THE TRANSIENT BEHAVIOUR OF THE QUEUEING SYSTEM $GI/M/1$

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Summary

We consider a single server queue for which the interarrival times are identically and independently distributed with distribution function $A(x)$ and whose service times are distributed independently of each other and of the interarrival times with distribution function $B(x) = 1 - e^{-x}$, $x \geqq 0$. We suppose that the system starts from emptiness and use the results of P. D. Finch [2] to derive an explicit expression for q_j^n, the probability that the $(n+1)$th arrival finds more than j customers in the system. The special cases $M/M/1$ and $D/M/1$ are considered and it is shown in the general case that q_j^n is a partial sum of the usual Lagrange series for the limiting probability $q_j = \lim_{n\to\infty} q_j^n$.

1. Introduction

Consider a queueing system whose service times are distributed independently of each other and of the arrival process with d.f. $B(x) = 1 - e^{-x}$, $x \geqq 0$. Suppose that the first $m+1$ arrivals occur at times $t_0, t_1, \cdots, t_m$ and define

$$\tau_j = t_{j+1} - t_j, \qquad j = 0, 1, 2, \cdots, m-1,$$

$\eta(t) =$ the number of customers in the system at time t including the one, if any, being served,

$\eta_j = \eta(t_j - 0)$.

Define also the following probabilities conditional upon $\tau_0, \tau_1, \cdots, \tau_m$:

$$(1) \quad Q_j^{m+1}(\tau_0, \tau_1, \cdots, \tau_m) = \Pr(\eta_{m+1} > j),$$

$$(2) \quad U_j^{m+1}(\tau_0, \tau_1, \cdots, \tau_m) = \Pr(\eta_1 > 0, \eta_2 > 0, \cdots, \eta_m > 0, \eta_{m+1} = j).$$

If $\tau_0, \tau_1, \cdots, \tau_m$ are random variables with joint distribution function $F_m(x_0, x_1, \cdots, x_m)$ then the corresponding unconditional probabilities can be written

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$$q_j^{m+1} = \int Q_j^{m+1}(x_0, x_1, \cdots, x_m) dF_m(x_0, x_1, \cdots, x_m), \tag{3}$$

$$u_j^{m+1} = \int U_j^{m+1}(x_0, x_1, \cdots, x_m) dF_m(x_0, x_1, \cdots, x_m). \tag{4}$$

In his study of the single server queue with Erlang service times and non-recurrent input process P. D. Finch [2] obtained the following expressions for Q_j^{m+1}, U_j^{m+1}:

$$Q_{m-j}^{m+1} = (-)^j \begin{vmatrix} -\phi_j^m & \frac{(-\phi_j^m)^2}{2!} & \cdots & \frac{(-\phi_j^m)^j}{j!} & e^{-\phi_j^m} \\ 1 & -\phi_{j-1}^m & \cdots & \frac{(-\phi_{j-1}^m)^{j-1}}{(j-1)!} & e^{-\phi_{j-1}^m} \\ 0 & 1 & \cdots & & \\ 0 & 0 & \cdots & & \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & e^{-\phi_0^m} \end{vmatrix}, \quad 0 \leqq j \leqq m, \tag{5}$$

$$U_{m+1-j}^{m+1} = (-)^j e^{-\phi_0^m} \begin{vmatrix} -\phi_j^m & \frac{(-\phi_j^m)^2}{2!} & \cdots & & \frac{(-\phi_j^m)^j}{j!} \\ 1 & -\phi_{j-1}^m & \cdots & & \frac{(-\phi_{j-1}^m)^{j-1}}{(j-1)!} \\ 0 & 1 & \cdots & & \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\phi_1^m \end{vmatrix}, \quad 0 \leqq j \leqq m. \tag{6}$$

where $\phi_j^m = \tau_m + \tau_{m-1} + \cdots + \tau_j$.

It is the purpose of this paper to evaluate the integrals (3) and (4) in the particular case when $F_m(x_0, \cdots, x_m) = \prod_{i=0}^m A(x_i)$.

