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NEARLY VARIANTS OF PROPERTIES AND ULTRAPOWERS

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Abstract. We show that if S is the maximal ideal space of certain ultrapowers of $C_0(L)$ spaces then: $C_0(S, \mathbb{C})$ allows polar decompositions while $C_0(S, \mathbb{R})$ does not, which answers a question of Greim and Rajalopagan [**4**]. Also, S is almost homogeneous but not transitive, which answers a question of Wood [**9**].

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0. Introduction. A Banach space X is called *transitive* if the group $G(X)$ of (linear surjective) isometries of X acts transitively on the unit sphere $S(X)$. The space X is called almost transitive if $G(X)$ acts with dense orbits (that is, almost transitively) on $S(X)$.

There are unexpected difficulties in deciding whether certain natural classes of Banach spaces contain a transitive (or an almost transitive) member or not. A recent survey on the topic is [**1**]. Apart from the notorious problem of S. Mazur, the most glaring example is a problem of G. V. Wood concerning the existence of an almost transitive $C_0(S)$ space. As usual, we denote by $C_0(S)$ or $C_0(S, \mathbb{K})$ the Banach space of continuous $\mathbb{K}$-valued functions on the locally compact space S vanishing at infinity, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ (we assume that S is not a singleton). In this paper, 'topological space' means 'completely regular Hausdorff space'.

Wood conjectured in [**9**] that all $C_0(S)$ spaces lack almost transitive norm. Some work has been done: in [**4**] it is proved that real $C_0(S)$ spaces lack almost transitive norm; in the opposite direction, both the real and complex spaces $C[0, 1]$ have equivalent almost transitive renormings [**2**]. Moreover, it can be proved that an almost transitive $C_0(S)$ space exists if and only if a transitive $C_0(S)$ space exists [**4**], which is equivalent to the existence of a separable almost transitive $C_0(S)$ space (this means that the one-point compactification of S is metrizable [**2**]). Nevertheless, the question remains open in the complex case:

Wood's problem. Does every complex-valued $C_0(S)$ lack almost transitive norm?

In [**4**], P. Greim and M. Rajalopagan give necessary conditions on S for the (complex) space $C_0(S)$ to have transitive norm.

Following [**4**], we will say that $C_0(S, \mathbb{K})$ allows polar decompositions if for every $f \in C_0(S, \mathbb{K})$ there is an isometry of $C_0(S, \mathbb{K})$ mapping $|f|$ to f. By the form of the isometries of $C_0(S)$ this means that there is a continuous function $\sigma : S \to \mathbb{K}$, with $|\sigma(s)| = 1$ for all $s \in S$ such that $f(s) = \sigma(s)|f(s)|$ for all $s \in S$.

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The existence of polar decompositions in $C_0(S)$ is related to topological properties of compact subsets of S. Recall from [7] that a topological space X is said to be an F-space if for every continuous $f: X \to \mathbb{R}$ there is a continuous g such that $f = g|f|$. Observe that such a real-valued g exists if and only if a complex-valued g exists. From a more topological point of view, X is an F-space if disjoint cozero sets of X are completely separated, i.e., if given disjoint cozero sets A and B there is a continuous g such that $g(s) = 0$ for all $s \in A$ and $g(s) = 1$ for all $s \in B$.

Greim and Rajalopagan proved in [4] that if $C_0(S, \mathbb{K})$ allows polar decompositions then every compact subspace of S is an F-space (i.e., S is locally an F-space). They posed the following questions.

Question 0.1. ([4], p. 77) Does $C_0(S, \mathbb{R})$ allow polar decompositions if $C_0(S, \mathbb{C})$ does?

Question 0.2. ([4], p. 80) Does $C_0(S, \mathbb{R})$ allow polar decompositions if every compact subset of S is an F-space?

By the results in [4], a positive answer to either of these questions would prove the conjecture of Wood.

The organization of the paper is as follows. In Section 1 we answer in the negative questions 0.1 and 0.2 above by showing that there even exists a compact space K such that $C(K, \mathbb{C})$ allows polar decompositions (hence all closed subspaces of K are F-spaces) while $C(K, \mathbb{R})$ does not allow polar decompositions. The basic idea is to consider suitable "nearly variants" of the properties and to study how it leads to ultraproducts.

