Orthogonal Trajectories in Vectorial Coordinates.

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(*Received December 1909. Read February 1910.*)

1. The position of a point in a plane may be determined by its distances (r, ρ) from two fixed points which may be termed foci. These distances are termed vectorial coordinates. The determination is unique if attention is confined to the half-plane bounded by the line of foci. In certain cases where the properties of a curve are defined with respect to two points, representation of the curve by an equation between its vectorial coordinates possesses certain advantages. For example, the equation of the ellipse takes the extremely simple form $r + \rho = 2a$.

2. We may pass from cartesian to vectorial coordinates and *vice versa* by the transformations

$$x = (r^2 - \rho^2 + a^2)/2a,$$

$$y^2 = (r + \rho + a)(r + \rho - a)(-r + \rho + a)(r - \rho + a)/4a^2$$

where a is the distance between the foci, and

$$r^2 = x^2 + y^2, \quad \rho^2 = (x - a)^2 + y^2.$$

We may pass from polar to vectorial coordinates and *vice versa*, by the transformations

$$r = r, \quad \cos\theta = (r^2 + a^2 - \rho^2)/(2ra)$$

and $$r = r, \quad \rho^2 = r^2 + a^2 - 2ra\cos\theta.$$

3. The particular purpose of this paper is to investigate the condition for orthogonality of two curves whose equations are given in vectorial coordinates, and to illustrate by a few examples.

4. If (Fig. 1) A and B are the foci and P a point on the curve,

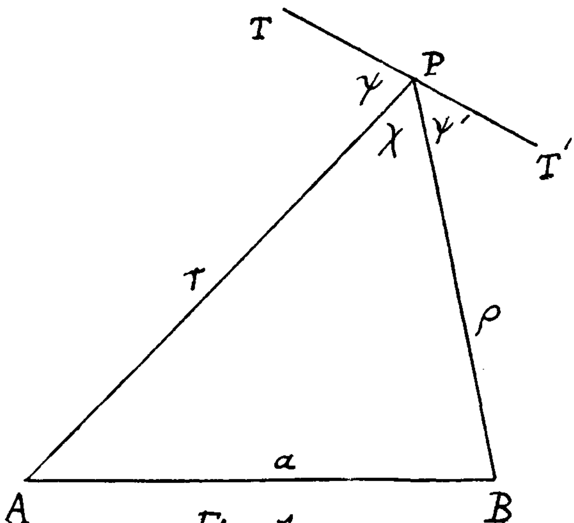

Fig 1

whose tangent is TPT', then, denoting the angles TPA, APB, BPT' by ψ, χ and ψ' respectively, we have

$$\cos\chi = (r^2 + \rho^2 - a^2)/(2r\rho) = -\cos(\psi + \psi') \qquad \text{(1).}$$

Also $dr = -ds\cos\psi$, $d\rho = +ds\cos\psi'$.

Hence $$\frac{dr}{d\rho} = -\frac{\cos\psi}{\cos\psi'} \qquad \text{(2).}$$

5. If tPt' is the tangent of a curve orthogonal to the original curve, then (from Fig. 2) we have

$$\phi = \psi - \tfrac{1}{2}\pi, \quad \phi' = \psi' + \tfrac{1}{2}\pi$$

$$\rho' = \rho \text{ and } r' = r$$

and $$\frac{dr'}{d\rho'} = -\frac{\cos\phi}{\cos\phi'} = +\frac{\sin\psi}{\sin\psi'} \qquad \text{(3)}$$

where ρ' and r' are the vectorial coordinates of the second curve.

6. *The orthogonal condition.*

$$\cos\chi = -\cos(\psi + \psi').$$

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Hence on multiplication throughout by $2\sin(\psi - \psi')$, we have

$$\begin{aligned} 2\sin(\psi - \psi')\cos\chi &= -2\sin(\psi - \psi')\cos(\psi + \psi) \\ &= \sin 2\psi' - \sin 2\psi \\ &= 2\sin\psi'\cos\psi' - 2\sin\psi\cos\psi. \end{aligned}$$

On division throughout by $2\sin\psi'\cos\psi'$, we get

$$\cos\chi(\sin\psi/\sin\psi' - \cos\psi/\cos\psi') = 1 - \sin\psi\cos\psi/(\sin\psi'\cos\psi')$$

or $$\left(\frac{dr}{d\rho} + \frac{dr'}{d\rho'}\right)\frac{r^2 + \rho^2 - a^2}{2r\rho} = 1 + \frac{dr}{d\rho}\cdot\frac{dr'}{d\rho'} \quad \text{..........................(4)}$$

This is the condition of orthogonality, and it must be remembered that r, ρ are to be put for r', ρ' respectively, after differentiation.

