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THE HARDY CLASS OF FUNCTIONS OF BOUNDED ARGUMENT ROTATION

SANFORD S. MILLER[1] and PETRU T. MOCANU

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Abstract

The Hardy classes for functions of bounded argument rotation and their derivatives are determined. In addition, if $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$, then growth conditions for a_n are obtained.

DEFINITION 1. Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc D, with $f(z)/z \neq 0$ in D, if $k \geqq 2$, and

$$\int_0^{2\pi} \left| \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \right| d\theta \leqq k\pi$$

for $z = re^{i\theta} \in D$, then $f(z)$ is said to be of bounded argument rotation. We denote the class of these functions by U_k. These functions were first studied by Tammi (1952). Note that $U_2 = S^*$, the class of starlike functions, but the classes U_k, for $k > 2$, are not univalent classes.

Closely related to the class U_k is the class V_k of functions of bounded boundary rotation.

DEFINITION 2. Let $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ be analytic in D with $f'(z) \neq 0$ in D. If $k \geqq 2$ and

$$\int_0^{2\pi} \left| \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| d\theta \leqq k\pi$$

for $z = re^{i\theta} \in D$ then $f(z)$ is said to be of bounded boundary rotation and is said to belong to the class V_k.

[1] This work was carried out while the first author was a U.S.A. - Romania Exchange Scholar.

These functions were first considered by Löwner (1917). V_2 is precisely the class K of normalized univalent functions that map D onto a convex domain. Paatero (1931) has shown that for $2 \leqq k \leqq 4$, V_k, consists only of univalent functions. This result is not true for V_k, if $k > 4$. Paatero (1931) has also shown that $f(z) \in V_k$ if and only if

(1) $$f'(z) = \exp\left\{\int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t)\right\}.$$

where $\mu(t)$ is of bounded variation and satisfies

$$\int_0^{2\pi} d\mu(t) = 2, \quad \int_0^{2\pi} |d\mu(t)| \leqq k.$$

There are many connections between the classes U_k and V_k of which we require the following results:

(2) $$g(z) \in U_k, \quad \text{iff}, \quad f(z) = \int_0^z \frac{g(\zeta)}{\zeta} d\zeta \in V_k, \quad \text{and}$$

(3) $$V_k \subsetneq U_k.$$

The first result is a simple calculation and the second result was obtained by Biernacki (1947–48).

We can write the measure $\mu(t)$ in (1) as the difference of two nondecreasing functions $\mu(t) = \nu(t) - \sigma(t)$. By using this together with (2) and the Herglotz representation for starlike functions we see that $g(z) \in U_k$ if and only if

(4) $$g(z) = [s_1(z)]^{(2+k)/4}[s_2(z)]^{(2-k)/4}$$

where $s_1(z)$, $s_2(z) \in S^*$ (see Brannan (1968–69)).

For $\lambda > 0$, we say that a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in D belongs to the Hardy class H^λ if

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} f(re^{i\theta})|^\lambda \, d\theta$$

exists (and is finite). Pinchuk (1973) has investigated the Hardy classes to which $f(z)$ and $f'(z)$ belong for $f(z) \in V_k$. In this paper we carry out a similar investigation for the encompassing class U_k, and deduce a growth condition for a_n.

In what follows, we denote by $h_\gamma(z)$ any function of the form $z(1 - e^{-i\tau}z)^{-2}$, where τ is a real constant.

We require the following lemmas.

LEMMA 1. *If $P(z)$ is analytic and $\operatorname{Re} P(z) > 0$ in D, then $P(z) \in H^\lambda$ for all $\lambda < 1$.*

LEMMA 2. *If* $f(z) \in H^\lambda$ $(0 < \lambda < 1)$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *then* $a_n = o(n^{1/\lambda - 1})$.

LEMMA 3. (i) *If* $f(z) \in V_k$ *then* $f'(z) \in H^\lambda$ *for all* $\lambda < 2/(k+2)$; (ii) *If* $f(z) \in V_k$ *and*

$$f(z) \neq \int_0^z [h_\tau(\tau)]^{(2+k)/4} s(\zeta)^{(2-k)/4} \zeta^{-1} \, d\zeta$$

where $s(z) \in S^*$, *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f'(z) \in H^{2/(k+2)+\varepsilon}$.

LEMMA 4. *If* $f(z) \in S^*$ *then* $f(z) \in H^\lambda$ *for all* $\lambda < \frac{1}{2}$ *and* $f'(z) \in H^\lambda$ *for all* $\lambda < \frac{1}{3}$.

LEMMA 5. *If* $f(z) \in S^*$ *and* $f(z) \neq h_\tau(z)$ *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f(z) \in H^{1/2+\varepsilon}$ *and* $f'(z) \in H^{1/3+\varepsilon}$.