Section 2 is a brief discussion of some topological properties of the locally compact spaces obtained as maximal ideal spaces of ultrapowers of $C_0(S)$ spaces.

Finally, Section 3 provides simple (and "new") counter-examples to a question of Wood [9, p. 181]: there are almost homogeneous connected locally compact spaces which are almost homogeneous but not transitive (see 3.1 for precise definitions).

1. Nearly variant of properties and ultraproducts. For the sake of simplicity, we use only $\mathbb{N}$ as "index" set, although the results are also true (with suitable modifications) for every "index" set. We refer the reader to [5] or [6] for general information about ultraproducts.

1.1. *The Banach space ultraproduct*. Let (X_n) be Banach spaces. Consider their Banach space product

$$\prod X_n = \{(x_n) : x_n \in X_n \text{ for all } n \text{ and } \sup_n ||x_n|| \text{ is finite}\},$$

endowed with the sup norm. Let U be a non-trivial ultrafilter on $\mathbb{N}$. Put $N_U = \{(x_n) : \lim_U ||x_n|| = 0\}$. The Banach space $\prod X_n / N_U$ (with the quotient norm) is called the (Banach space) ultraproduct of $\{X_n\}$ with respect to U and is denoted $(X_n)_U$. The class of (x_n) in will be denoted as $(x_n)_U$. The norm of $(X_n)_U$ is given by

$$||(x_n)_U|| = \lim_U ||x_n||.$$

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If all X_n coincide with some Banach space X, the ultraproduct $(X_n)_U$ is called the ultrapower of X with respect to U and it is denoted by $(X)_U$. There is a canonical isometric embedding of X into $(X)_U$ defined by $x \to (x)_U$.

When X_n are Banach algebras (respectively, commutative C^*-algebras), their ultraproduct $(X_n)_U$ is also a Banach algebra (respectively, a commutative C^*-algebra) equipped with the (well-defined) product

$$(x_n)_U(y_n)_U = (x_n y_n)_U.$$

Thus, in view of the Gelfand-Naimark theorem (see **[3]**, p. 11), when S_n are locally compact spaces, the Banach algebra $(C_0(S_n))_U$ is representable as a $C_0(S)$ space for some locally compact space S (which is obviously compact when all S_n are). In fact, S is the maximal ideal space of $(C_0(S_n))_U$, that is, the set of (continuous) multiplicative linear functionals on $(C_0(S_n))_U$ endowed with the relative weak* topology of $((C_0(S_n))_U)^*$.

We now introduce certain variants of the properties under consideration.

Definition 1.2. We will say that $C_0(S, \mathbb{K})$ allows nearly polar decompositions if for every $f \in C_0(S, \mathbb{K})$ and every $\varepsilon > 0$ there is $g \in C_0(S, \mathbb{K})$ such that $\|f - g\| \le \varepsilon$ and $g = \sigma|g|$, being $\sigma : S \to \mathbb{K}$ unimodular.

Lemma 1.3. *Let S be a closed bounded interval of the real line. Then:*
(a) *The space $C_0(S, \mathbb{K})$ does not allow polar decompositions for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.*
(b) *$C_0(S, \mathbb{C})$ allows nearly polar decompositions.*
(c) *$C_0(S, \mathbb{R})$ does not allow nearly polar decompositions.*

Proof. Without loss of generality assume $S = [-1, 1]$. The first part is immediate since the function defined by $f(t) = t\sin(1/t)$ if $t \ne 0$ and $f(0) = 0$ cannot be divided by $|f|$. Parts (b) and (c) follow easily from the obvious fact that the set of continuous $\mathbb{K}$-valued non-vanishing functions on S are dense in $C_0(S, \mathbb{K})$ if and only if $\mathbb{K} = \mathbb{C}$. □

The connection between polar and nearly polar decompositions appears now.

