GRAPHS WITH MAXIMAL EVEN GIRTH

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1. Introduction. In this paper we examine the class G of simple undirected, connected graphs of diameter $d > 1$, girth $2d$, and for any $g \in G$, if a pair of nodes are at distance d from each other, then that pair of nodes is connected by t distinct paths of length d, $t > 1$. (The girth of g is the length of the smallest circuit in g.)

We establish, in § 2, that for all $g \in G$, g is regular.

We establish necessary conditions for the existence of elements of G. If $g \in G$, we adopt the notation $g = g(d, t, v, n)$, where v is the valence of g and n is the number of nodes. It is of course possible for $g, h \in G$, $g \neq h$, and for given d, t, v, n to have both $g(d, t, v, n)$ and $h(d, t, v, n)$.

In particular, we show that if $d = 2$, $t \neq 2, 4$ or 6, then there is at most a finite number of graphs with a particular given t value.

We show that $g(2, 2, 10, 56)$, $g(2, 6, 22, 100)$ (**5**) and $g(3, 2, 4, 35)$ exist and are the only graphs with the stated parameters. We also show that $g(2, 4, 16, 77)$ is a subgraph of $g(2, 6, 22, 100)$. We examine the relations between these graphs and Balanced Incomplete Block Designs (BIBDs). A related problem dealing with graphs of diameter d and girth $2d + 1$ was considered in (**7**). An application of these graphs to the construction of transmission networks was given in (**1**) and the same concept can be easily modified to apply to the graphs considered here.

2. Regularity. We write $d(i, j) = k$ if the distance from node i to node j is k. We write (i, j) if i is adjacent to j, and $(i, \ldots, x, \ldots, j)$ for a path from i to j containing x.

LEMMA 2.1. *Let i be a node of g. Then there is a node j of g such that $d(i, j) = d$ and there are precisely t distinct paths from i to j of length d.*

Proof. If $d(i, j) = d$, there are exactly t paths of length d from i to j, by hypothesis.

Let i be given and let l be a circuit in g of length $2d$. If $i \in l$, take $j \in l$ such that $d(i, j) = d$. If $i \notin l$, then for all $x \in l$, $d(i, x) \leqq d$ by the hypothesis on the diameter of g. For some $x \in l$, let $d(i, x) = k$. If $k = d$, our proof is complete. Otherwise, $k < d$ and suppose that $k + r = d$. Let $z \in l$ be such that $d(x, z) = r$. Then $d(i, z) = d$.

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We write v_i for the valence of the node i.

LEMMA 2.2. *If $d(i, j) = d - 1$, then $v_i = v_j$.*

Proof. Let

$$X = \{x_k\colon (x_k, i) \text{ and } d(x_k, j) = d\},$$
$$Y = \{y_s\colon (y_s, j) \text{ and } d(y_s, i) = d\};$$

clearly,

$$|X| = v_i - 1, \qquad |Y| = v_j - 1.$$

We observe that the stated path (of length $d - 1$) joining i and j is unique; otherwise, there would be a circuit of length less than the girth. We distinguish the nodes of this unique path by $i = i_1, i_2, \ldots, i_{d-1} = j$. For each k, the path $(x_k, i_1, \ldots, i_{d-1})$ has the length d. Thus, by Lemma 2.1, there are $t - 1$ additional paths from x_k to $i_{d-1} = j$. The node i_{d-c}, $c = 1, \ldots, d - 1$, cannot be in such a path since if it were, we would have the circuit $(x_k, \ldots, i_{d-c}, \ldots, i, x_k)$ whose length would be $2(d - c) + 1 \leqq 2(d - 1) + 1 < 2d$, a contradiction. Thus, the $t - 1$ paths must be of the form $(x_k, \ldots, y_s, j)$. Since this holds for all $k = 1, \ldots, v_i - 1$, there are $(t - 1)(v_i - 1)$ such paths. By the same reasoning, there are $(t - 1)(v_j - 1)$ paths of length d of the form $(y_s, \ldots, x_k, i)$. However, each of the above numbers is the number of paths of length $d - 1$ joining nodes in X with nodes in Y. Thus, $(t - 1)(v_i - 1) = (t - 1)(v_j - 1)$, and since $t \geqq 2$, $v_i - 1 = v_j - 1$, and therefore $v_i = v_j$.

LEMMA 2.3. *Let $(i_0, i_1, \ldots, i_{2d-1}, i_0)$ be a circuit of length $2d$. Then*

(a) $$v_{i_j} = v_{i_k}, \qquad j, k = 0, 1, \ldots, 2d - 1 \quad \text{if } d \text{ is even},$$

(b) $$\left.\begin{matrix} v_{i_j} = v_{i_k}, & j, k = 0, 2, \ldots, 2d - 2 \\ v_{i_s} = v_{i_r}, & s, r = 1, 3, \ldots, 2d - 1 \end{matrix}\right\} \quad \text{if } d \text{ is odd}.$$

Proof of (a). If $d = 2$, the result is immediate. For $d > 2$, we have, from Lemma 2.2,

$$v_{i_0} = v_{i_{d-1}} = \ldots = v_{i_{md-m}} = \ldots. \tag{2.1}$$

However, $d - 1$ is relatively prime to $2d$ if d is even. Hence, the numbers $m(d - 1)$, $m = 0, 1, 2, \ldots,$ exhaust the residue classes modulo $2d$. Thus, (a) follows from (2.1).

Proof of (b). If d is odd, then the numbers $m(d - 1)$, $m = 0, 1, 2, \ldots,$ exhaust the even residue classes modulo $2d$ and the numbers $1 + m(d - 1)$ exhaust the odd residue classes modulo $2d$. Thus, (b) also follows from (2.1).

THEOREM 2.1. *Let $g \in G$. Then g is regular.*

Proof. There are two cases to consider.
(a) d is even,
(b) d is odd.

Proof of (a). Let $(i_0, \ldots, i_{2d-1}, i_0) = l$ be a circuit of length $2d$. By Lemma 2.3, each node in l has the same valence, say v. If $l = g$, our proof is

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complete. If not, let $x \in g$, $x \notin l$. As in the proof of Lemma 2.1, x is contained in a circuit of length $2d$ which also contains some node of l. Therefore $v_x = v$.

Proof of (b). We know from Lemma 2.3 that, if l is a circuit of length $2d$, then there are numbers v_1 and v_2 such that every node of l has valence v_1 or v_2; and, if i and j are adjacent nodes of l and $v_i = v_1$, then $v_j = v_2$. We now show that if i and j are any adjacent nodes of g, then their respective valencies are v_1 and v_2 or v_2 and v_1. Let (x_1, x_2), $x_1, x_2 \in g$, and let $l = (i_0, \ldots, i_{2d-1}, i_0)$.

Case 1. $x_1 \notin l$, $x_2 \in l$. Then, as in the proof of Lemma 2.1, x_1 and x_2 are adjacent nodes in a circuit of length $2d$ which includes at least two nodes of l. Hence, $v_{x_1} = v_1$ and $v_{x_2} = v_2$ or $v_{x_1} = v_2$ and $v_{x_2} = v_1$.

Case 2. $x_1, x_2 \notin l$ and $d(x_2, l) = d - j, j = 1, \ldots, d - 1$. If $j = 2, \ldots, d - 1$, then x_1 and x_2 are adjacent nodes in a circuit of length $2d$ containing at least two nodes of l, and the result follows from Case 1. We may now assume, without loss of generality, that $d(x_2, i_0) = d - 1$ and $v_{i_0} = v_2$. Thus, $(x_2, x_3, \ldots, i_0, i_1)$ is a path of length d, and thus part of a circuit l_1 of length $2d$, containing two adjacent nodes of l, and hence $v_{x_2} = v_{i_0} = v_2$ and $v_{x_3} = v_{i_1} = v_1$. Now, since $(x_1, x_2, x_3, \ldots, i_0)$ is a path of length d containing at least two nodes of l_1, we have $v_{x_1} = v_{x_3} = v_1$.

