k -stacks (even Deligne–Mumford k -stacks) is not known. For example, [17] only addresses the situation of a representable morphism $X \rightarrow S$, where X and S are Deligne–Mumford stacks. If one had a Poincaré lemma for algebraic k -stacks, one could probably prove Theorem 1.1(iv) more generally and without recourse to Property 5.1.

Proposition 6.8. *Let X be a smooth affine k -scheme. Then the p th-power Steenrod operations on $H_{\text{dR}}^*(X/k)$ are trivial.*

Proof. Because X is affine, we have $H^i(X, \Omega_{X/k}^j) = 0$ for all $i > 0$ and $j \geq 0$. Thus, we may apply Proposition 4.9 to \mathcal{S} the big étale (or Zariski) site of X , U the Čech nerve of the identity $X \rightarrow X$ and $A = \Omega_{X/k}$. We deduce that the Steenrod operations on $H_{\text{dR}}^*(X/k)$ are trivial if and only if the Steenrod operations on $H^*(\text{Tot } \check{C}(U, \Omega_{X/k}))$ are trivial. It is easy to see that $\Gamma(X, \Omega_{X/k})$ is a summand of $\text{Tot } \check{C}(U, \Omega_{X/k})$ and that the projection $\text{Tot } \check{C}(U, \Omega_{X/k}) \rightarrow \Gamma(X, \Omega_{X/k})$ is a quasi-isomorphism. The summand $\Gamma(X, \Omega_{X/k})$ arises as the totalisation of the cosimplicial submodule of $\check{C}_0(U, \Omega_{X/k})$ given by $\Gamma(X, \Omega_{X/k})$ in level zero, and zero in higher levels. Thus, there is a commutative square

$$\begin{CD} \mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, \Omega_{X/k})^{\otimes p} @>>> \text{Tot } \check{C}(U, \Omega_{X/k}) \\ @VVV @VVV \\ \mathcal{Z}(p) \otimes \Gamma(X, \Omega_{X/k})^{\otimes p} @>>> \Gamma(X, \Omega_{X/k}), \end{CD}$$

where the vertical arrows are the natural projections, the top horizontal arrow is (4.6) and the bottom arrow is a direct summand of the top arrow. Because $\Gamma(X, \Omega_{X/k})$ is the totalisation of the degree zero component of $\check{C}_0(U, \Omega_{X/k})$, it follows from the definitions that the bottom arrow is the tensor product of the augmentation of $\mathcal{Z}(p)$ and the p -fold

multiplication of $\Omega_{X/k}$. Pre-composing with the quasi-isomorphism $W \rightarrow \mathcal{Z}(p)$ of (4.2), we obtain a commutative square

$$\begin{CD} W \otimes \text{Tot } \check{C}(U, \Omega_{X/k})^{\otimes p} @>\check{\theta}>> \text{Tot } \check{C}(U, \Omega_{X/k}) \\ @VVV @VVV \\ W \otimes \Gamma(X, \Omega_{X/k})^{\otimes p} @>\epsilon^{\otimes m_p}>> \Gamma(X, \Omega_{X/k}). \end{CD}$$

Here $\check{\theta}$ is the map of (4.8), ϵ is the augmentation of W and m_p is the p -fold multiplication map. Because the projection $\text{Tot } \check{C}(U, \Omega_{X/k}) \rightarrow \Gamma(X, \Omega_{X/k})$ is a quasi-isomorphism, the conclusion will follow if we can show that the p th-power Steenrod operations on $H^*(\Gamma(X, \Omega_{X/k}))$ associated to the object $(\Gamma(X, \Omega_{X/k}), \epsilon \otimes m_p)$ of $\mathcal{C}(p)$ are trivial. This is true by Remark 2.2. □

6.4. Bockstein homomorphism

Lemma 6.9. *Let k be a field, let X be an algebraic stack over k and denote by $X_{\acute{E}T}$ and X_{SM} the big étale and big smooth sites of X , respectively.*

(a) *Let F be an étale sheaf of abelian groups on X . Then F is a smooth sheaf on X , and the canonical homomorphism*

$$H^*(X_{\acute{E}T}, F) \rightarrow H^*(X_{SM}, F)$$

is an isomorphism.

(b) *Assume that $\text{char } k = p > 0$ and that X is smooth over k . The canonical homomorphism*

$$H_{dR}^*(X/k) \rightarrow \mathbb{H}^*(X_{SM}, \Omega_{X/k})$$

is an isomorphism, compatible with Steenrod operations.

Proof. (a) This follows from the fact that a covering in the smooth topology can always be refined by a covering in the étale topology; see [20, Tags 055V, 00VX].

(b) By (a), $\Omega_{X/k}$ is a complex of smooth sheaves. We have a homomorphism between the étale hypercohomology spectral sequence for $\Omega_{X/k}$ to the smooth hypercohomology spectral sequence for $\Omega_{X/k}$. Applying (a) to $F = \Omega_{X/k}^j$ for every $j \geq 0$, we see that this homomorphism is an isomorphism between the E_2 -pages; hence, it induces an isomorphism of the abutments. The compatibility with Steenrod operations is a special case of Lemma 3.1. □

Let $(K, \theta) \in \mathcal{C}(p)$. Recall that one may define a Bockstein homomorphism on $H^*(K)$, assuming that (K, θ) is reduced; that is, that there exists $(\tilde{K}, \tilde{\theta}) \in \mathcal{C}(p, \mathbb{Z}/2\mathbb{Z})$ whose reduction modulo p is isomorphic to (K, θ) and such that \tilde{K} is flat over $\mathbb{Z}/p^2\mathbb{Z}$.

Let X be a smooth algebraic stack of finite type over a commutative ring R and assume that X has affine diagonal over R . By [20, Tag 04YA], there exists a smooth surjective morphism from a smooth affine R -scheme of finite type to X . Let U be the Čech nerve of a covering of X by a smooth affine R -scheme. Then $\Gamma(U_n, \Omega_{X/R}^j) = \Gamma(U_n, \Omega_{U_n/R}^j)$ is a flat R -module for every $n, j \geq 0$, and so $\text{Tot } \check{C}(U, \Omega_{X/R})$ is a complex of flat R -modules.

Let $R \rightarrow S$ be a ring homomorphism. By the theorem on cohomology and affine base change for quasicohherent sheaves [20, Tag 02KG], for all $n, j \geq 0$ the canonical map $\Omega^j_{U_n/R} \otimes_R S \rightarrow \Omega^j_{(U_n)_S/S}$ is an isomorphism of quasicohherent \mathcal{O}_{X_S} -modules. Moreover, the canonical maps are compatible with the de Rham differentials. We thus obtain an isomorphism of complexes of S -modules

$$\text{Tot } \check{C}(U, \Omega_{X/R}) \otimes_R S \xrightarrow{\sim} \text{Tot } \check{C}(U_S, \Omega_{X_S/S}). \tag{6.6}$$

Proposition 6.10. *Let k be a perfect field of characteristic p , and let X be a smooth algebraic stack over k . Assume that there exists a smooth algebraic stack \tilde{X} of finite type and with affine diagonal over $W_2(k)$, such that $X = \tilde{X} \times_{W_2(k)} k$. Then there exists a group homomorphism*

$$\check{\beta} : H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^{*+1}(X/k),$$

satisfying properties (i) and (ii) of Lemma 2.1.