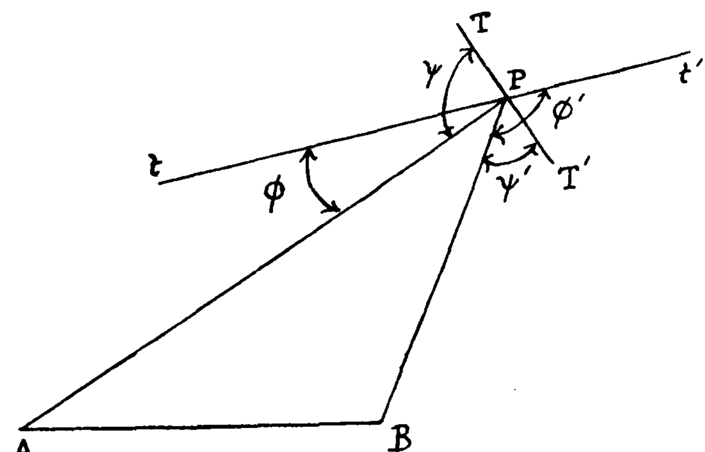

Fig. 2.

7. If the equation to one of the curves is given, (4) becomes the differential equation to the orthogonal trajectories, under suitable arrangements concerning the constants of the first curve. Thus if the equation to a curve is given in the form $F(r', \rho', b) = 0$, where by variation of b we pass from one member of the family to another, we must eliminate b between the equations $F = 0$, $\frac{\partial F}{\partial r'}\frac{dr'}{d\rho'} + \frac{\partial F}{\partial \rho'} = 0$ before substituting for $\frac{dr'}{d\rho'}$ in (4).

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If the equation of the curve is $F(r', \rho', b_1, b_2, \ldots b_n) = 0$, the genesis of the family of curves may arise in two ways, which may be looked on as fundamentally the same, but conveniently separated in practice. Thus $n-1$ of the constants may be absolute constants, while the n^{th} by its variability gives rise to the different members of the family, or we may be given $m \not> (n-1)$ equations between the constants. By elimination of m of the constants we arrive at an equation $\Phi(r', \rho', b_1, \ldots b_{n-m}) = 0$, which is a case of the former.

8. Equation (4) may be transformed and simplified as follows:— From it we deduce

$$\frac{dr}{d\rho} = \frac{2r\rho - (r^2 + \rho^2 - a^2)p}{r^2 + \rho^2 - a^2 - 2r\rho p} \text{ where } p = \frac{dr'}{d\rho'}.$$

Put now $r + \rho = \lambda$, $r - \rho = \mu$, and we obtain

$$\frac{d\lambda + d\mu}{d\lambda - d\mu} = \frac{\lambda^2 - \mu^2 - (\lambda^2 + \mu^2 - 2a^2)p}{\lambda^2 + \mu^2 - 2a^2 - (\lambda^2 - \mu^2)p}.$$

Hence

$$\frac{d\lambda}{d\mu} = \frac{(\lambda^2 - a^2)(p-1)}{(\mu^2 - a^2)(p+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This suggests the further substitutions $\lambda = a\coth u$, $\mu = a\coth v$, but for the present we leave the equation as it is above. The curves $\lambda =$ constant and $\mu =$ constant are confocal ellipses and hyperbolas respectively.

9. We pass now to some particular cases.

(i) *The Cassinian oval.* $r'\rho' = b^2$. Here $p = -r'/\rho'$.

$$\frac{d\lambda}{d\mu} = \frac{\lambda^2 - a^2}{\mu^2 - a^2}\,\frac{r+\rho}{r-\rho} = \frac{\lambda(\lambda^2 - a^2)}{\mu(\mu^2 - a^2)}.$$

By integration we find the equation to the orthogonal trajectories $(\lambda^2 - a^2)/\lambda^2 = c(\mu^2 - a^2)/\mu^2$, or in vectorial coordinates

$$[(r+\rho)^2 - a^2](r-\rho)^2 = k[(r-\rho)^2 - a^2](r+\rho)^2$$

which may be reduced to the cartesian equation

$$(x^2 + y^2)[(x-a)^2 + y^2] = c^2(x^2 - y^2 - ax)^2.$$

(ii) *Coaxal circles having the foci for limiting points.*

Here $r' = m\rho'$, $p = m$.

$$\frac{d\lambda}{d\mu} = \frac{(m-1)(\lambda^2 - a^2)}{(m+1)(r - a^2)} = \frac{r-\rho}{r+\rho}\cdot\frac{\lambda^2 - a^2}{\mu^2 - a^2} = \frac{\mu(\lambda^2 - a^2)}{\lambda(\mu^2 - a^2)}.$$

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Hence $\lambda^2 - a^2 + c^2(\mu^2 - a^2) = 0$ where c is the constant of integration. This may be written $(r+\rho)^2 - a^2 = c^2[a^2 - (r-\rho)^2]$, and it is easy to reduce it to the polar equation $r = a\cos\theta \pm k\sin\theta$, where $k = a(1-c^2)/2c$ and to the cartesian equation $x^2 + y^2 = ax \pm ky$. This represents a family of coaxal circles passing through the foci, as is well known.