Proposition 1.4. *Let K be a compact space. The space $C(K, \mathbb{K})$ allows nearly polar decompositions if and only if every (or some) non-trivial ultrapower allows polar decompositions.*

Proof. We first prove the "only if" part. Assume that $C(K, \mathbb{K})$ allows nearly polar decompositions. Let $(f_n)_U \in (C(K, \mathbb{K}))_U$. Then there are $g_n \in C(K, \mathbb{K})$ such that $\|f_n - g_n)\| \le 1/n$ and $g_n = \sigma_n|g_n|$, where $\sigma_n : S \to \mathbb{K}$ are unimodular. Hence (see **[6]**),

$$(f_n)_U = (g_n)_U = (\sigma_n|g_n|)_U = (\sigma_n)_U(|g_n|)_U = (\sigma_n)_U(|f_n|)_U = (\sigma_n)_U|(f_n)_U|.$$

Clearly, $(\sigma_n)_U$ is unimodular, which proves that $(C(K, \mathbb{K}))_U$ allows polar decompositions.

For the converse, suppose that $(C(K, \mathbb{K}))_U$ allows polar decompositions. Let $f \in C(K, \mathbb{K})$. Then one has

$$(f)_U = \sigma(|f|)_U,$$

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where σ is unimodular in $(C(K, \mathbb{K}))_U$. It is clear that σ can be written as $\sigma = (\sigma_n)_U$, where σ_n are unimodular in $C(K, \mathbb{K})$. Thus we have

$$\lim_U ||(f - \sigma_n|f|)|| = 0,$$

or, in other words, for each $\varepsilon > 0$, the set $\{n : ||(f - \sigma_n|f|)|| \leq \varepsilon\}$ belongs to U and therefore is non-empty. This completes the proof. □

PROPOSITION 1.5. *If K is the maximal ideal space of an ultrapower of $C(I, \mathbb{K})$, where I is a compact interval, then:*

(a) $C(K, \mathbb{C})$ *allows polar decompositions;*

(b) *every closed subset of K is an F-space;*

(c) *K is connected;*

(d) $C(K, \mathbb{R})$ *does not allow polar decompositions.*

Proof. Parts (a) and (b) follow immediately from the lemma and the proposition. Part (b) is a consequence of the fact that all compact spaces are normal and the Tietze-Urysohn's extension theorem for normal spaces. Part (c) is clear, since the maximal ideal space of an ultrapower of $C_0(S)$ is connected if and only if S is. □

REMARK 1.6. Let K be as in 1.5 and let F be a closed subset of K. Then $K \backslash F$ is a locally compact (not necessarily compact) space so that $C_0(K \backslash F, \mathbb{C})$ allows polar decompositions while $C_0(K \backslash F, \mathbb{R})$ does not allow polar decompositions.

2. Maximal ideal spaces of ultrapowers. This section has a preparatory character. Here we study properties of locally compact spaces obtained as maximal ideal spaces of ultraproducts of $C_0(S)$ spaces. We need some notation. $\mathrm{Sp}(A)$ stands for the maximal ideal space of a commutative C^*-algebra A. By αL we will denote the one-point compactification of the locally compact space L.

The maximal ideal spaces of ultraproducts of $C_0(S)$ spaces are related to the following construction.

2.1. *The set-theoretic ultraproduct*. ([**5**], [**6**]) Let $\{S_n\}$ a countable family of sets. Denote by $\prod S_n$ their direct product. Two families (s_n) and (t_n) are said to be equivalent with respect to the ultrafilter U if $\{n : s_n = t_n\}$ belongs to U. This defines an equivalence relation on $\prod S_n$ whose quotient is called the (set-theoretic) ultraproduct of (S_n) with respect to U and denoted by $(S_n)_U$. The equivalence class of (s_n) in $(S_n)_U$ will be denoted by $(s_n)_U$.

When S_n are topological spaces, the family of (canonically embedded) sets

$$\{(A_n)_U \subset (S_n)_U : A_n \text{ is open in } S_n \text{ for all } n \in \mathbb{N}\}$$

is the base of a topology on $(S_n)_U$. We will assume that $(S_n)_U$ is equipped with this topology.

Fix $(s_n)_U \in (S_n)_U$. The mapping

$$(f_n)_U \in (C_0(S_n))_U \to \lim_U f_n(s_n) \in \mathbb{K}$$

is a multiplicative linear (hence continuous) functional on $(C_0(S_n))_U$ which corresponds to a unique point of $\mathrm{Sp}(C_0(S_n))_U$. In this way, there is a mapping $(S_n)_U \to \mathrm{Sp}(C_0(S_n))_U$ which is injective, continuous and with dense range in $\mathrm{Sp}(C_0(S_n))_U$.