Proof. Let \tilde{U} be the Čech nerve associated to a smooth surjective morphism from a smooth affine $W_2(k)$ -scheme to \tilde{X} and define $U := \tilde{U} \times_{W_2(k)} k$. For every $n \geq 0$, \tilde{U}_n is a smooth and affine $W_2(k)$ -scheme. For all $n, j \geq 0$, letting $\eta_n : U_n \rightarrow X$ denote the natural projection map, we have an isomorphism $\Omega^j_{U_n/k} \cong \eta_n^{-1} \Omega^j_{X/k}$ of big étale sheaves on U_n . Because U_n is affine and $\Omega^j_{U_n/k}$ is a coherent sheaf on U_n for every $n, j \geq 0$, by Serre’s vanishing theorem we have $H^q(U_n, \Omega^j_{U_n/k}) = 0$ for all $q > 0$ and $n, j \geq 0$.

Let \mathcal{S} be the site whose objects are algebraic stacks over X , whose morphisms are 1-morphisms of algebraic stacks over X and whose covers are families of jointly surjective smooth morphisms, and let \mathcal{T} be the topos associated to \mathcal{S} . With the notation of Lemma 6.9, we have morphism of sites

$$X_{\text{ÉT}} \rightarrow X_{\text{SM}} \rightarrow \mathcal{S}.$$

Because every algebraic stack has a smooth cover by schemes, by Verdier’s comparison theorem [3, III, Théorème 4.1] the morphism on the right induces an equivalence of topoi. In particular, we have an induced isomorphism

$$\mathbb{H}^*(X_{\text{SM}}, \Omega_{X/k}) \xrightarrow{\sim} \mathbb{H}^*(\mathcal{S}, \Omega_{X/k}) = \mathbb{H}^*(\mathcal{T}, \Omega_{X/k}),$$

where we also denote by $\Omega_{X/k}$ the sheaf on \mathcal{S} induced by the de Rham complex on X_{SM} . This isomorphism is compatible with Steenrod operations by Lemma 3.1. Combining this with Lemma 6.9, we get a canonical isomorphism

$$H_{\text{dR}}^*(X/k) \xrightarrow{\sim} \mathbb{H}^*(\mathcal{T}, \Omega_{X/k}),$$

which is compatible with Steenrod operations.

Note that X is a terminal object in \mathcal{S} and that a smooth surjective morphism from a scheme to X is a cover in \mathcal{S} . Moreover, by Lemma 6.9 and [20, Tags 06W0, 0DGB, 03P2], for all $j, n \geq 0$ and $i \geq 1$ we have

$$\begin{aligned} H^i(\mathcal{S}/U_n, \eta_n^{-1} \Omega^j_{X/k}) &= H^i((\text{Sch}/U_n)_{\text{SM}}, \Omega^j_{U_n/k}) \\ &= H^i((\text{Sch}/U_n)_{\text{ÉT}}, \Omega^j_{U_n/k}) \end{aligned}$$

$$\begin{aligned}
 &= H^i((U_n)_{\acute{e}t}, \Omega_{U_n/k}^j) \\
 &= H^i(U_n, \Omega_{U_n/k}^j) = 0.
 \end{aligned}$$

In the last step we have used Serre’s vanishing theorem for the cohomology of quasicohherent sheaves on affine schemes. Thus, we may apply Proposition 4.9 to \mathcal{T} , $\Omega_{X/k}$ and U . We deduce the existence of an isomorphism

$$H^*(\text{Tot } \check{C}(U, \Omega_{X/k})) \xrightarrow{\sim} H_{\text{dR}}^*(X/k) \tag{6.7}$$

compatible with Steenrod operations.

The map (4.8) is compatible with extension of scalars; thus, (6.6) induces an isomorphism

$$(\text{Tot } \check{C}(\tilde{U}, \Omega_{\tilde{X}/W_2(k)}, \check{\theta})) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p \cong (\text{Tot } \check{C}(U, \Omega_{X/k}, \check{\theta}))$$

in $\mathcal{C}(p)$. For every $n \geq 0$, because \tilde{U}_n is smooth over $W_2(k)$, by [20, Tag 02G1] the $W_2(k)$ -module $\Omega_{\tilde{U}_n/W_2(k)}$ is finite locally free and, in particular, it is flat. Recall that $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$; in fact, it is the unique flat lifting of k over $\mathbb{Z}/p^2\mathbb{Z}$. Therefore, $\text{Tot } \check{C}(\tilde{U}, \Omega_{\tilde{X}/W_2(k)})$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, and so $(\text{Tot } \check{C}(U, \Omega_{X/k}, \check{\theta}))$ is reduced. As recalled at the end of Section 2.2, it now follows from [12, Proposition 2.3(v)] that a Bockstein homomorphism in the setting of May is defined on $H^*(\text{Tot } \check{C}(U, \Omega_{X/k}))$. Using (6.7), we obtain a Bockstein homomorphism on $\mathbb{H}^*(X, \Omega_{X/k})$. The conclusion now follows from Lemma 2.1. □

In the setting of Proposition 6.10, let $\iota : X \hookrightarrow \tilde{X}$ be the natural closed embedding. Following an idea of Totaro [21, Proof of Theorem 11.1], we may define a homomorphism $\beta : H_{\text{dR}}^*(X/k) \rightarrow H_{\text{dR}}^{*+1}(X/k)$ as follows. By [20, Tag 013T], there exists a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \iota_*\Omega_{X/k} & \longrightarrow & \Omega_{\tilde{X}/W_2(k)} & \longrightarrow & \iota_*\Omega_{X/k} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & I_3 \longrightarrow 0,
 \end{array} \tag{6.8}$$

where the vertical arrows are injective resolutions of sheaves of abelian groups and the rows are short exact sequences of complexes. Because I_1 is injective, passing to global sections the bottom row of (6.8) remains exact. We define β as the connecting homomorphism in the associated long exact sequence (see [20, Tag 0111]):

$$\dots \rightarrow H_{\text{dR}}^i(X/k) \rightarrow H_{\text{dR}}^i(\tilde{X}/W_2(k)) \xrightarrow{\iota^*} H_{\text{dR}}^i(X/k) \xrightarrow{\beta} H_{\text{dR}}^{i+1}(X/k) \rightarrow \dots$$

It is easy to verify that β does not depend on the choice of (6.8). For every $j \geq 0$, Totaro defines a Bockstein homomorphism on $H^*(X, \Omega_{X/k}^j)$ by considering a diagram of the form (6.8) but where $\Omega_{\tilde{X}/W_2(k)}$ and $\iota_*\Omega_{X/k}$ are replaced by $\Omega_{\tilde{X}/W_2(k)}^j$ and $\iota_*\Omega_{X/k}^j$.

Contrary to $\check{\beta}$ of Proposition 6.10, the Bockstein β does not fit in the setting of May [12]. Therefore, it is not automatic that β satisfies the conclusion of Theorem 1.1(v); we prove it by relating β to $\check{\beta}$.

Proof of Theorem 1.1(v). Let $\Omega_{X/k} \rightarrow I$ be an injective resolution of étale sheaves of abelian groups on X that is a monomorphism in every degree. The pushforward ι_* preserves injectives, and because $\iota : X \hookrightarrow \tilde{X}$ is a closed embedding, ι_* is exact. Thus, $\iota_*\Omega_{X/k} \rightarrow \iota_*I$ is an injective resolution of étale sheaves of abelian groups on \tilde{X} , and is a monomorphism in every degree.