Case 3. $x_1, x_2 \notin l$ and $d(x_1, i_j) = d(x_2, i_j) = d$, $j = 0, \ldots, 2d - 1$. Let $(x_2, x_3, x_4, \ldots, i_0)$ be a path of length d. Then by Case 2, $v_{x_2} = v_{x_4} = v_{i_1} = v_1$. Similarly, let $(x_1, x_5, x_6, \ldots, i_1)$ be a path of length d. Then, again, by Case 2, $v_{x_1} = v_{x_6} = v_{i_0} = v_2$.

We now count the nodes of g in two ways. From the fact that both ways count the same number, we will infer $v_1 = v_2$. We define (see **7**) a hierarchy of g as follows. Pick a node of g which we will call the distinguished node, and identify it by 0. We will say that 0 is on level 0 (tier 0) of the hierarchy (still to be defined). The valence of 0 is v_j, $j = 1, 2$. Then the nodes adjacent to 0 would have valence v_{j+1}, where $j + 1$ is an index modulo 2. These nodes are identified as $1, 2, \ldots, v_j$ and are said to be on level 1 of the hierarchy. Each node on level 1 is connected to $v_{j+1} - 1$ nodes other than 0, and this collection of nodes is said to be level 2 of the hierarchy. Clearly, in order not to violate the girth condition, the nodes on level i cannot be connected to each other unless $i = d$. If the arcs connecting nodes on level d to each other are removed, the residual graph is called a hierarchy of g. Clearly, there are at most two hierarchies, one with distinguished node having valence v_j, the other with distinguished node having valence v_{j+1}. We propose now to show that $v_j = v_{j+1}$, and thus there is but one hierarchy (and of course g will be regular). We display the hierarchy as Figure 2.1.

Let l_i be the set of nodes at a distance i from the distinguished node. If we assume that the distinguished node has valence v_j, $j = 1, 2$, and index $j + 1$ is

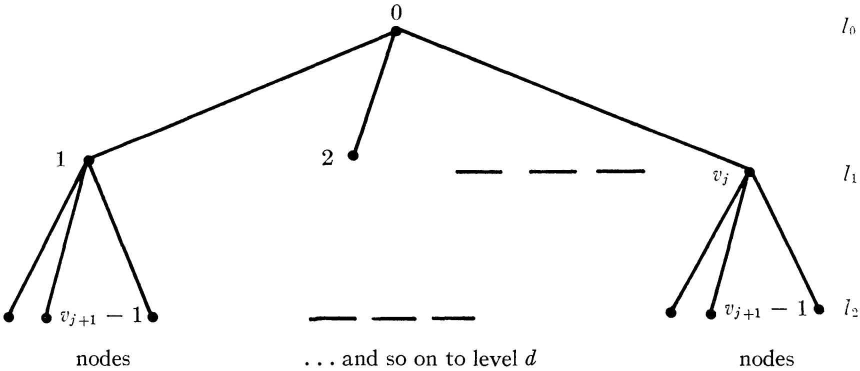

FIGURE 2.1

thought of modulo 2, then from our preceding discussion,

(2.2) $|l_0| = 1,$

(2.3) $|l_i| = v_j(v_j - 1)^{\frac{1}{2}(i-2)}(v_{j+1} - 1)^{\frac{1}{2}i}$ if i is even, $2 \leqq i \leqq d - 1,$

(2.4) $|l_i| = v_j(v_j - 1)^{\frac{1}{2}(i-1)}(v_{j+1} - 1)^{\frac{1}{2}(i-1)}$ if i is odd, $1 \leqq i \leqq d - 2.$

Since every node of l_d is at a distance d from the distinguished node, t of the edges from l_{d-1} must go to each node of l_d, and thus

$$|l_d| = v_j(v_j - 1)^{\frac{1}{2}(d-1)}(v_{j+1} - 1)^{\frac{1}{2}(d-1)}t^{-1}. \tag{2.5}$$

We now show that the sum of the cardinalities implied by (2.2)–(2.4) is the same regardless of whether v_j or v_{j+1} is picked as the valence of the distinguished node. To show this we need only show that $|l_{2h-1}| + |l_{2h}|$, $h = 1, \ldots, \frac{1}{2}(d - 1)$, is the same regardless of whether the distinguished node has valence v_j or v_{j+1}. Let the distinguished node have valence v_j. Then

$$\begin{aligned} |l_{2h-1}| + |l_{2h}| &= v_j(v_j - 1)^{h-1}(v_{j+1} - 1)^{h-1} + v_j(v_j - 1)^{h-1}(v_{j+1} - 1)^h \\ &= v_j(v_j - 1)^{h-1}(v_{j+1} - 1)^{h-1}v_{j+1}. \end{aligned} \tag{2.6}$$

If the distinguished node has valence v_{j+1}, then

$$\begin{aligned} |l_{2h-1}| + |l_{2h}| &= v_{j+1}(v_{j+1} - 1)^{h-1}(v_j - 1)^{h-1} + v_{j+1}(v_{j+1} - 1)^{h-1}(v_j - 1)^h \\ &= v_{j+1}(v_{j+1} - 1)^{h-1}(v_j - 1)^{h-1}v_j. \end{aligned} \tag{2.7}$$

However, (2.6) and (2.7) are the same. Thus, since the left-hand sides of (2.2) and (2.5) add up to the number of nodes of g, we have, from (2.5),

$$v_j(v_j - 1)^{\frac{1}{2}(d-1)}(v_{j+1} - 1)^{\frac{1}{2}(d-1)}t^{-1} = v_{j+1}(v_{j+1} - 1)^{\frac{1}{2}(d-1)}(v_j - 1)^{\frac{1}{2}(d-1)}t^{-1}.$$

Thus,

$$v_j = v_{j+1} = v.$$

COROLLARY 2.1. $|l_0| = 1$, $|l_i| = v(v - 1)^{i-1}$, $1 \leqq i \leqq d - 1$,

$$|l_d| = v(v - 1)^{d-1}t^{-1}.$$

3. Some necessary d, t, v, n **conditions.** Let d and t be as in § 2 and let v be the valence of g and n the number of nodes. We write $g = g(d, t, v, n)$. Clearly, $t \leqq v$. The case $t = v$ was thoroughly investigated by Singleton; see (**12**; **11**; **2**). We define $B \subset G$ to be the class studied by Singleton and R to be the complementary class, $R = G - B$. We use the symbols $g(d, t, v, n)$, $b(d, t, t, n)$, $r(d, t, v, n)$ for elements of G, B, and R, respectively, and $g(d, t, v, n)$ if the class is unspecified. Elements of G are bipartite if they belong to B, that is, using the language of (**7**), there are no re-entering arcs (arcs which connect nodes of l_d to each other) in the graph.

Let $i = 1, \ldots, v$. Then by $l_{d-j}(i)$, $j = 0, \ldots, d - 1$, we mean the set of nodes on level l_{d-j} of the hierarchy which are connected to node i of l_1 via the hierarchy arcs only (see Figure 3.1).