2. The expansion of Q_{m-j}^{m+1}

Expanding the determinantal expression, (5), about its last column it is found that

$$Q_{m-j}^{m+1} = \sum_{k=0}^{j} f_{j,k}^{m+1}(\phi_k^m, \phi_{k+1}^m, \cdots, \phi_j^m) e^{-\phi_k^m}, \qquad j = 0, 1, \cdots, m,$$

where the coefficients $f_{j,k}^{m+1}(\phi_k^m, \cdots, \phi_j^m)$ are to be determined. For this purpose we prove the following lemma.

LEMMA 1

(7) $$f_{j,0}^{m+1}(\phi_0^m, \cdots, \phi_j^m) = \sum_{r=1}^{j} \frac{(\phi_j^m)^r}{r!} f_{j-r,0}^{j}(\phi_0^{j-1}, \cdots, \phi_{j-r}^{j-1}), \qquad j = 1, 2, \ldots, m,$$

(8) $$f_{0,0}^{m+1}(\phi_0^m) = 1.$$

PROOF. The validity of (8) can be seen by inspecting the determinant given by (5).

To prove (7) we make the inductive hypothesis (denoted by $H(a)$) that (7) is true for $j = 1, 2, \cdots, a$, where a is a positive integer such that $a < m$, and furthermore that for $j = 0, 1, \cdots, a-1$,

$$f_{j,0}^{m+1}(\phi_0^m, \cdots, \phi_j^m) = \sum_{r=0}^{j} \frac{(\phi_k^m)^r}{r!} f_{j-r,0}^{k}(\phi_0^{k-1}, \cdots, \phi_{j-r}^{k-1}),$$

where k can take any one of the values $j+1, j+2, \cdots, m$.

We shall prove that $H(a)$ implies $H(a+1)$. The result (7) then follows by induction.

It follows from $H(a)$ that

$$\begin{aligned} f_{a,0}^{m+1}(\phi_0^m, \cdots, \phi_a^m) &= \sum_{r=1}^{a} \frac{(\phi_a^m)^r}{r!} f_{a-r,0}^{a}(\phi_0^{a-1}, \cdots, \phi_{a-r}^{a-1}) \\ &= \sum_{r=1}^{a} \sum_{s=0}^{r} \frac{(\phi_k^m)^s}{s!} \frac{(\phi_a^{k-1})^{r-s}}{(r-s)!} f_{a-r,0}^{a}(\phi_0^{a-1}, \cdots, \phi_{a-r}^{a-1}) \\ &= \sum_{u=0}^{a} \frac{(\phi_k^m)^u}{u!} \sum_{v=\delta_{u,0}}^{a-u} \frac{(\phi_a^{k-1})^v}{v!} f_{a-u-v,0}^{a}(\phi_0^{a-1} \cdots, \phi_{a-u-v}^{a-1}) \end{aligned}$$

where $\delta_{n,0}$ denotes Kronecker's delta.

Thus

(9a) $$f_{a,0}^{m+1}(\phi_0^m, \cdots, \phi_a^m) = \sum_{u=0}^{a} \frac{(\phi_k^m)^u}{u!} f_{a-u,0}^{k}(\phi_0^{k-1}, \cdots, \phi_{a-u}^{k-1}),$$

the last line following from the second part of the hypothesis $H(a)$.

Expanding the appropriate determinant about its first row it is found that

$$f_{a+1,0}^{m+1}(\phi_0^m, \cdots, \phi_{a+1}^m) = -\sum_{r=1}^{a+1} \frac{(-\phi_{a+1}^m)^r}{r!} f_{a+1-r,0}^{m+1}(\phi_0^m, \cdots, \phi_{a+1-r}^m).$$

Hence, using (9a) with $k = a+1$,

$$\begin{aligned} f_{a+1,0}^{m+1}(\phi_0^m, \cdots, \phi_{a+1}^m) &= -\sum_{r=1}^{a+1} \frac{(-\phi_{a+1}^m)^r}{r!} \sum_{s=0}^{a+1-r} \frac{(\phi_{a+1}^m)^s}{s!} f_{a+1-r-s,0}^{a+1}(\phi_0^a, \cdots, \phi_{a+1-r-s}^a) \\ &= -\sum_{u=1}^{a+1} \sum_{v=1}^{u} (-)^v \binom{u}{v} \frac{(\phi_{a+1}^m)^u}{u!} f_{a+1-u,0}^{a+1}(\phi_0^a, \cdots, \phi_{a+1-u}^a). \end{aligned}$$