(iii) *Concentric circles.* Take the foci collinear with, and at distances $\pm\frac{1}{2}a$ from, the centre, and let $2b$ be the radius of a circle. The equation of the family of circles is

$$r'^2 + \rho'^2 = \tfrac{1}{2}(a^2+b^2),\ \ p = dr'/d\rho' = -\rho'/r'.$$

Hence the differential equation to the trajectories is

$$\frac{d\lambda}{d\mu} = -\frac{\lambda(\lambda^2-a^2)}{\mu(\mu^2-a^2)}.$$ The primitive is $(\lambda^2-a^2)(a^2-\mu^2) = c^2\lambda^2\mu^2$.

It is not difficult to simplify this into the equation $y = c(2x-a)$, a family of straight lines through the middle of the focal line.

(iv) *Cartesian oval.* Here $r' + m\rho' = b$, and we have three cases, when m is the parameter and b is constant, when b is the parameter, and when there is a relation between m and b.

In the first case $\dfrac{dr'}{d\rho'} = -m = \dfrac{r'-b}{\rho'}$.

Hence $\dfrac{d\lambda}{d\mu} = \dfrac{\lambda^2-a^2}{\mu^2-a^2}\,\dfrac{\mu-b}{\lambda-b}$.

The integral of this is found without much difficulty to be

$$a\log[(\lambda^2-a^2)/(a^2-\mu^2)] - b\log[(\lambda-a)(a-\mu)(\lambda+a)^{-1}(a+\mu)^{-1}] = \text{const.}$$

Clearly not much will be gained by substituting for λ and μ in terms of r and ρ.

In the next case $r' + m\rho' = b$ and b is the parameter. Here

$$\frac{dr'}{d\rho'} = -m, \text{ and } \frac{d\lambda}{d\mu} = \frac{\lambda^2-a^2}{a^2-\mu^2}\cdot\frac{1+m}{1-m}$$

and the integral is

$$\left(\frac{\lambda-a}{\lambda+a}\right)^{1-m}\cdot\left(\frac{a-\mu}{a+\mu}\right)^{1+m} = \text{const.}$$

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[The two tri-confocal cartesians that pass through the point (r, ρ) are given by the equations

$$\left.\begin{array}{l} mr + l\rho = nc_3 \\ l^2c_1 + n^2c_3 = m^2c_2 \\ c_1 + c_3 = c_2 \end{array}\right\}$$

where c_1 and c_3 are the distances between the extreme and intermediate foci. (Williamson, Diff. Calc., London, 1887, p. 378.)

If p_1 and p_2 refer to these confocals, p_1 and p_2 are the roots of the equation in p, obtained by eliminating the ratios $l : m : n$ between these three equations and $mp + l = 0$, *i.e.* of the equation

$$p^2(\rho^2 + c_1c_3) - 2pr\rho + (r^2 - c_2c_3) = 0.$$

Hence $p_1 + p_2 = 2r\rho/(\rho^2 + c_1c_3)$, and $p_1p_2 = (r^2 - c_2c_3)/(\rho^2 + c_1c_3)$, and on substituting these values in (4), this equation is identically satisfied. It follows that the two confocal cartesians that pass through a point are orthotomic. (Williamson, op. cit., p. 381.)]*

(v) *Equipotential curves in a plane.* The equation when there are two charged points is $\frac{e_1}{r'} + \frac{e_2}{\rho'} = c$ where c varies from surface to surface.

Hence $\frac{dr'}{d\rho'}\frac{e_1}{r'^2} + \frac{e_2}{\rho'^2} = 0$ or $p = -\frac{e_2}{e_1}\frac{r^2}{\rho^2}$.

Hence $$\frac{d\lambda}{d\mu} = \frac{\lambda^2 - a^2}{a^2 - \mu^2}\frac{e_1\rho^2 + e_2r^2}{e_1\rho^2 - e_2r^2}$$

$$= \frac{\lambda^2 - a^2}{a^2 - \mu^2}\frac{e_1(\lambda - \mu)^2 + e_2(\lambda + \mu)^2}{e_1(\lambda - \mu)^2 - e_2(\lambda + \mu)^2}.$$

It is not difficult to show that the primitive of this differential equation is

$$\frac{e_1(a^2 + \lambda\mu)}{a(\lambda + \mu)} = \frac{e_2(a^2 - \lambda\mu)}{a(\lambda - \mu)} + c$$

where c is the constant of integration, and the equation is easily reducible to the well-known form for the equation to the lines of force—

$$e_1\cos\theta_1 - e_2\cos\theta_2 = \text{const.}$$

* Added March 4, 1910.

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