Canad. Math. Bull. Vol. 16 (2), 1973

EXISTENCE OF SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS IN A LOCAL SPACE

BY
M. A. MALIK(1)

1. Let H be a Hilbert space; $(\ ,\ )$ and $|\ |$ represent the scalar product and the norm respectively in H. Let A be a closed linear operator with domain D_A dense in H and A^* be its adjoint with domain D_{A^*}. D_A and D_{A^*} are also Hilbert spaces under their respective graph scalar product. $R(\lambda; A^*)$ denotes the resolvent of A^*; $\lambda=\sigma+i\tau \in \mathbb{C}$, complex plane. We write $L=D-A$, $L^*=D-A^*$; $D=(1/i)(d/dt)$.

By $\mathscr{H}^s(H)$, s arbitrary real number, we mean the space of H-valued tempered distributions u defined in R, (real line) whose Fourier transform $\hat{u}$ is a function and

$$\|u\|_s^2 = \int_R (1+|\sigma|^2)^s\, |\hat{u}(\sigma)|^2\, d\sigma < \infty. \tag{1.1}$$

The space $u \in \mathscr{H}^s(H)$ with compact support will be denoted by $\mathscr{H}_0^s(H)$. A sequence $u_n \to 0$ in $\mathscr{H}_0^s(H)$ if supp u_n; $n=1, 2, \ldots$ are contained in a fixed compact set and $u_n \to 0$ in the norms (1.1). The space $\mathscr{H}_{\text{loc}}^{-s}(H)$ is defined as the dual of $\mathscr{H}_0^s(H)$.

Taking D_{A^*} (resp. $\mathbb{C}$) instead of H in the above definitions, we define $\mathscr{H}_0^s(D_{A^*})$ (resp. $\mathscr{H}_0^s(\mathbb{C})$). It can be verified that $\mathscr{H}_{\text{loc}}^{-s}(H)$ is the space of continuous linear H-valued mappings defined on $\mathscr{H}_0^s(\mathbb{C})$ and if $u \in \mathscr{H}_{\text{loc}}^{-s}(H)$ and $\langle u, \psi\rangle \in D_A$ for all $\psi \in \mathscr{H}_0^s(\mathbb{C})$, then $u \in \mathscr{H}_{\text{loc}}^{-s}(D_A)$.

2. In view of imposing conditions on the resolvent we need:

Definition. Let F be a family of parallel lines

$$\{\text{Im}\,\lambda = \tau_n;\ \tau_n \to \infty \text{ as } n \to \infty,\ \tau_n \to -\infty \text{ as } n \to -\infty\}.$$

We shall say that the resolvent $R(\lambda; A^*)$ is of (k, Δ)-growth on F if the resolvent exists outside j intervals of length r on every line of F and for these λ

$$|R(\lambda; A^*)| \le \text{const. } |\lambda|^k e^{\Delta|Im\lambda|} \tag{2.1}$$

where k and Δ are nonnegative real numbers.

Throughout this paper 'const.' need not be the same constant.

For $\Delta=k=0$ in the above definition, Agmon and Nirenberg [1] have defined

Received by the editors June 24, 1971 and, in revised form, November 8, 1971.

(1) This work was done while the author was attending the Summer Research Institute of the Canadian Mathematical Congress.

https://doi.org/10.4153/CMB-1973-041-0 Published online by Cambridge University Press

the resolvent of (0, 0)-growth on F_+ (lying in the upper half-plane). Zaidman [5] considered the resolvent $R(\lambda; A^*)$ of (0, 0)-growth on F to prove:

THEOREM A. *If $R(\lambda; A^*)$ is of (0, 0)-growth on F, then for every $f \in L^2_{\text{loc}}(H)$, the abstract differential equation $Lu=f$ has a weak solution $u \in L^2_{\text{loc}}(H)$ i.e.*

$$\int_R (u(t), L^*\phi(t))\, dt = \int_R (f(t), \phi(t))\, dt \tag{2.2}$$

for all $\phi \in C_0^\infty(D_{A^*})$.