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Thus

$$f^{m+1}_{a+1,0}(\phi^m_0, \cdots, \phi^m_{a+1}) = \sum_{u=1}^{a+1} \frac{(\phi^m_{a+1})^u}{u!} f^{a+1}_{a+1-u,0}(\phi^a_0, \cdots, \phi^a_{a+1-u}). \tag{9b}$$

Inspection of (9a) and (9b) shows that $H(a)$ implies $H(a+1)$ as required. $H(1)$ can be readily verified and the result stated by the lemma then follows by induction.

From Lemma 1 we have a recurrence relation for $f^{m+1}_{j,0}(\phi^m_0, \cdots, \phi^m_j)$. In order to determine $f^{m+1}_{j,n}(\phi^m_n, \cdots, \phi^m_j)$, $n \neq 0$, we use the relation

$$f^{m+1}_{j,n}(\phi^m_n, \cdots, \phi^m_j) = f^{m+1}_{j-n,0}(\phi^m_n, \cdots, \phi^m_j) \tag{10}$$

whose validity can be seen by inspection of the appropriate determinants.

Another property of the coefficients which we shall use is that if we make the transformation

$$(\tau_n, \tau_{n+1}, \cdots, \tau_m) \to (\tau_0, \tau_1, \cdots, \tau_{m-n})$$

then

$$f^{m+1}_{j-n,0}(\phi^m_n, \phi^m_{n+1}, \cdots, \phi^m_j) \to f^{m+1-n}_{j-n,0}(\phi^{m-n}_0, \phi^{m-n}_1, \cdots, \phi^{m-n}_{j-n}). \tag{11}$$

3. The integration of Q^{m+1}_{m-j}

Suppose now that the first $m+1$ interarrival times are independently and identically distributed with d.f. $A(x)$. Then from equations (5) and (10) the unconditional probability, $q^{m+1}_{m-j} = \Pr(\eta_{m+1} > m-j)$ is given by

$$\begin{aligned} q^{m+1}_{m-j} &= \int \cdots \int \sum_{k=0}^{j} f^{m+1}_{j-k,0}(\phi^m_k, \cdots, \phi^m_j) e^{-\phi_k{}^m} dA(\tau_0) \cdots dA(\tau_m) \\ &= \int \cdots \int \sum_{k=0}^{j} f^{m+1-k}_{j-k,0}(\phi^{m-k}_0, \cdots, \phi^{m-k}_{j-k}) e^{-\phi_0{}^{m-k}} dA(\tau_0) \cdots dA(\tau_m), \end{aligned} \tag{12}$$

the last step being a consequence of (11). The problem has thus been reduced to the evaluation of the integral

$$I^{m+1}_j = \int \cdots \int e^{-\phi_0{}^m} f^{m+1}_{j,0}(\phi^m_0, \cdots, \phi^m_j) dA(\tau_0) \cdots dA(\tau_m).$$

This problem is dealt with in Lemma 2.

LEMMA 2

$$I^{m+1}_j = \frac{m-j+1}{m+1} \sum_{\sum i\alpha_i = j} \binom{m+1}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod(\alpha_i!)} \psi_0^{m+1-\sum\alpha_i} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_j^{\alpha_j},$$

summation being over all j-tuples $(\alpha_1, \cdots, \alpha_j)$ *of non-negative integers such that* $\sum i\alpha_i = j$, *and* ψ_k *being defined by*

$$\psi_k = \int_{x=0}^{\infty} \frac{x^k}{k!} e^{-x} dA(x).$$

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PROOF. Suppose that the lemma is true for $j = 0, 1, 2, \cdots, a-1$, $(a \leqq m)$. Then from Lemma 1

$$I_a^{m+1} = \sum_{r=0}^{a-1} \left\{ \int \cdots \int \frac{e^{-\phi_a{}^m} (\phi_a^m)^{a-r}}{(a-r)!} \, dA(\tau_a) \cdots dA(\tau_m) \right\} I_r^a$$