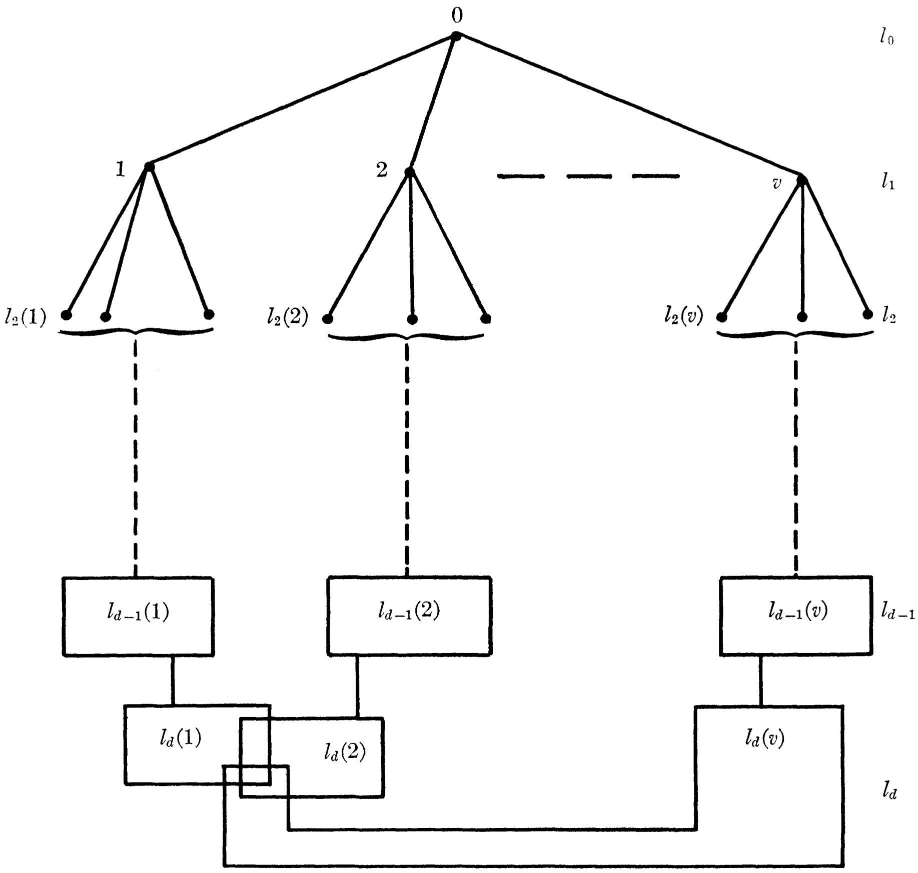

FIGURE 3.1

We list some results which we will use in subsequent sections.

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3.1. *A necessary condition for the existence of* $g(d, t, v, n)$, $v \neq 2$, *is*

$$t(2 - v)n = 2t - [v^2 + v(2 - t)](v - 1)^{d-1}.$$

Proof. $n = \sum_0^d |l_i| = 1 + v \sum_0^{d-2} (v - 1)^i + v(v - 1)^{d-1}t^{-1}$.

3.2. *Let* $m, n \in l_{d-1}(j)$ *and* $x \in l_d(i)$. *Then* (m, x) *and* (n, x) *implies* $m = n$.

3.3. $|l_{d-j}(i)| = (v - 1)^{d-1-j}$, $i = 1, \ldots, v$, $j = 0, \ldots, d - 1$.

3.4. *Let* $i, j = 1, \ldots, v$, $i \neq j$. *Then,*
(a) $|l_d(i) \cap l_d(j)| = (t - 1)(v - 1)^{d-2}$,
(b) $|l_d(i) \cup l_d(j)| = (2v - t - 1)(v - 1)^{d-2}$.

Proof. Let $m \in l_{d-1}(j)$. Then $d(m, i) = d$ (via the distinguished node). Thus, there must exist $t - 1$ other paths of length d, from m to i. Thus, m must be connected to $(t - 1)$ nodes of $l_d(i)$. $|l_{d-1}(j)| = (v - 1)^{d-2}$, and thus by § 3.2, (a) follows. For (b) we have

$$\begin{aligned}|l_d(i) \cup l_d(j)| &= |l_d(i)| + |l_d(j)| - |l_d(i) \cap l_d(j)| \\ &= 2(v - 1)^{d-1} - (t - 1)(v - 1)^{d-2}.\end{aligned}$$

3.5. *Let* $x \in \bigcap_{i \in I} l_d(i)$, $I \subset \{1, \ldots, v\}$, *and let* $y \in l_d$. *Then* (x, y) *implies that* $y \notin l_d(i)$ *for all* $i \in I$.

3.6. $x \in l_d$ *implies that* $x \in \bigcap_{j=1}^t l_d(i_j)$, $i_j \in \{1, \ldots, v\}$.

3.7. *Let* h *be the subgraph of* g *whose nodes are the nodes* $x \in l_d$ *(the arcs are the re-entering arcs). Then the valence of* x, *in* h, *is* $v - t$, *and if we define*

$$\langle x \rangle = \{y \in l_d\colon (x, y)\},$$

then $|\langle x \rangle| = v - t$.

3.8. *A necessary condition for the existence of* $r(d, t, v, n)$ *is* $v > 2t - 1$; see (**12**).

Proof. Let $x \in \bigcap_{j=1}^t l_d(i_j)$ and suppose that $y \in l_d$ is adjacent to x. (We know that y exists since our graph is in R. Observe that this argument is not valid for B graphs.) Then by § 3.5, $y \notin l_d(i_j)$, $j = 1, \ldots, t$. By § 3.6, $y \in l_d(k)$, $k \in \{1, \ldots, v\}$, for t distinct values of k. Thus, $v \geqq \sum j + \sum k = t + t = 2t$.

3.9. *If* t *is a prime, then a necessary condition for the existence of* $r(d, t, v, n)$ *is* $v = mt$ *or* $v = mt + 1$, $m > 1$, *an integer.*

Note. If $m = 1$, then either $v = t$, which is the case for B graphs, or $v = t + 1 \leqq 2t - 1$, which means, from § 3.8, no graph.

Proof. $|l_d| = v(v - 1)^{d-1}t^{-1}$, and when t is prime, this expression is integral only if $t|v$ or $t|v - 1$.

Following the line of argument in (**12**; **7**; **6**), we establish the existence of certain matrix polynomials $P(x)$ such that if $A = (a_{ij})$ is the $n \times n$ adjacency

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matrix of g, then $J = P(A)$, where J is the $n \times n$ matrix of all ones. Since (i, j) implies (j, i), A is symmetric, and since (i, i) does not exist, $\mathrm{tr}(A) = 0$. We also note that if $u' = (1, 1, \ldots, 1)$, then $Au = vu$ so that u is an eigenvector, and v the corresponding eigenvalue, of A. Let $J = uu'$ be the $n \times n$ matrix of all ones. Let I be the identity matrix (we will sometimes write $A^0 = I$).

We see that $A^p = (A^p)_{ij}$, $p = 0, 1, \ldots, d$, has the property that $(A^p)_{ij} = c$ if there are c paths, including paths in which arcs are retraced, of length p, from node i to node j. We observe that if $d(i, j) = d$, then $(A^d)_{ij} = t$. This follows from Lemma 2.1 and the above statement. We thus have the following result.

3.10. *A necessary condition for the existence of $g(2, t, v, n)$ with adjacency matrix A is:*

$$A^2 + tA + (t - v)I = tJ.$$

Proof. If $d = 2$, then for any $g \in G$ we have an adjacency matrix A. A^2 has the following three properties:

(1) If $d(i, j) = 2$, then $(A^2)_{ij} = t$;
(2) If $d(i, j) = 1$, then $(A^2)_{ij} = 0$ (note that in this case, $a_{ij} = 1$);
(3) $(A^2)_{ii} = v$ for all i.

In order to consider higher diameters we define

$$\begin{aligned} F_0(A) &= I, & G_0(A) &= I,\\ F_1(A) &= A, & G_1(A) &= A + I,\\ F_2(A) &= A^2 - vI, & &\\ F_{i+1}(A) &= AF_i(A) - (v-1)F_{i-1}(A), \quad i \geqq 2, & &\\ G_{i+1}(A) &= AG_i(A) - (v-1)G_{i-1}(A), \quad i \geqq 1. & & \end{aligned}$$

Observe the following result.

3.11. $G_i(A) = \sum_{j=0}^{i} F_j(A)$, $i \geqq 0$; see (**12**).