The author [2] proved the existence of weak solution of $Lu=f$ in the distribution space; also studied the uniqueness of Cauchy problem in [4].

In [2], he has

THEOREM B. *Let $R(\lambda, A^*)$ be of $(k, 0)$-growth on F, then for every $f \in D'(H)$ the equation $Lu=f$ has a weak solution $u \in D'(H)$ i.e.*

$$\langle u, L^*\phi\rangle = \langle f, \phi\rangle \tag{2.3}$$

for all $\phi \in C_0^\infty(D_{A^*})$.

In this paper, we study the existence of solutions of $Lu=f$ in $\mathscr{H}^s_{\text{loc}}(H)$; see Theorem 3. In fact, this is an improvement as well as a generalization of author's result [3] where he announced the existence of a weak solution in $\mathscr{H}^{-s}_{\text{loc}}(H)$ when the resolvent is of $(k, 0)$-growth; s and k were restricted to be nonnegative integer in this case.

3. **The following conditions will be needed.**

Hypothesis I. Let $K \subset R$ be a compact set. There exists a constant depending on K such that for all $\phi \in \mathscr{H}^s_0(D_{A^*})$ with supp $\phi \subset K$,

$$\|\phi\|_s \le \text{const. } (K)\, \|L^*\phi\|_{s+k}, \tag{3.1}$$

s is an arbitrary real number and k nonnegative real number but both are fixed.

Hypothesis II. Let $k \subset R$ be a compact set and supp $L^*\phi \subset K$ where $\phi \in \mathscr{H}^s_0(D_{A^*})$, s arbitrary real number. Then there exists a compact set K_1 such that supp $\phi \subset K_1$.

First we study the conditions on the resolvent $R(\lambda; A^*)$ so that Hypothesis I and Hypothesis II are satisfied.

THEOREM 1. *Let $R(\sigma; A^*)$ exist on P (the real axis R minus j intervals of length r) and for these $\sigma \in P$*

$$|R(\sigma; A^*)| \le \text{const. } |\sigma|^k \tag{2.1'}$$

where $k \ge 0$. Then Hypothesis I is satisfied.

LEMMA 1. *Let the condition of Theorem* 1 *be satisfied and* $K \subset R$ *be a compact set. Then for all* $\phi \in \mathscr{H}_0^s(H)$ *with supp* $\phi \subset K$,

$$\int_R (1+|\sigma|^2)^s |\hat{\phi}(\sigma)|^2 \, d\sigma \leq \text{const.} \int_P (1+|\sigma|^2)^s |\hat{\phi}(\sigma)|^2 \, d\sigma. \tag{3.2}$$

The const. depends on K, j *and* r *but not on the position of the intervals.*

Proof of Lemma 1. For $s=0$ the lemma is known [1]. Let $\{e_m\}$ be an orthonormal basis in H. For $\phi \in \mathscr{H}_0^s(H)$ we write $\phi=\sum_1^\infty \phi_m e_m$ where $\phi_m(t)=(\phi(t), e_m)$. By the continuity of scalar product, it can be easily verified that $\hat{\phi}_m(\sigma)=(\hat{\phi}(\sigma), e_m)$ and following Parseval's relation one has $|\hat{\phi}(\sigma)|_H=\sum_1^\infty |\hat{\phi}_m(\sigma)|^2$. So $\phi_m \in \mathscr{H}_0^s(\mathbb{C})$ and supp $\phi=$supp $\phi_m \subset K$.

