ON A NOTE BY H. SCHWERDTFEGER

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(received May 13, 1958)

Let x, y denote real column vectors with n elements. Let A be a regular symmetric real $n \times n$ matrix. Dashes indicate transposition.

If x is fixed, $x'Ax > 0$, the discriminant of A at x is the quadratic form

$y'Sy$ where $S = S(x) = x'Ax \cdot A - Axx'A$.

In Can. Math. Bull. 1, pp.175-179, Dr. Schwerdtfeger proved the equivalence of the following properties of A:

(i) A is of the congruence type $[+, -, \ldots, -]$.

(ii) $y'Sy \leqslant 0$ for all y, equality holding if and only if y is a multiple of x. His note is of particular interest because he also discusses the eigen-values of S. If only the quoted result is aimed at, the following procedure may be shorter.

Following Dr. Schwerdtfeger, we transform A into its congruence normal form J. If the image of the fixed vector x is again denoted by x, we have

$$(1) \qquad x'Jx = \sum_1^p x_k^2 - \sum_{p+1}^n x_k^2 > 0;$$

here $1 \leqslant p \leqslant n$.

Any vector y permits a unique decomposition

$$(2) \qquad y = z + \lambda x \qquad \text{where } x'Jz = 0.$$

On account of $Sx = 0$, this leads to $Sy = Sz + \lambda Sx = Sz$ and

$$(3) \qquad y'Sy = z'Sz = x'Jx \cdot z'Jz.$$

If A is positive definite, we have $p = n$ and $z'Jz > 0$. By (1) and (3), S will be non-negative definite.

Now let $p = 1$. Then by (1) and (2)

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$$x_1^2 > \sum_2^n x_k^2 \quad , \quad x_1 z_1 = \sum_2^n x_k z_k .$$

Hence bySchwarz's inequality

$$x_1^2 \cdot z'Jz = x_1^2 \left(z_1^2 - \sum_2^n z_k^2\right) = \left(\sum_2^n x_k z_k\right)^2 - x_1^2 \cdot \sum_2^n z_k^2$$

$$\leq \left(\sum_2^n x_k z_k\right)^2 - \left(\sum_2^k x_k^2\right) \cdot \left(\sum_2^k z_k^2\right) \leq 0$$

or $z'Jz \leqslant 0$, equality holding if and only if $z = 0$. Thus (3) shows that (i) implies (ii).

If $p = n-1$, (1) and (2) read

$$(4) \qquad x_n^2 - \sum_1^{n-1} x_k^2 < 0 \ , \ x_n z_n = \sum_1^{n-1} x_k z_k \quad .$$

Any non-trivial solution z of $\sum_1^{n-1} x_k z_k = 0$, $z_n = 0$ will satisfy (4) and

$$z'Jz = \sum_1^{n-1} z_k^2 - z_n^2 > 0 .$$

On the other hand,

$$z' = \left(x_n x_1, \ldots, x_n x_{n-1}, \sum_1^{n-1} x_k^2\right)$$

satisfies (4) and

$$z'Jz = x_n^2 \sum_1^{n-1} x_k^2 - \left(\sum_1^{n-1} x_k^2\right)^2 = \sum_1^{n-1} x_k^2 \left(x_n^2 - \sum_1^{n-1} x_k^2\right) < 0,$$

Finally, let $1 < p < n-1$. Then there are numbers

$$t_1, \ldots, t_p ; \ t_{p+1}, \ldots, t_n$$

satisfying

$$\sum_1^p x_i t_i = 0 \ , \ \sum_1^p t_i^2 > 0 \ ; \ \sum_{p+1}^n x_i t_i = 0 \ , \ \sum_{p+1}^n t_i^2 > 0 .$$

put

$$z' = (t_1, \ldots, t_p, \ \rho t_{p+1}, \ldots, \ \rho t_n).$$

Thus $x'Jz = 0$. The function

$$f(\rho) = z'Jz = \sum_1^p t_i^2 - \rho^2 \sum_{p+1}^n t_i^2$$

is positive for $\rho = 0$, negative for large ρ . Thus (ii) does not hold if (i) if not satisfied.

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