3.12. *$F_k(A) = (f_{ij}^{(k)})$ has the property that $f_{ij}^{(k)}$ is equal to the number of paths of length k from node i to node j.*

Proof. See (**12**).

THEOREM 3.1. *A necessary condition for the existence of $g(d, t, v, n)$ is given by*

$$F_d(A) + tG_{d-1}(A) = tJ.$$

Proof. By Lemma 2.1,

$$f_{ij}^{(d)} = \begin{cases} t & \text{if } d(i, j) = d,\\ 0 & \text{otherwise.} \end{cases}$$

From §§ 3.11, 3.12, and the hypothesis for $d(i, j) < d$ we have:

$$g_{ij}^{(d-1)} = \begin{cases} 1 & \text{if } f_{ij}^{(d)} = 0,\\ 0 & \text{if } f_{ij}^{(d)} = t. \end{cases}$$

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The theorem follows.

Since $Ju = nu$, we see that n is the eigenvalue of J which corresponds to the eigenvalue v of A. The other eigenvalue of J is 0, with multiplicity $n - 1$, and therefore, from Theorem 3.1, the other $n - 1$ roots α, of A, must satisfy

$$F_d(\alpha) + tG_{d-1}(\alpha) = 0;$$

see (**8**).

We have $a^2 = t^2 - 4t + 4v$, and thus $4v = a^2 - (t^2 - 4t)$.

THEOREM 3.2. *Let $d = 2$ and let t be fixed. Then there is a finite number o $r(2, t, v, n)$ graphs, except, possibly, for the cases $t = 2, 4,$ or 6.*

Proof. We have:

$$A^2 + tA + (t - v)I = tJ, \quad v^2 + (t - 1)v + t = tn, \quad \alpha^2 + t\alpha + (t - v) = 0,$$

Therefore $\alpha = [-t \pm (t^2 - 4t + 4v)^{\frac{1}{2}}]2^{-1}$.

LEMMA 3.1. *$(t^2 - 4t + 4v)^{\frac{1}{2}} = a$ is integral.*

Proof. Suppose the contrary. Then $(-t + a)2^{-1}$ and $(-t - a)2^{-1}$ have the same multiplicity x as roots of A. Since $\mathrm{tr}(A) = 0$, we have

$$v + (-t + a)2^{-1}x + (-t - a)2^{-1}x = 0$$

giving $x = vt^{-1}$. Since the total number of roots of A is n, we have $1 + x + x = n$ and by substituting for x we obtain $tn = 2v + t$, which implies that

$$v^2 + (t - 3)v = 0$$

(see Theorem 3.2), which is impossible.

Let x represent the multiplicity of $(-t + a)2^{-1}$ as a root. Therefore, since there are n roots and v has multiplicity 1, we have $(-t - a)2^{-1}$ is a root with multiplicity $n - x - 1$. Since $\mathrm{tr}(A) = 0$,

$$v + (-t + a)2^{-1}x + (-t - a)2^{-1}(n - 1 - x) = 0$$

or $2ax = (n - 1)(t + a) - 2v$ which by substitution yields

$$32tax = a^5 + ta^4 - (2t^2 - 12t + 4)a^3 - (2t^3 - 12t^2 + 12t)a^2 + (t^4 - 12t^3 + 36t^2 - 16t)a + t^2(t - 2)(t - 4)(t - 6).$$

We proved, in Lemma 3.1, that a is integral. The integral solutions a must be the factors of $t^2(t - 2)(t - 4)(t - 6)$, unless $t = 2, 4,$ or 6, in which case this constant term is 0. If $t^2(t - 2)(t - 4)(t - 6) \neq 0$, then there are at most a finite number of factors, and the theorem follows.

4. In this section we consider the existence of elements of G of the form $g(2, 2, v, n)$. Of course, we have $b(2, 2, 2, 4)$, and there are no other B graphs with $d = 2, t = 2$. The existence and uniqueness of $r(2, 2, 5, 16)$ (see Figure 4.1) is shown in (**12**). We will show the existence and uniqueness of $r(2, 2, 10, 56)$ and establish some necessary conditions for other values of v and n.

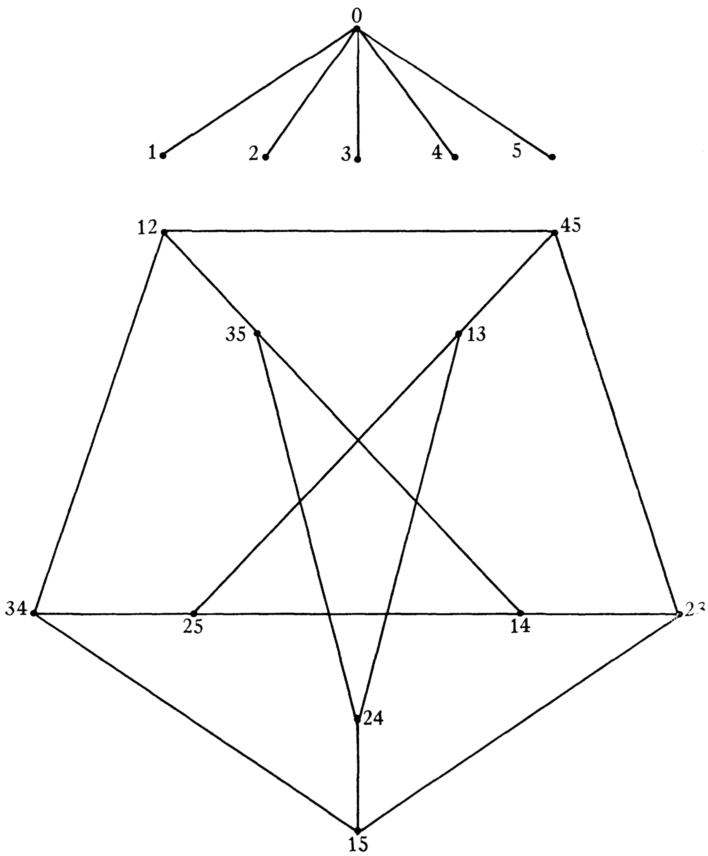

FIGURE 4.1

We have:

$$A^2 + 2A + (2 - v)I = 2J, \tag{4.1}$$

and thus

$$v^2 + v + 2 = 2n. \tag{4.2}$$

Letting $t = 2$ in Theorem 3.1, we see that $(-1 + v)^{\frac{1}{2}} = \bar{a}$ must be integral, and thus

$$v = \bar{a}^2 + 1 \tag{4.3}$$

and

$$4x = \bar{a}^4 + \bar{a}^3 + 3\bar{a}^2 + \bar{a} + 2. \tag{4.4}$$

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Thus, x is integral except when $\bar{a} \equiv 0$ (4). We have previously mentioned

$$b(2, 2, 2, 4), \tag{4.5}$$

the complete bipartite graph on four nodes and

$$r(2, 2, 5, 16). \tag{4.6}$$

THEOREM 4.1. *$r(2, 2, 10, 56)$ exists and is unique.*

Proof. We exhibit the hierarchy as Figure 4.2. We identify the nodes of l_1 as $1, 2, \ldots, 9, T$. From § 3 we see that the identification of the nodes of l_2 is unique. We identify a node in l_2 by the identifications of the two nodes in l_1 which connect to it. This identification is unordered, but in proving the theorem we will often establish order, as a convenience only. The bulk of the following argument will be concerned with the subgraph which consists of the nodes of l_2 and the re-entering arcs. We list the main lemmas, the proofs to appear in (**3**). (We illustrate the style of proof in Lemma 4.6.) Let B be the submatrix of A which is the adjacency matrix of l_2. Let $\langle ij \rangle = \{kl: b_{ij,kl} = 1\}$.