By [20, Tag 013T], there exists a diagram (6.8) where $I_1 = \iota_*I$. From [20, Proof of Tag 013T, Step 2], we see that we may also assume that $I_3 = \iota_*I$. (Here we use the fact that $\iota_*\Omega_{X/k} \rightarrow \iota_*I$ is a monomorphism in every degree.) In other words, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \iota_*\Omega_{X/k} & \longrightarrow & \Omega_{\tilde{X}/W_2(k)} & \longrightarrow & \iota_*\Omega_{X/k} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \iota_*I & \longrightarrow & J & \longrightarrow & \iota_*I \longrightarrow 0,
 \end{array} \tag{6.9}$$

for some injective resolution of sheaves of abelian groups $\Omega_{\tilde{X}/W_2(k)} \rightarrow J$. Let \tilde{U} be the Čech nerve associated to a smooth cover of \tilde{X} by a smooth affine $W_2(k)$ -scheme, and let $U := \tilde{U} \times_{W_2(k)} k$. Because \tilde{X} is smooth and with affine diagonal, so is every \tilde{U}_n . From (6.9) we deduce the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Tot } \check{C}(U, \Omega_{X/k}) & \longrightarrow & \text{Tot } \check{C}(\tilde{U}, \Omega_{\tilde{X}/W_2(k)}) & \longrightarrow & \text{Tot } \check{C}(U, \Omega_{X/k}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Tot } \check{C}(U, I) & \longrightarrow & \text{Tot } \check{C}(\tilde{U}, J) & \longrightarrow & \text{Tot } \check{C}(U, I) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \Gamma(X, I) & \longrightarrow & \Gamma(\tilde{X}, J) & \longrightarrow & \Gamma(X, I) \longrightarrow 0.
 \end{array}$$

Recall that we defined $\text{Tot } \check{C}(U, -)$ in Section 4.2. The two commutative squares on the top come from the functoriality of $\text{Tot } \check{C}(U, -)$. The commutativity of the two squares on the bottom follows from the naturality of (4.10).

Write $\beta' : H^*(\text{Tot } \check{C}(U, I)) \rightarrow H^{*+1}(\text{Tot } \check{C}(U, I))$ for the connecting homomorphism associated to the second line of the above diagram. Because connecting homomorphisms are compatible with homomorphisms of short exact sequences, for every $i \geq 0$ we have a commutative diagram

$$\begin{array}{ccccc}
 H_{\text{dR}}^i(X/k) & \xrightarrow{\sim} & H^i(\text{Tot } \check{C}(U, I)) & \xleftarrow{\sim} & H^i(\text{Tot } \check{C}(U, \Omega_{X/k})) \\
 \downarrow \beta & & \downarrow \beta' & & \downarrow \check{\beta} \\
 H_{\text{dR}}^{i+1}(X/k) & \xrightarrow{\sim} & H^{i+1}(\text{Tot } \check{C}(U, I)) & \xleftarrow{\sim} & H^{i+1}(\text{Tot } \check{C}(U, \Omega_{X/k})).
 \end{array}$$

The horizontal maps on the left are isomorphisms by Lemma 4.7 and those on the right by Proposition 4.9. Note that we apply both results with $R = \mathbb{Z}$. (The composition $H^*(\text{Tot } \check{C}(U, \Omega_{X/k})) \rightarrow H_{\text{dR}}^*(X/k)$ is the map of (6.7), where $A = \Omega_{X/k}$.) The conclusion now follows from Proposition 6.10. \square

Remark 6.11. If X is a scheme, one can define a Bockstein homomorphism even without assuming that X admits a lifting to $W_2(k)$, by using the crystalline Poincaré lemma and the short exact sequence

$$0 \rightarrow \iota_* \mathcal{O}_{X/k} \rightarrow \mathcal{O}_{X/W_2(k)} \rightarrow \iota_* \mathcal{O}_{X/k} \rightarrow 0.$$

If X is a smooth algebraic stack over k , the Hodge filtration for X is defined as the filtration on $H_{\text{dR}}^*(X/k)$ arising from the E_∞ -page of the Hodge spectral sequence for X .

Proposition 6.12. *Under the assumptions of Proposition 6.10, for all $i \geq 0$ the map $\beta : H_{\text{dR}}^i(X/k) \rightarrow H_{\text{dR}}^{i+1}(X/k)$ respects the Hodge filtration for X , and for all $0 \leq j \leq i$ the induced map $H^j(X, \Omega_{X/k}^{i-j}) \rightarrow H^{j+1}(X, \Omega_{X/k}^{i-j})$ coincides with the Bockstein defined by Totaro.*

Proof. Let \mathcal{T} be a topos and assume that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_A & \longrightarrow & I_B & \longrightarrow & I_C & \longrightarrow & 0, \end{array} \tag{6.10}$$

where A, B, C are cochain complexes of abelian sheaves on \mathcal{T} , concentrated in nonnegative degrees, the vertical arrows are Cartan–Eilenberg resolutions and the rows are exact. Let \mathcal{C} be the category of complexes of abelian sheaves on \mathcal{T} . Define $I_A[1]$ as the shift of I_A as a complex in \mathcal{C} : $I_A[1]^{i,j} = I_A^{i,j+1}$. The bottom row defines an element of $\text{Hom}_{D(\mathcal{C})}(I_C, I_A[1])$. By [20, Tag 05TG], the canonical map $\text{Hom}_{K(\mathcal{C})}(I_C, I_A[1]) \rightarrow \text{Hom}_{D(\mathcal{C})}(I_C, I_A[1])$ is an isomorphism. It follows that there exists a homomorphism of double complexes $\delta : I_C \rightarrow I_A[1]$ such that, for every integer i , the homomorphism

$$H^{i,*}(\Gamma(\mathcal{T}, \delta)) : H^*(\Gamma(\mathcal{T}, I_C^{i,*})) \rightarrow H^{*+1}(\Gamma(\mathcal{T}, I_A^{i,*}))$$

is the connecting map of the hypercohomology long exact sequence associated to

$$0 \rightarrow A^i \rightarrow B^i \rightarrow C^i \rightarrow 0.$$

The map δ induces a homomorphism from the first hypercohomology spectral sequence of C to that of A , the latter with vertical degrees shifted by 1. (In Section 6.3, we denoted these spectral sequences by $'E$.) Thus, δ induces a homomorphism between the E_∞ -pages. This means that for every integer i the map δ induces a homomorphism $\mathbb{H}^i(\mathcal{T}, C) \rightarrow \mathbb{H}^{i+1}(\mathcal{T}, A)$ that respects the filtrations and that induces the maps $H^{i,j}(\Gamma(\mathcal{T}, \delta))$ upon taking quotients of successive pieces of the filtrations.

To conclude, it suffices to let \mathcal{T} be the big étale site on \tilde{X} , and let (6.10) be as in (6.8). \square

7. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The naturality follows from Lemma 3.1. Properties (i), (ii), (iii) hold by Proposition 2.3. Property (iv) follows from Proposition 5.3, Proposition 6.3 and Proposition 6.5(a). Property (v) was proved in Proposition 6.10. \square

Proof of Theorem 1.2(a). We denote by $B_{\text{simp}}G$ the simplicial classifying space of G ; see [13, Chapter 16, §5]. Because G is a finite discrete group, $B_{\text{simp}}G$ is a simplicial set. It is the quotient of the contractible simplicial set $E_{\text{simp}}G$ by the free action of G . We write $B_{\text{top}}G$ and $E_{\text{top}}G$ for the geometric realisations of $B_{\text{simp}}G$ and $E_{\text{simp}}G$, respectively; they are CW complexes.