Suppose (3.2) with ϕ replaced by $\psi \in \mathscr{H}_0^s(\mathbb{C})$ is not true. So there exists a sequence

$$\{\psi_n;\, \text{supp } \psi_n \subset K\} \text{ with } \int_R (1+|\sigma|^2)^s |\hat{\psi}_n(\sigma)|^2 \, d\sigma = 1$$

and a sequence of axes P_n (with j intervals $I_{n1}, \ldots, I_{nj}$ removed) such that

$$\int_{P_n} (1+|\sigma|^2)^s |\hat{\psi}_n(\sigma)|^2 \, d\sigma \to 0. \tag{3.3}$$

For each n (large enough), on at least one interval labelled as I_{n1} (by appropriate displacement we may suppose $I_{11}=I_{21}=\cdots=I_{n1}=\cdots=I$) one has

$$\int_I (1+|\sigma|^2)^s |\hat{\psi}_n(\sigma)|^2 \, d\sigma \geq \frac{1}{2j}. \tag{3.4}$$

As $\|\psi_n\|_s=1$, the set $\{\psi_n\}$ is bounded and closed in D', so there exists a subsequence $\{\psi_{n_k}\}$ converging weakly in D', so strongly in D'. Since the supp ψ_{nk} are contained in a fixed compact set K, ψ_{nk} converges strongly in ξ'. But this immediately implies that their Fourier transforms which are analytic functions of exponential type converges uniformly to an analytic function ξ on every compact set of $\mathbb{C}$. So we conclude from (3.4) that

$$\int_I (1+|\sigma|^2)^s |\xi(\sigma)|^2 \, d\sigma \geq \frac{1}{2j}. \tag{3.5}$$

However, it follows from (3.3)—that $\hat{\psi}_{nk}$ converges to zero on some interval on the real axis. Hence $\xi \equiv 0$—contradiction. Consequently, we have

$$\int_R (1+|\sigma|^2)^s |\hat{\phi}_m(\sigma)|^2 \, d\sigma \leq \text{const.}\,(K) \int_P (1+|\sigma|^2)^s |\hat{\phi}_m(\sigma)|^2 \, d\sigma. \tag{3.6}$$

From the sequence of inequalities (3.6) and Parseval's relation, Lemma 1 is proved.

https://doi.org/10.4153/CMB-1973-041-0 Published online by Cambridge University Press

Proof of Theorem 1. Let $\phi \in \mathscr{H}_0^s(D_{A^*})$. Set

$$\frac{1}{i}\frac{d}{dt}\phi - A^*\phi \equiv L^*\phi. \tag{3.7}$$

Taking the Fourier transform of (3.7), we obtain

$$(\sigma I - A^*)\hat{\phi}(\sigma) = \widehat{L^*\phi}(\sigma). \tag{3.8}$$

For $\sigma \in P$ from (3.8), one gets

$$|\hat{\phi}(\sigma)| \leq \text{const.}\ |\sigma|^k\ |\widehat{L^*\phi}(\sigma)| \tag{3.9}$$

and so combined with Lemma 1,

$$\|\phi\|_s \leq \text{const.}\ \|L^*\phi\|_{s+k}.$$

The const. depends on the support of ϕ, as the constant in Lemma 1 is not independent of supp ϕ.

THEOREM 2. *Suppose the resolvent $R(\lambda; A^*)$ is of (k, Δ)-growth on F. Then Hypothesis* II *is satisfied.*

LEMMA 2. *If $R(\lambda; A^*)$ is of (k, Δ)-growth on F_+ and $u \in C^\infty(D_{A^*})$ is a solution of $L^*u(t)=0$ on $0 \leq t \leq T$ with $u(T)=0$. Then $u(t)=0$ for $t \geq 2\Delta$.*