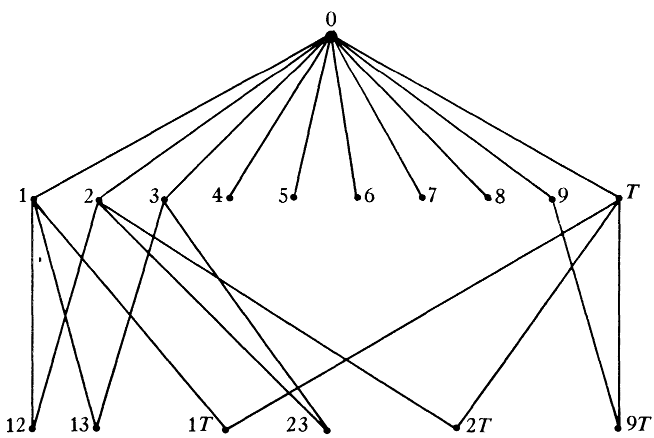

FIGURE 4.2

LEMMA 4.1. *$\langle ij \rangle$ has the following properties:*
(a) *$kl \in \langle ij \rangle$ implies that $k, l \neq i, j$;*
(b) *$kl \in \langle ij \rangle$ implies that there is an $m \neq l$ and an $s \neq k$ such that $km, sl \in \langle ij \rangle$;*
(c) *Suppose that $kl, km \in \langle ij \rangle$, $l \neq m$. Then for all $r \neq l, m$, $kr \notin \langle ij \rangle$;*
(d) *If $x \neq i, j$, then there exist m, n, $m \neq n$, such that $xm, xn \in \langle ij \rangle$.*

LEMMA 4.2. (a) *If $kl \in \langle ij \rangle$, then there is an m such that $kl \in \langle im \rangle$;*
(b) *If $s \neq m, j$, then $kl \notin \langle is \rangle$;*
(c) *Let $xy \in l_2$, then for every $i \neq x, y$ there is a j and a k such that $xy \in \langle ij \rangle$, $xy \in \langle ik \rangle$;*
(d) *If $k \neq j$, then $|\langle ij \rangle \cap \langle ik \rangle| = 1$.*

Let $ij \in \langle kl \rangle$, then with respect to $\langle kl \rangle$ we define

$$\langle \overline{ij} \rangle = \langle ij \rangle - \{kl, kx, ly: kx, ly \in \langle ij \rangle\}.$$

LEMMA 4.3. (a) $|\langle \overline{ij} \rangle| = v - 5$;
(b) *Let* $ij, im \in \langle kl \rangle$. *Then* $\langle \overline{ij} \rangle \cap \langle \overline{im} \rangle = \emptyset$.

LEMMA 4.4. *Let* $\langle xy \rangle$ *be given*, $ij, kl, ms \in \langle xy \rangle$, $i, j \neq k, l$, *then*
(a) $|\langle \overline{ij} \rangle \cap \langle \overline{kl} \rangle| = 1$;
(b) $|(\langle \overline{ij} \rangle \cap \langle \overline{kl} \rangle) \cap \langle \overline{ms} \rangle| = 0$.

LEMMA 4.5. *A necessary condition for the existence of* $r(2, 2, 10, 56)$ *is that* $\langle xy \rangle$ *be one of the following:*
(a) $\{ij, jk, kl, lm, mn, np, pq, iq\}$;
(b) $\{ij, jk, ik, lm, mn, np, pq, lq\}$;
(c) $\{ij, jk, kl, il, mn, np, pq, mq\}$.

There is no loss of generality in assuming that when $\langle ab \rangle = \langle 12 \rangle$ we have $i = 3, j = 4, k = 5, l = 6, m = 7, n = 8, p = 9, q = T$.

LEMMA 4.6. $\langle 12 \rangle \neq \{34, 45, 35, 67, 78, 89, 9T, 6T\}$.

Proof. The elements of l_2 available to fill the sets $\langle \overline{34} \rangle$, $\langle \overline{45} \rangle$, $\langle \overline{35} \rangle$, $\langle \overline{67} \rangle$, $\langle \overline{78} \rangle$, $\langle \overline{89} \rangle$, $\langle \overline{9\mathrm{T}} \rangle$, $\langle \overline{6\mathrm{T}} \rangle$ are 36, 37, 38, 39, $3T$, 46, 47, 48, 49, $4T$, 56, 57, 58, 59, $5T$, 68, 69, 79, $7T$, $8T$. By definition, no elements of the form $1x$ or $2y$ can be in the sets; neither can elements of $\langle 12 \rangle$, otherwise there would be triangles. By Lemma 4.3 we know that $|\langle \overline{34} \rangle| = |\langle \overline{35} \rangle| = 5$. We show that to meet this cardinality condition must cause a contradiction to Lemma 4.1. $\langle \overline{34} \rangle \subset \langle 34 \rangle$. Thus, by Lemma 4.1, no elements of the form $3x$ or $4x$ are in $\langle \overline{34} \rangle$. Furthermore, at most two elements of the form $5x$ are in $\langle \overline{34} \rangle$, and thus at least three of 68, 69, 79, $7T$, $8T$ are in $\langle \overline{34} \rangle$. By Lemma 4.3, this means that at most two of 68, 69, 79, $7T$, $8T$ are in $\langle \overline{35} \rangle$ and since no elements of the form $3x$ or $5x$ are in $\langle \overline{35} \rangle$, at least three elements of the form $4x$ must be contradicting Lemma 4.1.

LEMMA 4.7. $\langle 12 \rangle \neq \{34, 45, 56, 67, 78, 89, 9T, 3T\}$.

LEMMA 4.8. $\langle 12 \rangle = \{34, 45, 56, 36, 78, 89, 9T, 7T\}$.

In the proof of Lemma 4.8 we establish the memberships of the sets $\langle 1x \rangle$, $x = 2, \ldots, T$, and thus we have 260 of the 280 edges (**9**) in the graph. The twenty other edges which are not connected to any node of the form $1x$ or $2y$ are now easily obtained and shown to be unique. We illustrate by means of Figure 4.3 the adjacency matrix of $r(2, 2, 10, 56)$, where the rows and columns of the matrix are in the natural order. C. Sims (private correspondence) has, in the course of his study of primitive groups, independently verified the existence of $r(2, 2, 10, 56)$. Sims' representation is the following. Call the distinguished node $*$. The nodes of l_1 are the ten Sylow 3-subgroups of A_6. The nodes of l_2 are the 45 involutions of A_6. A node of l_2 is connected to a node of l_1 if the node of l_2 normalizes the node of l_1. The re-entering arcs are defined by the following rule. Let $x, y \in l_2$. Then (x, y) if, as involutions, the product xy has order 4. We show in Figure 4.4 the nodes as just defined and the corresponding node from our presentation. One notes that applying the permutation $(12)(79)(8T)$ to the graph would again yield a graph isomorphic to the original. (In Sims' representation, interchange the numbers 5 and 6.)

```
01111111111000000000000000000000000000000000000000000000
10000000000111111111000000000000000000000000000000000000
10000000000100000000111111110000000000000000000000000000
10000000000010000000100000001111111000000000000000000000
10000000000001000000010000001000000111111000000000000000
10000000000000100000001000000100000100000111110000000000
10000000000000010000000100000010000010000100001111000000
10000000000000001000000010000001000001000010001000111000
10000000000000000100000001000000100000100001000100100110
10000000000000000010000000100000010000010000100010010101
10000000000000000001000000010000001000001000010001001011
01100000000000000000000000001010000100000100000000101101
01010000000000000000000011000000000010001011000010000001
01001000000000000000000010010010000000000000101001000100
01000100000000000000000000110000110110000000000100100000
01000010000000000000000001100101000000110000010000001000
01000001000000000000001100001000001000100000100010000010
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01000000010000000000110000000001000001000101000001000010
01000000001000000000100100000000100100010010000100010000
00110000000000000011000000000000000000101000011100100000
00101000000000000110000000001000010000000010000110001000
00100100000000001100000000000000011000100100000000010000001
00100010000000001001000000000100010001001001000000000100
00100001000011000000000000000000010000010100010100000010
00100000100010010000000000000000001101000000100001010000
00100000010000110000000000001000100000001010001000000010
00100000001001100000000000000011000000100010000100010000
00011000000100001000000000100000000000000001010101010000
00010100000001010000010100000000000000000000000000110011
00010010000100000100000000010000000001010010100000000010
00010001000000010010001000010000000000010000001000000100
00010000100001000001001000100000000001000100000000100100
00010000010000100100000110000000000000100100000000100100
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00001100000100000001000001000001010000000000000010100000010
00001010000010100000101000000000000000000000000000011110
00001001000000100010000101000010100000000000010000000001
00001000100000011000100000010001000000000110000000000001
00001000010000010001001010000010001000000001000000100000
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00000010001001000010001001001000100000000100000000100000
00000001100100100000100000000100000000011000000001100000
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00000001001100010000010000000000110010000001100000000000
00000000110101000000000100000001001010000010010000000000
00000000101000001010000010100110000110000000000000000000
00000000011110000000001000000100000001100000011000000000
```