The proofs of the following lemmata are left to the reader.

LEMMA 2.2. *Let S_n be locally compact spaces and let U be an ultrafilter. The space* $\mathrm{Sp}(C(\alpha S_n))_U$ *is (canonically homeomorphic to) the one-point compactification of* $\mathrm{Sp}(C_0(S_n))_U$. □

LEMMA 2.3. *Let S_n be locally compact spaces. If A is a neighborhood of $(s_n)_U$ in the set-theoretic ultraproduct $(S_n)_U$, then the closure of the image of A in* $\mathrm{Sp}(C_0(S_n))_U$ *is a neighborhood of $(s_n)_U$ in* $\mathrm{Sp}(C_0(S_n))_U$. □

Now let us consider the points of $(S_n)_U$ as points in $\mathrm{Sp}(C_0(S_n))_U$. Recall from [**7**, p. 37] that a point of a topological space is called a P-point if every G_δ-set containing the point is a neighborhood of the point. For completely regular spaces (such as maximal ideal spaces) one can replace 'G_δ-set' by 'zero-set'.

PROPOSITION 2.4. *Let S_n be locally compact spaces.*

(a) *All points of $(\alpha S_n)_U$ are P-points in* $\alpha\mathrm{Sp}(C_0(S_n))_U = \mathrm{Sp}(C(\alpha S_n))_U$.

(b) *All points of $(S_n)_U$ are P-points in* $\mathrm{Sp}(C_0(S_n))_U$.

Proof. Clearly, it suffices to prove the first part. Let $(s_n)_U$ be a point in $(\alpha S_n)_U$. Choose $f = (f_n)_U \in (C(\alpha S_n))_U$ so that $f((s_n)_U) = 0$, i.e., $\lim_U f_n(s_n) = 0$. Let (ε_n) be a sequence of positive numbers converging to 0. Put

$$A_n = \{t \in S_n : |f_n(s_n) - f_n(t)| < \varepsilon_n\}.$$

The set $(A_n)_U$ is a neighborhood of $(s_n)_U$ in $(\alpha S_n)_U$. Moreover, given $(t_n)_U \in (A_n)_U$, one has

$$|f((t_n)_U)| = \left|\lim_U f_n(t_n)\right| \leq \lim_U |f_n(t_n)| \leq \lim_U [|f_n(s_n)| + \varepsilon_n] = |f((s_n)_U)| + \lim_U \varepsilon_n = 0.$$

Hence f vanishes on $(A_n)_U$ and also in its closure in $\mathrm{Sp}(C(\alpha S_n))_U$ which is, by lemma 2.3, a neighborhood of $(s_n)_U$. Thus, $(s_n)_U$ is a P-point and the proof is complete. □

The following proposition and corollary should be compared to corollary 4.1, proposition 4.4 and corollary 4.2 of [**4**].

PROPOSITION 2.5. *If $S = \mathrm{Sp}(C_0(S_n))_U$, then every $f \in C_0(S)$ has compact support.*

Proof. The assertion is just that the infinity point of αS is a P-point and follows from proposition 2.4. □

REMARK 2.6. The attentive reader will have noticed that proposition 2.5 can be derived from the fact that all functions in $C_0(S_n)$ have "nearly compact support" reasoning as in the first section.

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From proposition 2.5 and basic general topology (see [**8**], p. 125–126, 17F and 17J) it follows the following.

COROLLARY 2.7. *Let* $S = \mathrm{Sp}(C_0(S_n))_U$.
(a) *Every open* K_σ *is relatively compact.*
(b) *The union of countably many compacts subsets of* S *is relatively compact.*
(c) S *is countably compact (i.e., each sequence has a cluster point in* S*).*
(d) S *is pseudocompact (i.e., every continuous function on* S *is bounded).* □

3. Almost homogeneous spaces. In his study of symmetries in Banch spaces [**9**], Wood introduced several homogeneity conditions on a locally compact (Hausdorff) space S which are related to the "size" of the group of (linear surjective) isometries of the Banach space $C_0(S)$.

DEFINITIONS 3.1. (a) S is called homogeneous if, for every pair of finite subsets F and E with the same number of elements, there is a homeomorphism φ of S such that $\varphi(F) = E$. (b) S is almost homogeneous if, for every finite F and open A, there is a homeomorphism φ of S such that $\varphi(F) \subset A$. (c) S is transitive (respectively, almost transitive) if its group of homeomorphisms acts transitively (respectively, with dense orbits) on S.