$$= \sum_{r=0}^{a} (1 - r/a) \left\{ \sum_{\sum i\beta_i = a-r} \binom{m-a+1}{\sum \beta_i} \frac{(\sum \beta_i)!}{\prod (\beta_i!)} \psi_0^{m-a+1-\sum \beta_i} \psi_1^{\beta_1} \cdots \psi_{a-r}^{\beta_{a-r}} \right\}$$

$$\times \left\{ \sum_{\sum h\gamma_h = r} \binom{a}{\sum \gamma_h} \frac{(\sum \gamma_h)!}{\prod (\gamma_h!)} \psi_0^{a-\sum \gamma_h} \psi_1^{\gamma_1} \cdots \psi_r^{\gamma_r} \right\}.$$

The latter expression is the coefficient of z^a in the expansion of

$$\left\{ \left(\sum_{k=0}^{\infty} \psi_k z^k \right)^{m+1} - \left(\sum_{k=0}^{\infty} k\psi_k z^k \right) \left(\sum_{k=0}^{\infty} \psi_k z^k \right)^m \right\}$$

and can therefore also be written as

$$\frac{m-a+1}{m+1} \sum_{\sum i\alpha_i = a} \binom{m+1}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod (\alpha_i!)} \psi_0^{m+1-\sum \alpha_i} \psi_1^{\alpha_1} \cdots \psi_a^{\alpha_a}.$$

The lemma is certainly true for $j = 0, 1$ and hence by induction it is true for $j = 0, 1, \cdots, m$.

Applying Lemma 2 to equation (12) it is found that

$$(13) \quad q_j^{m+1} = (j+1) \sum_{n=j+1}^{m+1} \frac{1}{n} \sum_{\sum i\alpha_i = n-j-1} \binom{n}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod (\alpha_i!)} \psi_0^{n-\sum \alpha_i} \psi_1^{\alpha_1} \cdots \psi_{n-j-1}^{\alpha_{n-j-1}}.$$

Equation (13) defines the transient queue size distribution at an arrival.

4. The busy period probabilities for $GI/M/1$

In the terminology of § 3 we have from equation (6) that

$$U_{m+1-j}^{m+1} = f_{j,0}^{m+1}(\phi_0^m, \phi_1^m, \cdots, \phi_j^m) e^{-\phi_0{}^m}.$$

Applying Lemma 2 in order to determine the corresponding probabilities u_{m+1-j}^{m+1} we find that

$$(14) \quad u_{m+1-j}^{m+1} = \frac{m+1-j}{m+1} \sum_{\sum i\alpha_i = j} \binom{m+1}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod (\alpha_i!)} \psi_0^{m+1-\sum \alpha_i} \psi_1^{\alpha_1} \cdots \psi_j^{\alpha_j}, \quad 0 \leqq j \leqq m.$$

The probability that $(m+1)$ customers are served in a busy period is given by

$$(15) \quad u_0^{m+1} = \sum_{j=1}^{m} u_j^m - \sum_{j=1}^{m+1} u_j^{m+1}, \qquad m \geqq 1,$$

$$u_0^1 = 1 - u_1^1.$$

5. The special cases $M/M/1$, $D/M/1$

(*a*) $M/M/1$. We have in this case $A(x) = 1-e^{-\rho x}$, $x \geqq 0$, and hence $\psi_k = \rho/(1+\rho)^{k+1}$. Substituting in equation (13) and using the identity,

$$\binom{m+j}{j} = \sum_{\sum i\alpha_i = j} \binom{m+1}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod (\alpha_i !)},$$

which can be proved by considering the power series expansion of $(1-x)^{-m-1}$ about $x = 0$, we find that

$$(16) \quad q_j^{m+1} = \frac{\rho^{j+1}}{(1+\rho)^{j+1}} \sum_{i=0}^{m-j} (j+1) \frac{(j+2i)!}{(j+i+1)!\,i!} \frac{\rho^i}{(1+\rho)^{2i}}, \qquad 0 \leqq j \leqq m.$$

It follows that

$$\lim_{m\to\infty} q_j^{m+1} = \frac{\rho^{j+1}}{(1+\rho)^{j+1}} \left[\frac{1-\sqrt{1-\dfrac{4\rho}{(1+\rho)^2}}}{\dfrac{2\rho}{(1+\rho)^2}} \right]^{j+1}$$

$$= \begin{cases} \rho^{j+1} & \text{if } \rho < 1 \\ 1 & \text{if } \rho \geqq 1 \end{cases}$$

and this is a well known result for the system $M/M/1$.