FIGURE 4.3

l_0	$*$	0		(12)(34)	12	(16)(24)	$3T$
	(125)(346)	1		(15)(34)	13	(14)(56)	45
	(126)(345)	2		(12)(36)	14	(12)(56)	46
	(156)(234)	3		(25)(34)	15	(14)(36)	47
	(124)(356)	4		(12)(46)	16	(24)(35)	48
l_1	(134)(256)	5		(15)(36)	17	(14)(35)	49
	(123)(456)	6		(15)(46)	18	(24)(36)	$4T$
	(145)(236)	7		(25)(46)	19	(13)(56)	56
	(135)(246)	8		(25)(36)	$1T$	(14)(26)	57
	(146)(235)	9		(16)(34)	23	(13)(26)	58
	(136)(245)	T		(12)(35)	24	(14)(25)	59
			l_2	(26)(34)	25	(13)(25)	$5T$
				(12)(45)	26	(23)(45)	67
				(26)(45)	27	(13)(46)	68
				(26)(35)	28	(23)(46)	69
				(16)(35)	29	(13)(45)	$6T$
				(16)(45)	$2T$	(15)(26)	78
				(24)(56)	34	(14)(23)	79
				(34)(56)	35	(36)(45)	$7T$
				(23)(56)	36	(35)(46)	89
				(15)(23)	37	(13)(24)	$8T$
				(15)(24)	38	(16)(25)	$9T$
				(16)(23)	39		

FIGURE 4.4

5. In this section we establish some possible parameters for

$$r(2, 3 \leqq t \leqq 10, v, n) \text{ graphs.}$$

Higman and Sims (**5**) have shown the existence of $r(2, 6, 22, 100)$ which has as a subgraph $r(2, 4, 16, 77)$. In the next section we show the uniqueness of $r(2, 6, 22, 100)$. For parameters not listed in this section, $r(2, 3 \leqq t \leqq 10, v, n)$ cannot exist. We note, in particular, that for $t = 3$, 5 or 8, $r(2, 3, 21, 162)$, $r(2, 5, 55, 650)$ and $r(2, 8, 136, 2432)$ are the only open cases. For other values of t there is more than one undecided case. For $t = 3$ we go through the proof and for $4 \leqq t \leqq 10$ we list the results. We also examine $g(d \geqq 3, t, v, n)$ graphs and exhibit the unique graph $r(3, 2, 4, 35)$.

THEOREM 5.1. *$r(2, 3, 21, 162)$ is the only possible $r(2, 3, v, n)$ graph.*

Proof. We have:

$$A^2 + 3A + (3 - v)I = 3J, \tag{5.1}$$

$$v^2 + 2v + 2 = 3n, \tag{5.2}$$

$$a^2 + 3 = 4v, \tag{5.3}$$

$$96ax = a^5 + 3a^4 + 14a^3 + 18a^2 + 33a + 27. \tag{5.4}$$

The values of a for which integral x are possible are the factors of 27, namely 1, 3, 9, 27. From (5.3), if $a = 1$, then $v = 1$ which does not give a graph. If $a = 3$, $(x = 4)$, then $v = 3$ and $n = 6$ which we know is $b(2, 3, 3, 6)$. When $a = 9$ $(x = 105)$, we have $v = 21$ and $n = 162$. If $a = 27$, then $x = \frac{1}{3}(18788)$ which is not integral, and therefore $a = 27$ cannot be used.

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THEOREM 5.2. *For* $4 \leqq t \leqq 10$, *necessary parameters for the existence of* $r(2, 4 \leqq t \leqq 10, v, n)$ *graphs are:*

(a) *For* $t = 4$,

$$v = a^2, \qquad a > 1, \tag{5.5}$$

$$v^2 + 3v + 4 = 4n; \tag{5.6}$$

(b) *For* $t = 6$,

$$v = a^2 - 3, \qquad a \geqq 3, \qquad a \not\equiv 0, 4, 8 \ (12) \tag{5.7}$$

$$v^2 + 5v + 6 = 6n; \tag{5.8}$$

(c) *For* $t = 5, 7, 8, 9, 10$ *we list the possible graphs:*

$$r(2, 5, 55, 650), \tag{5.9}$$

$$r(2, 7, 105, 1666), \tag{5.10}$$

$$r(2, 7, 301, 12202), \tag{5.11}$$

$$r(2, 7, 2646, 1002457), \tag{5.12}$$

$$r(2, 8, 136, 2432), \tag{5.13}$$

$$r(2, 9, 45, 266), \tag{5.14}$$

$$r(2, 9, 99, 1178), \tag{5.15}$$

$$r(2, 9, 171, 3402), \tag{5.16}$$

$$r(2, 9, 495, 27666), \tag{5.17}$$

$$r(2, 9, 981, 107802), \tag{5.18}$$

$$r(2, 9, 2745, 839666), \tag{5.19}$$

$$r(2, 9, 8919, 8846658), \tag{5.20}$$

$$r(2, 9, 24795, 68134018), \tag{5.21}$$

$$r(2, 10, 21, 64), \tag{5.22}$$

$$r(2, 10, 85, 800), \tag{5.23}$$

$$r(2, 10, 385, 15170), \tag{5.24}$$

$$r(2, 10, 885, 74720), \tag{5.25}$$

$$r(2, 10, 3585, 1288450). \tag{5.26}$$

We observe in passing that when $t = 11$ there are ten possible r graphs and when $t = 12$ there are six possibilities.

We have

$$F_3(A) + tG_2(A) = tJ. \tag{5.27}$$

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Therefore

$$A^3 + tA^2 + (t - 2v + 1)A + (t - tv)I = tJ, \tag{5.28}$$

and thus

$$v^3 + (t - 2)v^2 + v + t = tn. \tag{5.29}$$

If $t = 2$, using the techniques of previous sections, we obtain the following result.

LEMMA 5.1. *Necessary conditions for the existence of* $g(3, 2, v, n)$ *are*

$$v^3 + v + 2 = 2n, \tag{5.30}$$

$$8v = a^2 + 7, \qquad a \equiv 1, 3, 5, 7 \ (8). \tag{5.31}$$

THEOREM 5.3. $r(3, 2, 4, 35)$ *exists and is unique. We define the hierarchy in the following manner:*

The distinguished node is 0;

The nodes of l_1 *are named* 1, 2, 3, 4;

The nodes of $l_2(i)$ *are named* $i1, i2, i3, i = 1, \ldots, 4$;

The nodes of l_3 *are named as follows. We observe that each element of* l_3 *is connected to two elements of* l_2. *If the two elements of* l_2 *are* $ia, jb, i, j = 1, \ldots, 4, a, b = 1, \ldots, 3$, *then the element of* l_3 *is given by* $kia, mjb, k, m = 1, \ldots, 3$. *We display the hierarchy, and the re-entering arc subgraph (note that this subgraph is bipartite and consists of three disjoint circuits of length the girth) in Figure* 5.1.