Let $\mathbb{F}_p \rightarrow I$ be an injective resolution of sheaves of \mathbb{F}_p -vector spaces on $B_{\text{top}}G$. Let $\theta : W \otimes \Gamma(B_{\text{top}}G, I)^{\otimes p} \rightarrow \Gamma(B_{\text{top}}G, I)$ be a π -equivariant homomorphism of $\mathbb{F}_p[\pi]$ -complexes such that $(\Gamma(B_{\text{top}}G, I), \theta)$ is an element of $\mathcal{C}(p)$, as constructed in Section 2. By Lemma 3.4 and [9, Theorem p. 206], the induced Steenrod operations on the sheaf cohomology ring $H^*(B_{\text{top}}G, \mathbb{F}_p)$ are the classical topological Steenrod operations.

Let X be a paracompact Hausdorff topological space (for example, a CW complex). Let $\text{Op}(X)$ denote the site of open embeddings of X , and let $\text{Homeo}(X)$ be the site whose objects are local homeomorphisms $Y \rightarrow X$, whose morphisms are continuous maps $Y \rightarrow Y'$ over X (they are automatically local homeomorphisms) and whose covers are families $\{U_i \rightarrow Y\}$ of jointly surjective local homeomorphisms. Because open embeddings are local homeomorphisms, there is an obvious morphism of sites $\text{Op}(X) \rightarrow \text{Homeo}(X)$, which by Verdier’s comparison theorem [3, III, Théorème 4.1] induces an equivalence of topoi; see also the beginning of [2, XI, §4]. By Lemma 3.1, we may compute the Steenrod operations on $H^*(B_{\text{top}}G, \mathbb{F}_p)$ while viewing \mathbb{F}_p as a sheaf in either $\text{Op}(B_{\text{top}}G)$ or $\text{Homeo}(B_{\text{top}}G)$. It follows that we may apply Proposition 4.9 to the topos associated to $\text{Homeo}(B_{\text{top}}G)$ and to the cover $E_{\text{top}}G \rightarrow B_{\text{top}}G$. Let U denote the Čech nerve of $E_{\text{top}}G \rightarrow B_{\text{top}}G$. We have a morphism of simplicial topological spaces $U \rightarrow B_{\text{simp}}G$, given by the unique map $E_{\text{top}}G \rightarrow \{*\}$ in level 0 and by projections onto the G^{n-1} factor in level n , for all $n \geq 1$.

Recall that the big étale topos of BG is the topos associated to the site whose objects are schemes over BG , whose arrows are morphisms of schemes over BG and whose covers are families of jointly surjective étale morphisms of schemes over BG . By Verdier’s comparison theorem, it is also the topos associated to the site whose objects are representable morphisms of algebraic stacks $X \rightarrow BG$, whose arrows are morphisms of algebraic stacks over BG and whose covers are families of jointly surjective étale morphisms. Because BG is a terminal object in the second site, we may apply Proposition 4.9 to the big étale topos of the algebraic stack BG and to the universal G -torsor $\text{Spec}\mathbb{F}_p \rightarrow BG$. The Čech nerve of this morphism is exactly $B_{\text{simp}}G$.

By Remark 4.3, the projection $U \rightarrow B_{\text{simp}}G$ and the inclusion $\mathbb{F}_p \hookrightarrow \Omega_{BG/\mathbb{F}_p}$ induce the commutative squares of π -equivariant maps:

$$\begin{array}{ccc} \mathcal{Z}(p) \otimes \text{Tot } \check{C}(U, \mathbb{F}_p)^{\otimes p} & \longrightarrow & \text{Tot } \check{C}(U, \mathbb{F}_p) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{Z}(p) \otimes \text{Tot } \check{C}(B_{\text{simp}}G, \mathbb{F}_p)^{\otimes p} & \longrightarrow & \text{Tot } \check{C}(B_{\text{simp}}G, \mathbb{F}_p) \end{array}$$

and

$$\begin{CD} \mathcal{Z}(p) \otimes \text{Tot } \check{C}(B_{\text{simp}} G, \mathbb{F}_p)^{\otimes p} @>>> \text{Tot } \check{C}(B_{\text{simp}} G, \mathbb{F}_p) \\ @VV\wr V @VV\wr V \\ \mathcal{Z}(p) \otimes \text{Tot } \check{C}(B_{\text{simp}} G, \Omega_{BG/\mathbb{F}_p})^{\otimes p} @>>> \text{Tot } \check{C}(B_{\text{simp}} G, \Omega_{BG/\mathbb{F}_p}). \end{CD}$$

The vertical maps in the first square are isomorphisms because $E_{\text{top}} G$ is connected and \mathbb{F}_p is a constant sheaf. Because G is discrete, we have $\eta_j^{-1} \Omega_{BG/\mathbb{F}_p}^q = \Omega_{G^j/\mathbb{F}_p}^q = 0$ for all $q > 0$ and $j \geq 0$, and $\eta_j^{-1} \mathcal{O}_{BG/\mathbb{F}_p} = \mathcal{O}_{G^j/\mathbb{F}_p}$; see [20, Tag 06TU]. (Recall that the η_j^{-1} are pullbacks of sheaves on big étale sites. They have been defined in Section 4.2.) Thus, the vertical maps in the second square are isomorphisms, too. Combining the two squares and pre-composing with the quasi-isomorphism $W \rightarrow \mathcal{Z}(p)$ of (4.2), we obtain a square of π -equivariant maps

$$\begin{CD} W \otimes \text{Tot } \check{C}(U, \mathbb{F}_p)^{\otimes p} @>\check{\theta}_t>> \text{Tot } \check{C}(U, \mathbb{F}_p) \\ @VV\wr V @VV\wr V \\ W \otimes \text{Tot } \check{C}(B_{\text{simp}} G, \Omega_{BG/\mathbb{F}_p})^{\otimes p} @>\check{\theta}_a>> \text{Tot } \check{C}(B_{\text{simp}} G, \Omega_{BG/\mathbb{F}_p}), \end{CD}$$

where $\check{\theta}_t$ and $\check{\theta}_a$ are given by (4.8). We conclude that the second projection $U \rightarrow B_{\text{simp}} G$ induces an isomorphism

$$\text{Tot}(\check{C}(U, \mathbb{F}_p), \check{\theta}_t) \cong (\text{Tot } \check{C}(B_{\text{simp}} G, \Omega_{BG/\mathbb{F}_p}), \check{\theta}_a)$$

in $\mathcal{C}(p)$, and so (1.1) is compatible with Steenrod operations. In particular, we have $\text{Sq}^0 = \text{Id}$ if $p = 2$, and $\text{P}^0 = \text{Id}$ if $p > 2$. From Lemma 2.1 we deduce that $\beta = \beta \circ \text{Sq}^0 = \text{Sq}^1$ if $p = 2$, and $\beta = \beta \circ \text{P}^0 = \beta \text{P}^0$ if $p > 2$. By the previous part of the proof, we know that (1.1) is compatible with Sq^1 and βP^0 . Because the same formulas hold for the topological Bockstein, we conclude that (1.1) is also compatible with Bockstein homomorphisms. \square

We now turn to proving Theorem 1.2(b) and (c). The following is a special case of the Künneth formula in de Rham cohomology.