Proof of Lemma 2. Fix a positive $\alpha < T$ and let $\xi(t)$ be a nonnegative C^∞ function of t which vanishes for $t \leq \alpha/2$ and is equal to 1 for $t \geq \alpha$. Set $v(t) = e^{\tau_n t}\xi(t)u(t)$; note $\tau_n \geq 0$. Setting $(L^* + i\tau_n)v(t) = h(t)$, we see that on taking Fourier transform

$$(\sigma + i\tau_n - A^*)\hat{v}(\sigma) = \hat{h}(\sigma). \tag{3.10}$$

As $R(\lambda; A^*)$ is of (k, Δ)-growth, for all real σ except on j intervals of length s

$$\begin{aligned} |\hat{v}(\sigma)|^2 &\leq \text{const.}\ |\sigma + i\tau_n|^{2k} e^{2\Delta\tau_n}\ |\hat{h}(\sigma)|^2 \\ &\leq \text{const.}\ |\tau_n|^{2k} e^{2\Delta\tau_n} \sum_0^k |\sigma|^{2k}\ |\hat{h}(\sigma)|^2. \end{aligned} \tag{3.11}$$

There is no loss of generality in assuming here that k is a nonnegative integer. Integrating (3.11) on P and using Lemma 1 with $s=0$ and Parseval's theorem, we have

$$\int_{-\infty}^{\infty} |v(t)|^2\, dt \leq \text{const.}\ \tau_n^{2k} e^{2\Delta\tau_n} \int_{-\infty}^{\infty} |h^{(k)}(t)|^2\, dt. \tag{3.12}$$

which leads to

$$\int_\alpha^T |e^{\tau_n t}u(t)|^2\, dt \leq \text{const.}\ \tau_n^{2k} e^{(2\Delta+\alpha)\tau_n}. \tag{3.13}$$

A choice of $\beta<\alpha$ in (3.13) implies that

$$\int_{\beta}^{T} |u(t)|^2 \, dt \leq \text{const.}\, \tau_n^{2k} e^{(2\Delta+\alpha-\beta)\tau_n}. \tag{3.14}$$

As $\tau_n \to \infty$, we observe that $u(t)=0$ for $t>2\Delta+\alpha$. Making $\alpha\to 0$, $u(t)=0$ for $t>2\Delta$. Lemma 1 is proved.

In Lemma 2, the interval $[0, T]$ may be replaced by any other interval $[a, b]$.

LEMMA 3. *If* $R(\lambda; A^*)$ *is of* (k, Δ)*-growth on* F *and* supp $L^*u \subset [a, b]$ *where* $u \in C_0^\infty(D_{A^*})$. *Then* supp $u \subset [a-2\Delta, b+2\Delta]$.

The proof of Lemma 3 is immediate after Lemma 2.

Proof of Theorem 2. Let $K=[a, b]$. Consider a delta convergent sequence α_n, supp $\alpha_n \subset [-1/n, 1/n]$ and take $u*\alpha_n$. It is obvious that $u*\alpha_n \in C_0^\infty(D_{A^*})$ and supp $L^*(u*\alpha_n) \subset [a-(1/n), b+(1/n)]$. From Lemma 3 supp $u*\alpha_n \subset [a-2\Delta-(1/n), b+2\Delta+(1/n)]$. Making $n\to\infty$, we have Theorem 2.

REMARK 1. In Hypothesis II, the existence of the resolvent $R(\lambda; A^*)$ on Im $\lambda=0$ is not assumed. It may also be pointed out that the purpose of this paper is to determine what one can say when one requires of the family F that $\tau_n=0$ for some n.

THEOREM 3. *Suppose* Hypothesis I *and* Hypothesis II *are satisfied. Then for* $f \in \mathscr{H}^s_{\text{loc}}(H)$ *the abstract differential equation*

$$\frac{1}{i}\frac{du}{dt} - Au = f \tag{3.15}$$

has at least one solution $u \in \mathscr{H}^{s-k-1}_{\text{loc}}(D_A)$; $k\geq 0$ *and* s *is an arbitrary real number.*

REMARK 2. Let F be a family of parallel lines $\{\text{Im}\,\lambda=\tau_n;\ \tau_0=0,\ \tau_n\to\infty$ as $n\to\infty,\ \tau_n\to-\infty$ as $n\to-\infty\}$. If $R(\lambda; A^*)$ is of (k, Δ)-growth on F, then Hypotheses I and II are satisfied.