That the adjacencies between l_2 and l_3 are correct is given by the results of § 3. That they are unique is clear, if we note that, given the adjacencies of the nodes $11, 21 \in l_2$ in l_3, if 22 was adjacent to any node of l_3 of the form $112abc$ or $113xyz$, then there would be three paths of length 3 from 2 to 11. That the adjacencies using the re-entering arcs are the bipartite ones shown in Figure 5.1 is the result of § 3.

From (5.29), if $t = 3$ we have that a necessary condition for the existence of $g(3, 3, v, n)$ is

$$v^3 + v^2 + v + 3 = 3n \tag{5.32}$$

which implies that $v \equiv 0, 1 \ (3)$. From § 3.8, we deduce that

$$r(3, 3, 4, 29) \notin R \tag{5.33}$$

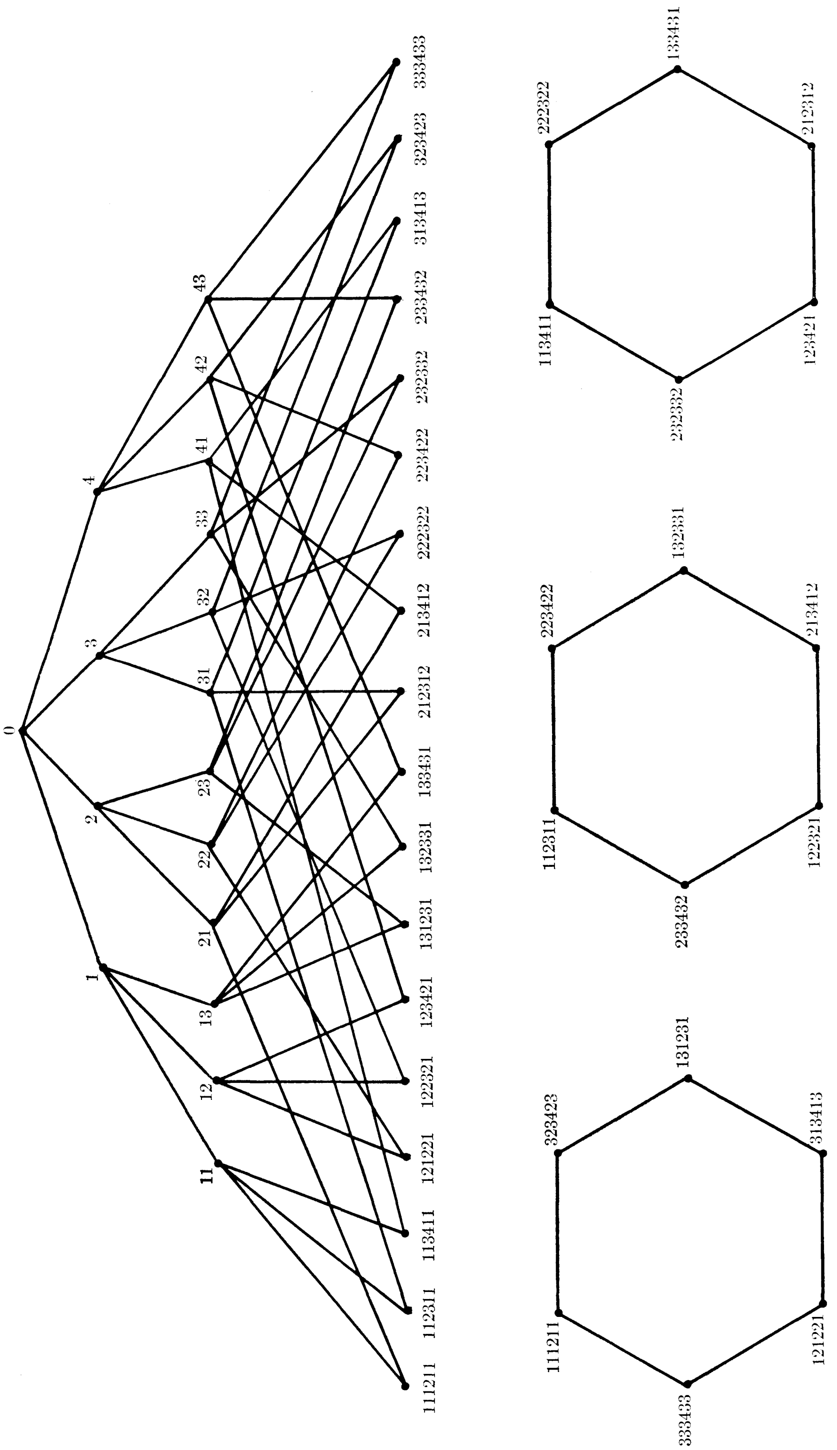
(even though the parameters satisfy (5.32)).

For $d > 3$, we know (**12**; **2**) that B graphs exist for $d = 4$ and $d = 6$ and for no other values of d, provided that $t > 2$. Of course, for $t = 2$, we know that $b(d, 2, 2, 2d)$ always exists. For $d = 4$ we have, by § 3.1, that a necessary condition for the existence of $r(4, t, v, n)$ is that

$$v^4 + (t - 3)v^3 + (3 - t)v^2 + (t - 1)v + t = tn. \tag{5.34}$$

Thus, if $t = 2$, a necessary condition for the existence of $r(4, 2, v, n)$ is that

$$v^4 - v^3 + v^2 + v + 2 = 2n. \tag{5.35}$$


0
1
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112311
113411
121221
122321
123421
131231
132331
133431
212312
213412
222322
223422
232332
233432
313413
323423
333433
222322
133431
212312
113411
232332
123421
223422
132331
213412
112311
233432
122321
323423
131231
313413
111211
333433
121221

FIGURE 5.1

However, by (3.8) we have

$$r(4, 2, 3, 34) \notin R. \tag{5.36}$$

Similarly, a necessary condition for $r(4, 3, v, n)$ to exist is that

$$v^4 + 2v + 3 = 3n, \tag{5.37}$$

and thus $v \equiv 0, 1\ (3)$.

Similar necessary conditions can be written for any d and t. Further, study of the polynomials of the graphs is indicated as being a way to impose stiffer necessary conditions.

6. BIBDs. In this section we examine the relationship that exists between the nodes of l_d and BIBDs. A BIBD can be thought of as a collection of b sets (blocks) with k elements (varieties) in each set, the varieties to be picked from a set with v elements, each variety to appear in exactly r blocks, and each pair of varieties to appear together in exactly λ blocks. v, b, k, r, λ are called the parameters of the BIBD. It is well known, (**10**), that the parameters of a BIBD satisfy

$$vr = bk, \tag{6.1}$$

$$r(k - 1) = \lambda(v - 1). \tag{6.2}$$

Hanani (**3**) proved that (6.1) and (6.2) are sufficient for $k = 3$ or 4 and any λ, and also for $k = 5$ and $\lambda = 4$. We view $x \in l_d$ as a block of a BIBD, whose varieties are the t nodes i_j of § 3.6. By Corollary 2.1, and the results of § 3 we have the following lemma.

LEMMA 6.1. *The nodes $x \in l_d$ are the blocks of a BIBD with parameters b, k, v, r and λ, where*

(1) $v = v$, *the valence of* g,
(2) $b = |l_d| = v(v-1)^{d-1}t^{-1}$,
(3) $k = t$,
(4) $r = |l_d(i)| = (v-1)^{d-1}$,
(5) $\lambda = (t-1)(v-1)^{d-2}$.

COROLLARY 6.1. *If $d = 2$, the nodes of l_2 are the blocks of a BIBD with parameters given by*

$$v,\ r = v - 1, \quad k = t, \quad \lambda = k - 1, \quad b = v(v-1)t^{-1}. \tag{6.3}$$

If the nodes of some given l_d give rise to a BIBD, then the BIBD will be called an associated design of l_d. Many associated designs of l_d can exist, for a given l_d. If a design is an associated design of l_d, we will write BIBD(l_d).