Lemma 7.1. *Let k be a field, and let G and H be linear algebraic k -groups. Assume that the Hodge spectral sequences for BG and BH degenerate. Then*

- (a) *the Hodge spectral sequence for $B(G \times_k H)$ degenerates and*
- (b) *the projections $G \times_k H \rightarrow G$ and $G \times_k H \rightarrow H$ induce an isomorphism of k -algebras*

$$H_{\text{dR}}^*(BG/k) \otimes H_{\text{dR}}^*(BH/k) \xrightarrow{\sim} H_{\text{dR}}^*(B(G \times_k H)/k).$$

Proof. (a) By the Künneth formula in Hodge cohomology [21, Proposition 5.1], the projections of $G \times_k H$ onto G and H induce an isomorphism

$$H_{\mathbb{H}}^*(BG/k) \otimes H_{\mathbb{H}}^*(BH/k) \xrightarrow{\sim} H_{\mathbb{H}}^*(B(G \times_k H)/k), \tag{7.1}$$

where $H_{\mathbb{H}}^i(-/k) := \oplus_l H^l(-, \Omega^{i-l})$ is Hodge cohomology. This isomorphism is multiplicative and respects the bigrading in Hodge cohomology.

The Hodge spectral sequence is a spectral sequence of k -algebras, and its E_1 -page is Hodge cohomology. By assumption, all differentials are trivial on elements coming from $H_{\mathbb{H}}^*(BG/k)$ and $H_{\mathbb{H}}^*(BH/k)$, and hence on all elements of $H_{\mathbb{H}}^*(B(G \times_k H)/k)$.

(b) By Proposition 5.3, BG and BH satisfy Property 5.1. Let $d \geq 0$. Then we may find morphisms $g_d : X_d \rightarrow BG$ and $h_d : Y_d \rightarrow BH$, where X_d and Y_d are smooth k -schemes of finite type, and the induced maps

$$g_d^* : H_{\text{dR}}^*(BG/k) \rightarrow H_{\text{dR}}^*(X_d/k), \quad h_d^* : H_{\text{dR}}^*(BH/k) \rightarrow H_{\text{dR}}^*(Y_d/k)$$

are injective in degrees $\leq d$. We obtain a commutative diagram

$$\begin{CD} H_{\text{dR}}^*(BG/k) \otimes H_{\text{dR}}^*(BG/k) @>>> H_{\text{dR}}^*(B(G \times_k H)/k) \\ @V g_d^* \otimes h_d^* VV @VV (g_d \times h_d)^* V \\ H_{\text{dR}}^*(X_d/k) \otimes H_{\text{dR}}^*(Y_d/k) @>\sim>> H_{\text{dR}}^*(X_d \times_k Y_d/k). \end{CD}$$

The bottom horizontal map is an isomorphism by the Künneth formula for the de Rham cohomology of smooth k -schemes; see [20, Tag 0FM9]. Because k is a field, the map $g_d^* \otimes h_d^*$ is injective in degrees $\leq d$. Because d was arbitrary, we deduce that the top horizontal map is injective. Because the top horizontal map respects the grading, in order to prove that it is an isomorphism, it suffices to check that

$$\dim_k H_{\text{dR}}^n(B(G \times_k H)/k) = \sum_{i+j=n} \dim_k H_{\text{dR}}^i(BG/k) \cdot \dim_k H_{\text{dR}}^j(BH/k)$$

for all $n \geq 0$. This immediately follows from (7.1) and (a). □

In order to prove Theorem 1.2(b) and (c), we need some auxiliary computations.

Lemma 7.2. *Let $H_{\text{dR}}^*(\mathbb{P}^n/\mathbb{F}_p) = \mathbb{F}_p[x]/(x^{n+1})$, where x has degree 2.*

- (a) *If $p = 2$, then $\text{Sq}(x) = x^2$ and $\beta(x) = 0$.*
- (b) *If $p > 2$, then $\text{P}(x) = x^p$ and $\beta(x) = 0$.*

Proof. For a computation of $H_{\text{dR}}^*(\mathbb{P}^n/\mathbb{F}_p)$, see [20, Tag 0FMJ]. By Theorem 1.1(iv), the negative Steenrod operations on \mathbb{P}^n are trivial. Recall that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is zero for $i > 0$. By Theorem 1.1(iv), we deduce that P^0 and Sq^0 are zero on $H_{\text{dR}}^i(\mathbb{P}^n/\mathbb{F}_p)$ for all $i > 0$. Because $\beta(x)$ has degree 3, necessarily $\beta(x) = 0$. □

Lemma 7.3. *Let $H_{\text{dR}}^*(B\mathbb{G}_m/\mathbb{F}_p) = \mathbb{F}_p[x]$, where x has degree 2.*

- (a) *If $p = 2$, then $\text{Sq}(x) = x^2$ and $\beta(x) = 0$.*
- (b) *If $p > 2$, then $\text{P}(x) = x^p$ and $\beta(x) = 0$.*

Proof. Because $\beta(x)$ has degree 3 and $H_{\text{dR}}^*(B\mathbb{G}_m/\mathbb{F}_p)$ is concentrated in even degrees, we have $\beta(x) = 0$.

Let $f : \mathbb{P}^1 \rightarrow B\mathbb{G}_m$ be the morphism corresponding to the \mathbb{G}_m -torsor $\mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$. We may fix an isomorphism $H_{\text{dR}}^*(\mathbb{P}^1/\mathbb{F}_p) \cong \mathbb{F}_p[x]/(x^2)$ so that the induced ring homomorphism $f^* : H_{\text{dR}}^*(B\mathbb{G}_m/\mathbb{F}_p) \rightarrow H_{\text{dR}}^*(\mathbb{P}^1/\mathbb{F}_p)$ is given by reduction modulo x^2 . By Lemma 3.1, f^* is compatible with Steenrod operations.

(a) By Theorem 1.1(i) and (iv), we may write $Sq(x) = ax + x^2$ for some $a \in \mathbb{F}_2$. By Lemma 7.2(a), we have $Sq^0(f^*(x)) = 0$. Thus,

$$0 = Sq^0(f^*(x)) = f^*(Sq^0(x)) = f^*(ax) = af^*(x).$$

This implies that $a = 0$, and hence $Sq(x) = x^2$.

(b) By Theorem 1.1(i) and (iv), we may write $P(x) = ax + x^p$ for some $a \in \mathbb{F}_p$. By Lemma 7.2(b), we have $P^0(f^*(x)) = 0$. Then

$$0 = P^0(f^*(x)) = f^*(P^0(x)) = f^*(ax) = af^*(x),$$

and hence $a = 0$ and $P(x) = x^p$. □

Let k be a field, let G be a connected reductive k -group and let p be a prime number. Assume first that k is algebraically closed. Let T be a maximal torus of G , let \hat{T} be the character group of T , let B be a Borel subgroup of G containing T and let $n := \dim G/B$. We have a natural group homomorphism $\hat{T} \rightarrow CH^1(G/B)$. We obtain a homomorphism

$$\text{Sym}^n(\hat{T}) \rightarrow CH^n(G/B) \xrightarrow{\text{deg}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z}, \tag{7.2}$$

where deg is the degree map and π is the projection modulo p . We say that p is a torsion prime for G if the composition (7.2) is zero. If k is an arbitrary field, we say that p is a torsion prime for G if p is a torsion prime for $G_{\bar{k}}$. This is the definition given in [21, p. 1592], to which we refer for other equivalent formulations.