Proof of Theorem 3. Let $f \in \mathscr{H}^s_{\text{loc}}(H)$ be given. On the subspace $X=\{L^*\phi;\ \phi \in \mathscr{H}_0^{-s}(D_{A^*})\} \subset \mathscr{H}_0^{-s+k}$ we define a functional F by the relation

$$\langle F, L^*\phi\rangle = \langle f, \phi\rangle. \tag{3.16}$$

The linearity of F is obvious. To verify its continuity, suppose $y_n=L^*\phi_n\to 0$ in $\mathscr{H}_0^{-s+k}(H)$. From Hypothesis II, supp ϕ_n are contained in a fixed compact set and Hypothesis I implies that $\|\phi_n\|_{-s}\to 0$. Consequently, F is a continuous linear functional on X and therefore by Hahn-Banach theorem, can be extended to u defined on the whole space $\mathscr{H}_0^{-s+k}(H)$. It is clear $u \in \mathscr{H}^{s-k}_{\text{loc}}(H)$ and satisfies

$$\langle u, D\phi - A^*\phi\rangle = \langle f, \phi\rangle \tag{3.17}$$

for all $\phi \in \mathscr{H}_0^{-s}(D_{A^*})$.

https://doi.org/10.4153/CMB-1973-041-0 Published online by Cambridge University Press

Now we are going to prove that u satisfies (3.15). Let $\phi=\psi\otimes x$ where $\psi\in\mathscr{H}_0^{-r}(\mathbb{C})$, $r=s-k-1$ and $x\in D_{A^*}$. Treating f and u belonging to $\mathscr{H}^r_{\text{loc}}(H)$ from (3.17), we observe that

$$(3.18)\qquad (\langle Du-f, \psi\rangle, x) = (\langle u, \psi\rangle, A^*x)$$

for all $x\in D_{A^*}$. Thus $\langle u, \psi\rangle\in D_{A^{**}}=D_A$; A is closed with dense domain. As $\langle u, \psi\rangle\in D_A$ for all $\psi\in\mathscr{H}_0^{-r}(\mathbb{C})$, $u\in\mathscr{H}^r_{\text{loc}}(D_A)$ and verifies the relation

$$(3.19)\qquad \langle Du-Au, \psi\otimes x\rangle = \langle f, \psi\otimes x\rangle$$

for all $\psi\otimes x\in\mathscr{H}_0^{-r}(\mathbb{C})\otimes D_A$. We also know that

$$(3.20)\qquad \mathscr{H}_0^{-r}(\mathbb{C})\otimes D_A\subset\mathscr{H}_0^{-r}(D_A)\subset\mathscr{H}_0^{-s}(H)(r<s)$$

and the embedding is dense. Consequently, from (3.19), we conclude

$$Du-Au=f$$

where $f\in\mathscr{H}^s_{\text{loc}}(H)$ and $u\in\mathscr{H}^{s-k-l}_{\text{loc}}(D_A)$. The proof is complete.

We immediately have the following:

COROLLARY. *Suppose* Hypothesis I *and* Hypothesis II *are satisfied. Then for* $f\in C^\infty(H)$ *the abstract differential equation* $Lu=f$ *has at least one solution* $u\in C^\infty(D_A)$.

REFERENCES

1. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16** (1963), 121–239.

2. M. A. Malik, *Weak generalized solutions of abstract differential equations*, J. Math. Anal. Appl., **40** (1972), 763–768.

3. ——, *Uniqueness of solutions of abstract differential equations*, Canad. Math. Bull. (to appear).

4. ——, *Weak solutions of abstract differential equations*, Proc. Nat. Acad. Sci. U.S.A., **57** (1967), 883–884.

5. S. Zaidman, *A global existence theorem for some differential equations in Hilbert spaces*, Proc. Nat. Acad. Sci. U.S.A., **51** (1964), 1019–1022.

SIR GEORGE WILLIAMS UNIVERSITY,
MONTREAL, QUEBEC