We now proceed to show that the existence of a BIBD(l_d) and the suggestion of possible parameters by the eigenvalue argument are not sufficient for the existence of r graphs.

THEOREM 6.1. $r(2, 4, 9, 28) \notin R$; see (5.5) and (5.6).

Proof. The BIBD parameters are $v = 9$, $b = 18$, $k = 4$, $r = 8$, and $\lambda = 3$.

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Let $(abcd)$ be an arbitrary node of l_2. We have:

$$|l_2| = 18, \tag{6.4}$$

$$|l_2(a) \cup l_2(b)| = 13, \tag{6.5}$$

$$|l_2(c)| = 8, \tag{6.6}$$

$$|l_2(c) \cap l_2(i)| = 3 \quad \text{for all } i \neq c, \tag{6.7}$$

and in particular, (6.7) holds if $i = a$ or b. One of the nodes in the intersection $l_2(c) \cap l_2(a)$ is $(abcd)$, and thus there are exactly two other nodes in this intersection, say α and β. Similarly, $(abcd) \in l_2(c) \cap l_2(b)$, and thus there are exactly two other nodes in this intersection, (possibly α and β), call them δ and γ. In any case, there are at most five nodes in $l_2(c)$ that have the letters a or b in their identification namely $(abcd), \alpha, \beta, \delta$, and γ. Thus, there are at least, by (6.6), three nodes of $l_2(c)$ which do not have a or b in their identification. This fact, together with (6.5), yields

$$|[l_2(a) \cup l_2(b)] \cup l_2(c)| \geqq 16. \tag{6.8}$$

Therefore, from (6.4) the number of nodes in l_2 of the form $(efgh)$, where $e, f, g, h \neq a, b, c$ is 0, 1 or 2. We have:

$$|\langle abcd \rangle| = 5. \tag{6.9}$$

$(efgh) \in \langle abcd \rangle$ implies that $e, f, g, h \neq a, b, c, d$, and since there are at most two such nodes, (6.9) cannot be satisfied.

COROLLARY 6.2. $r(2, 10, 21, 64) \notin R$.

Given the parameters $v = 16$, $r = 15$, $k = 4$, $\lambda = 3$, and $b = 60$, we know that a BIBD exists. In fact, in (**3**), Hanani gives a construction technique. Using this method, one obtains a design with the following property. Distinguish a variety 1 and let the other varieties be $b, c, d, e, \ldots, p$. Then the blocks $(1bcd)$, $(1efg)$, $(1hij)$, $(1klm)$, $(1nop)$ are each repeated three times. Using such a design, it is trivial to show that $r(2, 4, 16, 77) \notin R$ even though the eigenvalue argument ((5.5) and (5.6)) suggests their use as parameters. In fact, we have the following result.

THEOREM 6.2. *If any two blocks have three varieties the same (i.e., if* $(abcd)(abce) \in l_2$*), then* $r(2, 4, 16, 77) \notin R$.

Proof. As in Theorem 6.1, we can construct a design but it will not affect the proof.

Using the techniques of Theorem 6.1, we have

$$|\langle abci \rangle| = 12 \quad (\text{in particular, for } i = d, e), \tag{6.10}$$

$$|l_2(a) \cup l_2(b)| = 27, \tag{6.11}$$

$$|l_2| = 60, \tag{6.12}$$

$$|l_2(i)| = 15, \tag{6.13}$$

$$|[(l_2(a) \cup l_2(b)) \cup l_2(c)] \cup l_2(i)| \geqq 46 \quad \text{for } i = d \text{ or } e. \tag{6.14}$$

From (6.14) and (6.12), we have that the number of nodes that can belong to

$\langle abcd \rangle$ is no more than 14. Suppose that the twelve nodes required for $\langle abcd \rangle$ by (6.10) have been selected from the fourteen available and now let us examine the set $\langle abce \rangle$. Again, from (6.14) and (6.12), there are at most fourteen nodes that can belong to $\langle abce \rangle$.

The number of paths, of length 2, from $(abcd)$ to $(abce)$, via the hierarchy nodes, is three (via a, b, and c). Thus, there must be another path of length 2 between these nodes, and thus there is at most one node of l_2 adjacent to $(abcd)$ and $(abce)$. (None, if $d = e$.) This node, say α, is certainly in $\langle abcd \rangle$. There are, in $\langle abcd \rangle$, exactly four nodes of the form $(exyz)$. The other seven (or eight) nodes of $\langle abcd \rangle$ do not have e in their identification, and thus were counted in the fourteen nodes that were potentially members of $\langle abce \rangle$. However, since they are nodes in $\langle abcd \rangle$ and we already have α as the only node in both $\langle abcd \rangle$ and $\langle abce \rangle$, we see that there are only seven (or six) other possible nodes for $\langle abce \rangle$, which implies that $|\langle abce \rangle| \leqq 8$, contradicting (6.10).

If $v = 25$, $r = 24$, $k = 4$, $\lambda = 3$, and $b = 150$, Hanani's construction criteria again forces the distinguishing of a variety 1 and the repeating of each block containing this 1, three times, as above. That is, if the other varieties are $a, b, c, d, \ldots, x$, then the blocks which are repeated three times are $(1abc)$, $(1def)$, $(1ghi)$, $(1jkl)$, $(1mno)$, $(1pqr)$, $(1vwx)$. We will now designate any BIBD with sets of repeated blocks as shown, as a design of Hanani-type. We then have the following result.

THEOREM 6.3. *A necessary condition for the possible existence of* $r(2, 4, 25, 176)$ *is that the nodes of* l_2 *not be blocks of a Hanani-type design.*

Proof. Once again note that (7.5) and (7.6) suggest the possible existence of this graph.

To prove the theorem one need only observe that the three distinct nodes each labeled $(1abc)$ can have no common adjacencies (other than in the hierarchy) since they are connected to each other by four paths of length 2 via the hierarchy nodes 1, a, b, and c. Thus, for instance, the three distinct sets $\langle 1234 \rangle$, $\langle 1234 \rangle$, $\langle 1234 \rangle$ contain 63 distinct nodes. There are 87 nodes with 1, 2, 3, or 4 in their identifications, and thus there is no choice in picking the 63 nodes. Similarly, if we examine $\langle 1567 \rangle$, $\langle 1567 \rangle$, $\langle 1567 \rangle$ we see that there is no choice; however, using the methods of Theorems 6.1 and 6.2 we see that some of the 63 possible nodes are in a $\langle 1234 \rangle$ set and cannot be used for a $\langle 1567 \rangle$ set.

One might note that the full strength of the hypothesis was not used. The hypothesis could have been weakened to include all BIBDs which have $(1abc)$ and $(1def)$ repeated three times, each as blocks.

THEOREM 6.4. $r(2, 6, 22, 100)$ *is unique.*

Proof. In (**5**), the existence of such an r is shown. In (**13**), Witt proved the uniqueness of the BIBD(l_2) which has no three varieties appearing together in any two distinct blocks. By Corollary 6.2, $r(2, 6, 22, 100)$ is thus unique.

THEOREM 6.5. *There exists an* $r(2, 4, 16, 77)$ *which is a subgraph of* $r(2, 6, 22, 100)$.

Proof. The nodes of $r(2, 4, 16, 77)$ can be taken to be the nodes of l_2 of $r(2, 6, 22, 100)$.

Two nodes are connected if and only if they have no varieties in common (each variety is disjoint from sixteen others).

COROLLARY 6.3. *There exists a block design with parameters* $v = 16$, $b = 60$, $k = 4$, $r = 15$, $\lambda = 3$ *such that no two blocks have three varieties in common.*

Proof. See Theorem 6.2.

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