Wood raises the question of whether every almost transitive locally compact space is transitive. The answer is negative, since $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$ (the growth of $\mathbb{N}$ in its Stone-Cech compactification) is almost homogeneous (immediate) and non-transitive (delicate: Frolik, see [**7**, p. 92]). The abundance of homeomorphisms in $\mathbb{N}^*$ seems to be related to the existence of many clopen sets. There are other examples whose nature is completely different, as we show now.

The following result is somewhat surprising.

LEMMA 3.2. *Given locally compact spaces* S_n *and a non-trivial ultrafilter* U*, the maximal ideal space of* $(C_0(S_n))_U$ *is not transitive (unless it is finite).*

Proof. Observe that homeomorphisms must preserve P-points. So, proposition 2.4 (b) implies that if $\mathrm{Sp}(C_0(S_n))_U$ is transitive then it contains only P-points. According to [**7**, p. 37], every pseudocompact space in which all points are P-points must be finite. □

As a preparation for the following example, let K be a compact space. The canonical inclusion $j : C(K) \to (C(K))_U$ given by $jf = (f)_U$ is an algebra homomorphism. The adjoint map j^* induces a continuous mapping from $\mathrm{Sp}(C(K))_U$ onto K. This mapping is given by $j^*(s_n)_U = \lim_U s_n$ for points in the set-theoretic ultraproduct $(K)_U$.

LEMMA 3.3. *Let* s *be a point of* $\mathrm{Sp}(C(K))_U$ *and let* $j^* : \mathrm{Sp}(C(K))_U \to K$ *be as above. Let* B *be a neighborhood of* j^*s *in* K*. Then* s *lies in the closure of* $(B)_U$ *in* $\mathrm{Sp}(C(K))_U$. □

EXAMPLE 3.4. *There is a compact connected space which is almost homogeneous but not transitive.*

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Proof. Let K be a compact connected space having the following property:
(*) Given points $(s_i)_{i=1}^k$ of K there exist neighborhoods B_i of s_i such that, for every open subset A of K, there is an homeomorphism φ of K satisfying $\varphi(B_i) \subset A$ for $1 \leq i \leq k$.

For instance, choose K as the unit circle.

Let S be the maximal ideal space of $(C(K))_U$. Then S has the required properties. First of all observe that, by lemma 3.2, S cannot be transitive. That S is compact and connected is clear.

We prove now that S is almost homogeneous. Let A be an open set of S and let $F = \{s_i\}_{i=1}^k$ be a finite subset. Let j^* be as in Lemma 3.3. Choose neighborhoods B_i of $j^* s_i$ $(1 \leq i \leq k)$ so that (*) holds. Observe that, by Lemma 3.3, each s_i belongs to the closure of $(B_i)_U$ in S.

Obviously, A contains the closure of a set of the form $(A_n)_U$, where A_n are open subsets of K. By (*), there is a sequence (φ_n) of homeomorphisms of K such that $\varphi_n(B_i) \subset A_n$ for all $1 \leq i \leq k$ and all n. Define an homeomorphism $(\varphi_n)_U$ on $(K)_U$ by putting

$$(\varphi_n)_U((u_n)_U) = (\varphi_n(u_n))_U.$$

Clearly, $(\varphi_n)_U((B_i)_U) \subset (A_n)_U$ for $1 \leq i \leq k$. This homeomorphism extend to all of S as follows: consider the operator L on $(C(K))_U$ defined by $L(f_n)_U = (f_n \circ \varphi_n^{-1})_U$. It is evident that L is an (algebra) automorphism, hence L^* defines an homeomorphism on the maximal ideal space of $(C(K))_U$ which clearly extends $(\varphi_n)_U$ to the whole S. Plainly,

$$L^*(F) = L^*(\{s_i\}_{i=1}^k) \subset \bigcup_{i=1}^k L^*(\overline{(B_i)_U}^S) \subset \bigcup_{i=1}^k \{\overline{(\varphi_n)_U(B_i)_U}^S\} \subset \overline{(A_n)_U}^S \subset A.$$

This completes the proof. $\square$

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