Proof of Theorem 1.2(b). Because $H_{\text{dR}}^*(BG/\mathbb{F}_p)$ is concentrated in even degrees and β has degree 1, necessarily $\beta = 0$. It thus suffices to show that all Steenrod operations are trivial.

Let T be a maximal torus of G , let \mathfrak{g} and \mathfrak{t} be the Lie algebras of G and T , \mathfrak{g}^* and \mathfrak{t}^* be their duals and $\text{Sym}(\mathfrak{g}^*)$ and $\text{Sym}(\mathfrak{t}^*)$ be the symmetric \mathbb{F}_p -algebras on \mathfrak{g} and \mathfrak{t} . Fix an integer $i \geq 0$. We have a commutative diagram

$$\begin{CD} H^0(G, \text{Sym}^i(\mathfrak{g}^*)) @<\sim<< H^i(BG, \Omega_{BG/\mathbb{F}_p}^i) @>> H_{\text{dR}}^{2i}(BG/\mathbb{F}_p) \\ @VVV @VVV @VVV \\ H^0(T, \text{Sym}^i(\mathfrak{t}^*)) @<\sim<< H^i(BT, \Omega_{BT/\mathbb{F}_p}^i) @>> H_{\text{dR}}^{2i}(BT/\mathbb{F}_p), \end{CD} \tag{7.3}$$

where the vertical maps are pullbacks, the horizontal maps on the left are the isomorphisms of [21, Corollary 2.2] and the horizontal maps on the right arise in the Hodge spectral sequence for BG and BT ; see [21, Lemma 8.2].

Let T be a split maximal torus of G , and let B be a Borel subgroup of G containing T . We have a spectral sequence of k -algebras

$$E_2^{ij} := H_{\mathbb{H}}^i(BG/\mathbb{F}_p) \otimes H_{\mathbb{H}}^j((G/B)/\mathbb{F}_p) \Rightarrow H_{\mathbb{H}}^{i+j}(BT/\mathbb{F}_p),$$

where $H_{\mathbb{H}}^i(-/\mathbb{F}_p) := \bigoplus_l H^l(-, \Omega^{i-l})$; see [21, Proposition 9.3]. Because p is not a torsion prime for G , the spectral sequence degenerates; see the last paragraph of the proof of [21, Theorem 9.1]. In particular, the pullback map $H^i(BG, \Omega_{BG/\mathbb{F}_p}^i) \rightarrow H^i(BT, \Omega_{BT/\mathbb{F}_p}^i)$ appearing in (7.3) is injective.

It follows from [21, Corollary 2.2, Theorem 4.1] that $H^i(BT, \Omega^j_{BT/\mathbb{F}_p}) = 0$ when $i \neq j$. Thus, $H^i(BG, \Omega^j_{BG/\mathbb{F}_p}) = 0$ when $i \neq j$. Thus, in the Hodge spectral sequences

$$E_1^{ij} := H^j(X, \Omega^i_{X/\mathbb{F}_p}) \Rightarrow H_{\text{dR}}^{i+j}(X/\mathbb{F}_p)$$

for $X = BG, BT$, the E_1 -pages are concentrated in the diagonal $i = j$. This implies that the spectral sequences degenerate. From this we obtain that $H_{\text{dR}}^j(BG/\mathbb{F}_p) = 0$ for odd j and that the right horizontal maps in (7.3) are isomorphisms.

Summarising, we have shown that all horizontal arrows in (7.3) are isomorphisms and all vertical arrows are injective. Because $H_{\text{dR}}^j(BG/\mathbb{F}_p) = 0$ for odd j , this shows that the pullback $H_{\text{dR}}^*(BG/\mathbb{F}_p) \rightarrow H_{\text{dR}}^*(BT/\mathbb{F}_p)$ is injective.

We fix an isomorphism $T \cong \mathbb{G}_m^n$, for some $n \geq 0$. By Lemma 7.1(b), this induces a ring isomorphism $H_{\text{dR}}^*(BT/\mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_n]$, where x_i is the pullback of the generator of $H_{\text{dR}}^2(B\mathbb{G}_m/\mathbb{F}_p)$ along the i th projection $B\mathbb{G}_m^n \rightarrow B\mathbb{G}_m$. By Lemma 7.3 and Lemma 3.1, the Steenrod operations are trivial on the x_i and so, by the Cartan formula, they are trivial on $H_{\text{dR}}^*(BT/\mathbb{F}_p)$. □

Remark 7.4. One could also prove Theorem 1.2(b) by establishing the injectivity of $H^0(G, \text{Sym}^i(\mathfrak{g}^*)) \rightarrow H^0(T, \text{Sym}^i(\mathfrak{t}^*))$, as follows. It is enough to treat the case when G is semisimple. Then the injectivity follows from [21, Theorem 8.1] (due to T. A. Springer and R. Steinberg [19, Section II.3.17] and P.-E. Chaput and M. Romagny [5, Theorem 1.1]) when $p > 2$, or when $p = 2$ and \mathfrak{g} does not have factors of the form \mathfrak{Sp}_{2n} . In the remaining cases, it is a consequence of results of Chaput–Romagny; see the last paragraph of the proof of [21, Theorem 9.2].

The de Rham cohomology of $B\mu_p$ was computed for every prime p by Totaro [21, Proposition 10.1]. We have

$$H_{\text{dR}}^*(B\mu_p/\mathbb{F}_p) = \mathbb{F}_p[t, v]/(v^2), \quad |t| = 2, \quad |v| = 1. \tag{7.4}$$

Lemma 7.5. *Let Sq and P be the total Steenrod operations on $H_{\text{dR}}^*(B\mu_p/\mathbb{F}_p)$. With respect to (7.5), we have the following:*

- (a) *If $p = 2$, then $\text{Sq}(t) = t^2$, $\beta(t) = 0$, $\text{Sq}(v) = 0$ and $\beta(v) = t$.*
- (b) *If $p > 2$, then $\text{P}(t) = t^p$, $\beta(t) = 0$, $\text{P}(v) = 0$ and $\beta(v) = t$.*

Proof. From the proof of [21, Proposition 10.1], t is defined as the pullback of a generator of $H_{\text{dR}}^2(B\mathbb{G}_m/\mathbb{F}_p)$ along the inclusion $\mu_p \hookrightarrow \mathbb{G}_m$. By Lemma 7.3, we deduce that $\beta(t) = 0$, that $\text{Sq}(t) = t^2$ if $p = 2$ and that $\text{P}(t) = t^p$ if $p > 2$.

Because $|v| = 1$ and $v^2 = 0$, we have $\text{Sq}(v) = \text{Sq}^0(v)$ and $\text{P}(v) = \text{P}^0(v)$. From the proof of [21, Proposition 10.1], we see that v belongs to the image of the differential $H^0(B\mu_p, \Omega^1_{B\mu_p/\mathbb{F}_p}) \rightarrow H_{\text{dR}}^1(B\mu_p/\mathbb{F}_p)$ arising from the Hodge spectral sequence for $B\mu_p$. By Proposition 6.5(b), we deduce that $\text{Sq}^0(v) = 0$ and $\text{P}^0(v) = 0$.