Lemma 1 is well known, Lemma 2 is in Hardy and Littlewood (1932; Theorem 28), Lemma 3 is in Pinchuk (1973), and Lemmas 4 and 5 are in Eenigenburg and Keogh (1970).

By combining (2) with Lemma 3 we immediately obtain our first result.

THEOREM 1. (i) *If* $g(z) \in U_k$ *then* $g(z) \in H^\lambda$ *for all* $\lambda < 2/(k+2)$; (ii) *If* $g(z) \in U_k$ *and*

$$g(z) \neq [h_\tau(z)]^{(2+k)/4} [s(z)]^{(2-k)/4}$$

where $s(z) \in S^*$, *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $g(z) \in H^{2/(k+2)+\varepsilon}$.

In general, if an analytic function belongs to some Hardy class its derivative need not belong to any Hardy class. This is true even if the function is also univalent (Lohwater, Piranian and Rudin (1955)). We next show that this is not true for functions of bounded argument rotation.

THEOREM 2. (i) *If* $g(z) \in U_k$ *then* $g'(z) \in H^\lambda$ *for all* $\lambda < 2/(4+k)$; (ii) *If* $g(z) \in U_k$ *and*

$$g(z) \neq [h_\tau(z)]^{(2+k)/4} [s(z)]^{(2-k)/4}$$

where $s(z) \in S^*$, *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $g'(z) \in H^{2/(4+k)+\varepsilon}$.

PROOF. If $k = 2$, then $U_2 = S^*$ and the result follows from Lemmas 4 and 5. We assume $k > 2$.

(i) From (4) we see that $g(z) \in U_k$ if and only if

$$g(z) = [s_1(z)]^{(2+k)/4} [s_2(z)]^{(2-k)/4}$$

where $s_1(z)$, $s_2(z) \in S^*$. By differentiating we obtain

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$$g'(z)=\tfrac{1}{4}[s_2(z)]^{(2-k)/4}\left\{(2+k)[s_1(z)]^{(k-2)/4}s_1'(z)\right.$$
$$\left.+\frac{(2-k)}{z}[s_1(z)]^{(2+k)/4}\left[\frac{zs_2'(z)}{s_2(z)}\right]\right\}.$$

Since $s_2(z)\in S^*$ we use the distortion theorem $|s_2(z)|>|z|(1+|z|)^{-2}$ to obtain

$$\int_0^{2\pi}|g'(z)|^\lambda\,d\theta$$
$$\leqq\left(\frac{1}{4}\right)^\lambda\left[\frac{r}{(1+r)^2}\right]^{(2-k)\lambda/4}\left\{(2+k)^\lambda\int^{2\pi}|[s_1(s)]^{(k-2)/4}s_1'(s)|^\lambda\,d\theta\right.$$
$$\left.+\left(\frac{k-2}{r}\right)^\lambda\int_0^{2\pi}\left|[s_1(z)]^{(2-k)/4}\left[\frac{zs_2'(z)}{s_1(z)}\right]\right|^\lambda d\theta\right\},$$

where $z=re^{i\theta}$ and $0\leqq\lambda\leqq 1$. If $0<r_0\leqq r<1$, then we have

$$\int_0^{2\pi}|g'(z)|^\lambda\,d\theta\leqq c_1\int_0^{2\pi}|[s_1(z)]^{(k-2)/4}s_1'(z)|^\lambda\,d\theta$$
$$+c_2\int_0^{2\pi}\left|[s_1(z)]^{(2+k)/4}\left[\frac{zs_2'(z)}{s_2(z)}\right]\right|^\lambda d\theta\equiv c_1I_1(r)+c_2I_2(r)$$

where c_1 and c_2 are constants.

To investigate $I_1(r)$, we apply Hölder's inequality and obtain

$$I_1(r)=\int_0^{2\pi}|s_1(z)|^{\lambda(k-2)/4}|s_1'(z)|^\lambda\,d\theta$$
$$\leqq\left[\int_0^{2\pi}|s_1(z)|^{\lambda p(k-2)/4}\,d\theta\right]^{1/p}\left[\int_0^{2\pi}|s_1'(z)|^{\lambda q}\,d\theta\right]^{1/q}\equiv J_1(r)\cdot J_2(r)$$

where $1/p+1/q=1$ and $p>1$. Since $s_1(z)$ is starlike, we can use Lemma 4 and deduce that $\lim_{r\to 1}J_1(r)$ exists if

(5) $$\frac{\lambda p(k-2)}{4}<\frac{1}{2},$$

and $\lim_{r\to 1^-}J_2(r)$ exists if

(6) $$\lambda q<\frac{1}{3}.$$

The inequalities (5) and (6) will be satisfied for $\lambda<2/(4+k)$. $p=(4+k)/(k-2)$ and $q=(4+k)/6$. Thus $\lim_{r\to 1}I_1(r)$ exists provided that