Substituting for ψ_k in equation (14) we obtain the following expression for the busy period probabilities,

$$(17) \quad u_j^{m+1} = \frac{j}{m+1} \binom{2m+1-j}{m} \frac{\rho^{m+1}}{(1+\rho)^{2m+2-j}}, \qquad 1 \leqq j \leqq m+1.$$

(*b*) $D/M/1$. For this system $A(x) = 1$, $x \geqq c$, and hence $\psi_k = c^k/k!\, e^{-c}$. Substituting in equation (13) and using the identity,

$$m^j = \sum_{\sum i\alpha_i = j} \binom{m}{\sum a_i} \frac{(\sum \alpha_i)!\, j!}{\prod [\alpha_i!\,(i!)^{\alpha_i}]},$$

which can be proved by expanding $(\phi_1^m)^j$ and putting each τ equal to 1, we find that

$$(18) \quad q_j^{m+1} = (j+1) \sum_{r=0}^{m-j} \frac{(r+j+1)^{r-1}}{r!} c^r e^{-c(r+j+1)}, \qquad 0 \leqq j \leqq m.$$

Treating the busy period probabilities in the same way we find that

$$(19) \quad u_j^{m+1} = \frac{jc^{m+1-j}}{(m+1-j)!} (m+1)^{m-j} e^{-mc-c}, \qquad 1 \leqq j \leqq m+1.$$

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6. The limiting distribution for $GI/M/1$

We define

$$\psi(z) = \int_{\tau=0}^{\infty} e^{-\tau(1-z)} dA(\tau) = \sum_{k=0}^{\infty} \psi_k z^k$$

for real z and $|z| \leqq 1$.

It is well known that if $\psi'(1) > 1$ a limiting distribution of queue size at an arrival exists and is given by

$$q_j = \theta^{j+1},$$

where θ is the only root of $z = \psi(z)$ inside the circle $|z| = 1$.

The Lagrange expansion of q_j is given by

$$\begin{aligned} q_j &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(D^{n-1}[(j+1)z^j\{\psi(z)\}^n] \right)_{z=0} \\ &= \sum_{n=j+1}^{\infty} \frac{j+1}{n} \text{ (coefficient of } z^{n-1-j} \text{ in the expansion of } \{\psi(z)\}^n) \\ &= \sum_{n=j+1}^{\infty} \frac{j+1}{n} \sum_{\sum i\alpha_i = n-j-1} \binom{n}{\sum \alpha_i} \frac{(\sum \alpha_i)!}{\prod \alpha_i!} \psi_0^{n-\sum\alpha_i} \psi_1^{\alpha_1} \cdots \psi_{n-j-1}^{\alpha_{n-j-1}}. \end{aligned}$$

Comparing this expression with equation (13) we see that q_j^{m+1} is a partial sum of the series for q_j as conjectured by Finch [2]. This interesting result provides us with a means of examining the rate at which the system approaches statistical equilibrium.

7. Relationships with previously obtained results

The transient behaviour of $M/G/1$ has been studied by Pollaczek [7] and Finch [3]. Finch determines the transient distribution of queue size at a departure and for the case $\eta_0 = 0$ relates this to the transient distribution of queue size at an arrival. He also deals with the special case $M/M/1$ and from his equations (34) and (21) the result (16) can be deduced. Takács [6] uses a different method to determine the transient queue size distribution of $GI/M/1$ obtaining his results in the form of a generating function from which the explicit probabilities q_j^{m+1} found in this paper can be deduced. Takács, however, does not determine the probabilities u_j^{m+1}.

The distribution of the number of customers served in a busy period has been determined for $GI/M/1$ by Conolly [1], for $M/G/1$ by Prabhu [5], and for $GI/G/1$ by Finch [4]. It can be verified that the results obtained in this paper for the special cases $M/M/1$ and $D/M/1$ agree with those obtained in the papers mentioned.

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