To prove that $\beta(v) = t$, one may apply the Čech-theoretic interpretation given in Section 6.4 to the presentation of $B\mu_p$ as $[\mathbb{G}_m/\mathbb{G}_m]$, where \mathbb{G}_m acts on itself via the

p th-power map. The Čech complex for $\Omega^1_{B\mu_p/\mathbb{F}_p}$ starts as

$$k[x^{\pm 1}]dx \rightarrow k[y_1^{\pm 1}, y_2^{\pm 1}]\langle dy_1, dy_2 \rangle \rightarrow \dots$$

The homomorphism on the left is given by

$$f(x)dx \mapsto f(y_1^p y_2) d(y_1^p y_2) - f(y_1) dy_1.$$

Using this formula, a simple computation shows that dx/x is a generator for $H^0(B\mu_p, \Omega^1_{B\mu_p/\mathbb{F}_p})$ and that $\beta_H(dx/x) = dy_1/y_1$. Here β_H is the Bockstein in Hodge cohomology. The canonical inclusion $\mu_p \hookrightarrow \mathbb{G}_m$ induces a commutative diagram of Čech nerves:

$$\begin{array}{ccccc} B\mu_p & \longleftarrow & \mathbb{G}_m & \xleftarrow{\quad} & \mathbb{G}_m^2 \\ \downarrow & & \downarrow & & \downarrow \text{pr}_1 \\ B\mathbb{G}_m & \longleftarrow & \text{Spec } k & \xleftarrow{\quad} & \mathbb{G}_m \end{array}$$

where pr_1 denotes the first projection. If we use the coordinate u for the \mathbb{G}_m in the bottom right corner, then $H^1(B\mathbb{G}_m, \Omega_{B\mathbb{G}_m/k})$ is generated by du/u , and $\text{pr}_1^*(du/u) = dy_1/y_1$. This shows that $\beta_H(dx/x) = \text{pr}_1^*(du/u)$. By [21, Proposition 10.1], the Hodge spectral sequence for $B\mu_p$ degenerates and identifies dx/x with v and $\text{pr}_1^*(du/u)$ with t . By Proposition 6.12, it follows that $\beta(v) = t$, as desired. \square

Proof of Theorem 1.2(c). We start with BO_2 . We have

$$H_{\text{dR}}^*(BO_2/\mathbb{F}_2) = \mathbb{F}_2[u_1, u_2], \quad |u_1| = 1, |u_2| = 2.$$

We think of O_2 as the isometry group of the quadratic form q on $\mathbb{A}_{\mathbb{F}_2}^2$ given by $q(x, y) = xy$. There is a subgroup H of O_2 , isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mu_2$, where $\mathbb{Z}/2\mathbb{Z}$ acts by switching x and y and μ_2 acts by scaling on $\mathbb{A}_{\mathbb{F}_2}^2$. By Theorem 1.2(a) we have

$$H_{\text{dR}}^*(B(\mathbb{Z}/2\mathbb{Z})/\mathbb{F}_2) = \mathbb{F}_2[s], \quad |s| = 1, \quad \text{Sq}(s) = s + s^2,$$

and by Lemma 7.5(a) we have

$$H_{\text{dR}}^*(B\mu_2/\mathbb{F}_2) = \mathbb{F}_2[t, v]/(v^2), \quad |t| = 2, \quad |v| = 1, \quad \text{Sq}(t) = t^2, \quad \text{Sq}(v) = 0.$$

By Lemma 7.1, we have

$$H_{\text{dR}}^*(BH/\mathbb{F}_2) = \mathbb{F}_2[s, t, v]/(v^2),$$

compatibly with the projection maps $BH \rightarrow B(\mathbb{Z}/2\mathbb{Z})$ and $BH \rightarrow B\mu_2$. Applying Lemma 3.1 to the projections $BH \rightarrow B\mu_2$ and $BH \rightarrow B(\mathbb{Z}/2\mathbb{Z})$, we obtain

$$\text{Sq}(s) = s + s^2, \quad \text{Sq}(t) = t^2, \quad \text{Sq}(v) = 0.$$

The induced homomorphism $H^*(BO_2/\mathbb{F}_2) \rightarrow H^*(BH/\mathbb{F}_2)$ sends $u_1 \mapsto s$ and $u_2 \mapsto t + sv$, and in particular it is injective; see [21, p. 1602]. Note that $\text{Sq}(t + sv) = t^2$ and that $u_2^2 \mapsto t^2$. It follows that

$$\text{Sq}(u_1) = u_1 + u_1^2, \quad \text{Sq}(u_2) = u_2^2.$$

For every $r \geq 1$, by Lemma 7.1 we may write

$$H_{\text{dR}}^*(BO_2^r/\mathbb{F}_2) = \mathbb{F}_2[s_1, \dots, s_r, t_1, \dots, t_r].$$

Here s_i and t_i are the pullback of the classes u_1 and u_2 in $H^*(BO_2/\mathbb{F}_2)$ along the i th projection $BO_2^r \rightarrow BO_2$, respectively. Applying Lemma 3.1 to the projections, we see that

$$\text{Sq}(s_i) = s_i + s_i^2, \quad \text{Sq}(t_i) = t_i^2.$$

By [21, Theorem 11.1, Lemma 11.3], we have

$$H_{\text{dR}}^*(BO_{2r}/\mathbb{F}_2) = \mathbb{F}_2[u_1, u_2, \dots, u_{2r}],$$

where $|u_i| = i$ for every $i = 1, \dots, 2r$. Moreover, there exists a closed subgroup embedding $\iota: O_2^r \hookrightarrow O_{2r}$ such that the pullback map

$$\iota^*: H_{\text{dR}}^*(BO_{2r}/\mathbb{F}_2) \rightarrow H_{\text{dR}}^*(BO_2^r/\mathbb{F}_2)$$

is injective and is given by

$$\iota^*(u_{2a}) = \sum_{1 \leq i_1 < \dots < i_a \leq r} t_{i_1} \cdots t_{i_a}, \quad \iota^*(u_{2a+1}) = \sum_{m=1}^r s_m \sum_{1 \leq i_1 < \dots < i_a \leq r, i_h \neq m} t_{i_1} \cdots t_{i_a}.$$

Because $\text{Sq}(t_i) = t_i^2$ and ι^* is injective, we have $\text{Sq}(u_{2a}) = u_{2a}^2$. One proves by ascending induction on $0 \leq d \leq a$ that

$$\iota^*(u_{4a+1} + \sum_{t=2a-2d}^{2a-1} u_{2a-t}u_{2a+1+t}) = \sum_{m=1}^r s_m \sum_{j_h \neq m} t_{j_1} \cdots t_{j_d} \sum_{i_h \neq m} t_{i_1} \cdots t_{i_{2a-d}}, \tag{7.5}$$

$$\iota^*(u_{4a+1} + \sum_{t=2a-2d-1}^{2a-1} u_{2a-t}u_{2a+1+t}) = \sum_{m=1}^r s_m \sum_{\exists h: j_h = m} t_{j_1} \cdots t_{j_d} \sum_{i_h \neq m} t_{i_1} \cdots t_{i_{2a-d}}. \tag{7.6}$$

In the sums, h is arbitrary, $1 \leq j_1 < \dots < j_d \leq r$ and $1 \leq i_1 < \dots < i_{2a-d} \leq r$. We define $t_{j_h}, s_{i_h} := 0$ when $h > a$. When $d = 0$, (7.5) coincides with the previous expression of $\iota^*(u_{4a+1})$ in terms of the s_i and t_i and so is true. One then uses (7.5) for d to prove (7.6) for d and (7.6) for d to prove (7.5) for $d + 1$. By induction, this proves the formulas for all $0 \leq d \leq a$. When $d = a$, (7.5) becomes

$$\iota^*(u_{4a+1} + \sum_{t=0}^{2a-1} u_{2a-t}u_{2a+1+t}) = \sum_{m=1}^r s_m \sum_{i_h \neq m} t_{i_1}^2 \cdots t_{i_a}^2 = \iota^*(\text{Sq}(u_{2a+1})) + \iota^*(u_{2a+1})^2.$$

Because ι^* is injective, this implies Theorem 1.2(c) for BO_{2r} .