(7) $$\lambda<\frac{2}{4+k}.$$

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Since $s_2(z) \in S^*$, if we let $P(z) = zs_2'(z)/s_2(z)$, then $\operatorname{Re} P(z) > 0$ and $I_2(r)$ can be written as

$$I_2(r) = \int_0^{2\pi} |s_1(z)|^{\lambda(2+k)/4} |P(z)|^\lambda \, d\theta$$

$$\leqq \left[\int_0^{2\pi} |s_1(z)|^{\lambda p(2+k)/4} \, d\theta\right]^{1/p} \left[\int_0^{2\pi} |P(z)|^{\lambda q} \, d\theta\right]^{1/q} \equiv L_1(r) L_2(r)$$

where $1/p + 1/q = 1$ and $p > 1$. Since $s_1(z) \in S^*$ we can use Lemma 4 and deduce that $\lim_{r \to 1^-} L_1(r)$ exists if

(8) $$\frac{\lambda p(2+k)}{4} < \frac{1}{2}.$$

Since $\operatorname{Re} P(z) > 0$ we have by Lemma 1 that $\lim_{r \to 1^-} L_2(r)$ exists if

(9) $$\lambda q < 1.$$

The inequalities (8) and (9) will be satisfied provided that $\lambda < 2/(4+k)$, $p = (4+k)/(2+k)$ and $q = (4+k)/2$. Thus $\lim_{r \to 1^-} I_2(r)$ exists if

(10) $$\lambda < \frac{2}{4+k}$$

Combining the results for $I_1(r)$ and $I_2(r)$ we have shown that $\lim_{r \to 1^-} \int_0^{2\pi} |g'(z)|^\lambda \, d\theta$ exists provided that $\lambda < 2/(4+k)$.

(ii) If $g(z)$ is not of the form $[h_\tau(z)]^{(2+k)/4}[s(z)]^{(2-4)/4}$ then in the above proof $s_1(z) \neq h_\tau(z)$. Hence we can use Lemma 5 instead of Lemma 4 and conditions (5) and (6) will be replaced by

(5′) $$\frac{\lambda p(k-2)}{4} \leqq \frac{1}{2} + \varepsilon,$$

(6′) $$\lambda q \leqq \frac{1}{3} + \varepsilon,$$

which results in (7) being replaced by

(7′) $$\lambda \leqq \frac{2}{4+k} + \varepsilon,$$

for not necessarily the same ε as in (5′).

Similarly (8) can be replaced by a stronger condition which results in (10) being replaced by

(10′) $$\lambda \leqq \frac{2}{4+k} + \varepsilon.$$

By using (7′) and (10′) we have $g \in H^{2/(4+k)+\varepsilon}$.

That the results in Theorems 1 and 2 are best possible may easily be seen by choosing $s_1(z) = h_0(z)$ and $s_2(z) = z$, i.e. $g(z) = z/(1-z)^{(2+k)/2}$.

Our next result shows that in the class V_k even the second derivative belongs to some Hardy class.

THEOREM 3. (i) *If* $f(z) \in V_k$ *then* $f''(z) \in H^\lambda$ *for all* $\lambda < 2/(k+4)$. (ii) *If* $f(z) \in V_k$ *and*

$$f(z) \neq \int_0^z [h_\tau(\zeta)]^{\frac{2+k}{4}} [s(\zeta)]^{\frac{2-k}{4}} d\zeta$$

where $s(z) \in S^*$, *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f''(z) \in H^{2/(k+4)+\varepsilon}$.

PROOF. From (2) we obtain $f'(z) = g(z)z^{-1}$ and $f''(z) = g'(z)z^{-1} - g(z)z^{-2}$, where $g(z) \in U_k$. Then for $0 \leqq \lambda < 1$ and $z = re^{i\theta}$ we have

$$r^2 \int_0^{2\pi} |f''(z)|^\lambda d\theta \leqq r \int_0^{2\pi} |g'(z)|^\lambda d\theta + \int_0^{2\pi} |g(z)|^\lambda d\theta$$

and our result follows from Theorem 2.

Our final result concerns a growth condition on the Taylor coefficients of $f(z)$. Combining Theorem 1 with Lemma 2 we obtain the following theorem.

THEOREM 4. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in U_k$ *then*:

(i) $a_n = o(n^{1/\lambda - 1})$ for $\lambda < 2/(k+2)$

(ii) *If* $f(z) \neq [h_\tau(z)]^{\frac{2+k}{4}} [s_1(z)]^{\frac{2-k}{4}}$ *then* $a_n = o(n^{k/2})$.

Note that for $k = 2$ this coincides with a result of Eenigenburg and Keogh (1970). An alternate proof of this theorem has been given by Noonan (1972).

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State University of New York
Brockport, N. Y. 14420
U.S.A.

The Babes-Bolyai University
Cluj
Romania.