ON AN ARITHEMTICAL INEQUALITY

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Here we extend an arithmetical inequality about multiplicative functions obtained by K. Alladi, P. Erdős and J. D. Vaaler, to include also the case of submultiplicative functions. Also an alternative proof of an extension of a result used for this purpose is given.

1. Introduction. Let U_k, for integral k, denote the set $\{1,2,\ldots,k\}$, and V_k denote the collection of all subsets of U_k. In the following, all unspecified sets like $A,\ldots$, are assumed to be subsets of U_k. Let $\sigma=\{S_i\}$ and $\tau=\{T_j\}$ be two given collections of subsets of U_k. Set

$$A_{\sigma,\tau}=\{A:A\supseteq S_i\cup T_j \text{ for some } i,j\},$$

$$B_{\sigma,\tau}=\{A:S_i\subseteq A,\ T_j\cap A=\phi \text{ for some } i,j\}$$

and

$$C_\tau=\{A:A\subseteq T_j, \text{ for some } j\}.$$

Let $'$ denote complementation in U_k (but for in the proof of (3) where it denotes complementation in C). For any collection ρ of subsets of U_k, let ρ' denote the collection of the complements of members of ρ.

Let $h(A)$ denote a non-negative (real) valued function on V_k satisfying (i) $h(\phi)\neq 0$, (ii) $h(A\cup B)\leq h(A)h(B)$, if $A\cap B=\phi$ and (iii) $h(A)\leq c^{|A|}$ for some fixed constant c, and $\hat{h}(A)$ denote the corresponding function defined by $\hat{h}(A)=c^{|A|}$. Set

$$h_\tau^*=\Big(\sum_{A\in C_\tau}h(A)\Big)\Big(\sum_A h(A)\Big)^{-1},$$

and let $\hat{h}_\tau^*$ be similarly defined. Then we have the following result.

THEOREM 1. *There holds* $h_\tau^*\geq\hat{h}_\tau^*$.

We use the letter p, with or without affixes, to denote primes. For a given squarefree number $n=p_1\ldots p_k$, we define arithmetic functions $h,\ldots$, by $h(d)=h(\{j:p_j\mid d\}),\ldots$, over divisors d of n. Let $0<a<1$. Now, by taking $\tau=\tau_\alpha$ as the collection of $A_d=\{j:p_j\mid d\}$, with $d\leq n^\alpha$, we obtain immediately from Theorem 1, the following.

THEOREM 2. *Let $h(m)$ be a non-negative submultiplictive function defined on natural numbers satisfying $h(p)\leq c$, for some fixed $c>0$. Then, for squarefree numbers n and $0<\alpha<1$, we have*

$$\Big(\sum_{d\leq n^\alpha}{}^{*}h(d)\Big)\Big(\sum_{d}{}^{*}h(d)\Big)^{-1}\geq\Big(\sum_{d\leq n^\alpha}{}^{*}c^{\omega(d)}\Big)\Big(\sum_{d}{}^{*}c^{\omega(d)}\Big)^{-1}, \tag{1}$$

*with * signifying the condition that d is a divisor of n and $\omega(d)$ denoting the number of (distinct) prime divisors of d.*

If c is the reciprocal of an integer, and $\alpha=c/(c+1)$ we see that (1) extends Theorem 3 of **[2]** to include the case of submultiplicative functions h, and in view of the result in **[6]** we obtain the following stronger result.

THEOREM 3. *Let $h(m)$ be a submultiplicative function satisfying $0 \le h(p) \le 1/(k-1)$. Then, for squarefree n and $k = 2, 3, 4, \ldots$,*

$$\sum_{d|n} h(d) \le k \sum_{d|n, d \le n^{1/k}} h(d);$$

and, for rational $k > 1$ we have the same inequality but with the factor k replaced by a constant defined in terms of the partial quotients in the continued fraction of k.

In this context see the theorem on p. 6 of [**1**] and also [**5**]. For the proof of Theorem 1, it suffices to use Theorem 1 of [**1**]; however, we note that the latter theorm is contained in the following Lemma 0 due to Erdős, Herzog and Schőnheim, which also in turn is included in sharper versions from Marica and Berge (cf., pp. 103–4 of [**3**]).