The homomorphism

$$H_{\text{dR}}^*(BO_{2r}/\mathbb{F}_2) \rightarrow H_{\text{dR}}^*(BSO_{2r}/\mathbb{F}_2)$$

induced by the canonical inclusion $SO_{2r} \hookrightarrow O_{2r}$ sends $u_1 \mapsto 0$ and $u_i \mapsto u_i$ for $i \geq 2$. This implies Theorem 1.2(c) for SO_{2r} .

Consider now the composition of closed embeddings

$$\mathrm{SO}_{2r+1} \hookrightarrow \mathrm{O}_{2r+1} \hookrightarrow \mathrm{O}_{2r+2},$$

where O_{2r+1} is embedded as the stabiliser subgroup of a nonzero vector of the standard $(2r + 2)$ -dimensional representation of O_{2r+2} . The induced homomorphism $H_{\mathrm{dR}}^*(\mathrm{O}_{2r+2}/\mathbb{F}_2) \rightarrow H_{\mathrm{dR}}^*(\mathrm{SO}_{2r+1}/\mathbb{F}_2)$ sends $u_1, u_{2r+2} \mapsto 0$ and $u_i \mapsto u_i$ for $2 \leq i \leq 2r + 1$, proving Theorem 1.2(c) for SO_{2r+1} .

Finally, recall that $\mathrm{O}_{2r+1} \cong \mathrm{SO}_{2r+1} \times \mu_2$. The Künneth isomorphism

$$H_{\mathrm{dR}}^*(B\mathrm{SO}_{2r+1}/\mathbb{F}_2) \otimes H_{\mathrm{dR}}^*(B\mu_2/\mathbb{F}_2) \xrightarrow{\sim} H_{\mathrm{dR}}^*(B\mathrm{O}_{2r+1}/\mathbb{F}_2) \tag{7.7}$$

sends $u_i \mapsto u_i$ for all i , $v \mapsto v_1$ and $c \mapsto c_1$ (we are using the notation of Lemma 7.5 for the cohomology of $B\mu_2$). Taking Lemma 7.5 into account, this proves the formulas for Sq on $B\mathrm{O}_{2r+1}$.

We now consider β . Totaro defines u_{2a+1} so that $\beta_{\mathrm{H}}(u_{2a}) = u_{2a+1} + u_1 u_{2a}$, where β_{H} is the Bockstein in Hodge cohomology. (Here we identify the generators for Hodge and de Rham cohomology, because the Hodge spectral sequence degenerates.) The formula for $\beta(u_{2a})$ follows from Proposition 6.12. Applying β to both sides of the formula for $\beta(u_{2a})$, we get the formula for $\beta(u_{2a+1})$. In the case of O_{2r+1} , the formulas for c_1 and v_1 follow from Lemma 7.5 and (7.7). □

Acknowledgments I thank Ben Williams for a number of useful discussions on this topic, Benjamin Antieau and Samuel Bach for helpful conversations and my advisor Zinovy Reichstein for giving me direction at various stages of this work. I am grateful to the anonymous referee for useful comments and suggestions.

References

- [1] B. ANTIEAU, B. BHATT AND A. MATHEW, Counterexamples to Hochschild-Kostant-Rosenberg in characteristic p , Preprint, 2019, arXiv:1909.11437. To appear in *Forum of Mathematics, Sigma*.
- [2] M. ARTIN, A. GROTHENDIECK AND J. L. VERDIER, *Théorie des topos et cohomologie étale des schémas. Tome 3: Lecture Notes in Mathematics*, Vol. **305** (Springer, Berlin, 1973). Avec la collaboration de P. Deligne et B. Saint-Donat.
- [3] M. ARTIN, A. GROTHENDIECK AND J.-L. VERDIER, *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos, Lecture Notes in Mathematics*, Vol. **269** (Springer, Berlin, 1972). Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [4] P. BERTHELOT AND A. OGUS, *Notes on Crystalline Cohomology* (Princeton University Press, Princeton, NJ, 1978).
- [5] P.-E. CHAPUT AND M. ROMAGNY, On the adjoint quotient of Chevalley groups over arbitrary base schemes, *J. Inst. Math. Jussieu*, **9**(4) (2010), 673–704.
- [6] J. DE JONG, Notes on algebraic de Rham cohomology. (2007), http://math.columbia.edu/~dejong/seminar/note_on_algebraic_de_Rham_cohomology.pdf.
- [7] R. A. DRURY, Steenrod operations on algebraic de Rham cohomology, Hodge cohomology, and spectral sequences, 2019. URL: <https://escholarship.org/uc/item/75z0f0x4>.

- [8] D. B. A. EPSTEIN, Semisimplicial objects and the Eilenberg-Zilber theorem. *Invent. Math.*, **1** (1966), 209–220.
- [9] D. B. A. EPSTEIN, Steenrod operations in homological algebra, *Invent. Math.*, **1** (1966), 152–208.
- [10] P. G. GOERSS AND J. F. JARDINE, *Simplicial Homotopy Theory* (Birkhäuser, Basel, 2009).
- [11] A. GROTHENDIECK, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Vol. **4** of *Documents Mathématiques (Paris)* [Mathematical Documents (Paris)] (Société Mathématique de France, Paris, 2005).
- [12] J. P. MAY, A general algebraic approach to Steenrod operations, in *The Steenrod Algebra and Its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*, *Lecture Notes in Mathematics*, Vol. **168** (Springer, Berlin, 1970), 153–231.
- [13] J. P. MAY, *A Concise Course in Algebraic Topology, Chicago Lectures in Mathematics*, (University of Chicago Press, Chicago, IL, 1999).
- [14] J. P. MAY, Operads and sheaf cohomology. Unpublished manuscript. 2004, URL: <http://www.math.uchicago.edu/~may/PAPERS/Esheaf.pdf>.
- [15] J. S. MILNE, *Étale cohomology*, Vol. **33** of *Princeton Mathematical Series* (Princeton University Press, Princeton, NJ, 1980).
- [16] M. OLSSON, *Algebraic Spaces and Stacks, Vol. 62 of American Mathematical Society Colloquium Publications* (American Mathematical Society, Providence, RI, 2016).
- [17] M. OLSSON, Crystalline cohomology of algebraic stacks and Hyodo-Kato cohomology, *Astérisque*, **316** (2007) 412 pp. ISBN: 978-2-85629-249-5.
- [18] E. PRIMOZIC, Computations of de Rham cohomology rings of classifying stacks at torsion primes. *New York J. Math.* **26** (2020), 1002–1027.
- [19] T. A. SPRINGER AND R. STEINBERG, Conjugacy classes, in *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, *Lecture Notes in Mathematics*, Vol. **131** (Springer, Berlin, 1970), 167–266.
- [20] THE STACKS PROJECT AUTHORS, Stacks Project. URL: <http://stacks.math.columbia.edu>.
- [21] B. TOTARO, Hodge theory of classifying stacks, *Duke Math. J.*, **167**(8) (2018), 1573–1621.