LEMMA 0. *There exists a bijective mapping Φ of C_τ into itself, satisfying $A \cap \Phi(A) = \phi$ for all $A \in C_\tau$.*

Next, we shall see that, in view of Hall's theorem ([**4**]), this is equivalent to the following *universal* result.

LEMMA 0′. *There holds $|B_{\sigma,\tau}| \le |A_{\sigma,\tau}|$.*

In Section 3 we give a (non-inductive) proof of Lemma 0′.

2. Proof of Theorem 1. Let us define $C_\tau(r) = \{A \in C_\tau : |A| = r\}$; $C'_\tau(r) = \{A \notin C_\tau : |A| = r\}$ and $M_r = |C_\tau(r)|$, $M'_r = |C'_\tau(r)|$.

Set $h(A) = h_0(A)\hat{h}(A)$, so that (by the conditions on h) $1 \ge h_0(A) \ge h_0(B)$ for $A \subseteq B$. Now it is easily checked that the theorem follows if we show that, for every $J \ge 0$,

$$\sum_{r+s=J} H_r M'_s \ge \sum_{r+s=J} H'_r M_s, \tag{2}$$

where

$$H_r = \sum_{A \in C_\tau(r)} h_0(A), \qquad H'_r = \sum_{A \in C'_\tau(r)} h_0(A).$$

Next we view (2) as

$$\sum^{(J)} \{h_0(A) - h_0(B)\} \ge 0,$$

where (J) denotes summing over $(A, B) : A \in C_\tau(r)$ and $B \in C'_\tau(J-r)$ with $0 \le r \le J$. Now this inequality can be written as

$$\sum_{D \subseteq C} \left\{ \sum^{(J)}_{A \cup B = C; A \cap B = D} (h_0(A) - h_0(B)) \right\} \ge 0.$$

Let $H_J(C, D)$ denote the inner sum here. In fact, we shall show that

$$H_J(C, D) \ge 0, \qquad \text{for } D \subseteq C. \tag{3}$$

(Necessarily $|C| + |D| = J$.) At this point we note that it suffices to prove (3) with $D = \phi$, for, any pair (A, B) occurring in the summation determines a pair (A_1, B_1) defined

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through $A_1 = A - D$, $B_1 = B - D$ and *vice versa*, and we can work with $h_1(A_1) := h_0(A)$ and assume $h_1(\phi) \neq 0$. Further,

$$H_J(C, \phi) = \sum_{A \subseteq C} h_0(A) - \sum_{A \subseteq C} h_0(A').$$

Here, by using Lemma 0 with $\tau = \{C\}$ and regarding C as U_J, we have

$$\sum_{A \subseteq C} h_0(A) = \sum_{A \subseteq C} h_0(\Phi(A)) \geq \sum_{A \subseteq C} h_0(A'),$$

since the bijective Φ satisfies $\Phi(A) \subseteq A'$, for $A \subseteq C$. This proves (3) and thus completes the proof of Theorem 1.

3. Proof of Lemma 0. For $S \in C_\tau$, let $D(S)$ denote the set of all $A \in C_\tau$ such that $A \cap S = \phi$. Then, by Theorem 1 of [**4**], we know that a bijective Φ specified in the lemma exists if and only if

$$|D(S_1) \cup \ldots \cup D(S_r)| \geq r, \tag{4}$$

for any choice of r (distinct) subsets $S_1, \ldots, S_r$ from C_τ. Now (4) can be rewritten as

$$|\{A : S_i \subseteq A \subseteq T_j, \text{ for some } i, j\}| \leq |\{A : A \cap S_i = \phi, A \subseteq T_j, \text{for some } i, j\}|, \tag{5}$$

by considering the set on the left as the choice of r subsets. By changing τ in (5) to τ' and considering the complement of the set on the right side we obtain the statement of Lemma 0′.

For the proof of (5) we use the following extension of the classical Chebyshev's inequality.

Lemma *. *Suppose that two given set of real numbers x_A, y_A $(A \subseteq U_K)$ satisfy*

$$x_A \geq x_B, \qquad y_A \geq y_B; \qquad A \subseteq B \subseteq U_k. \tag{6}$$

Then, we have

$$|V_k| \Big(\sum_{A \subseteq U_k} x_A y_A \Big) \geq \Big(\sum_{A \subseteq U_k} x_A \Big) \Big(\sum_{A \subseteq U_k} y_A \Big). \tag{7}$$

Lemma * is a special case of FKG-inequality (cf., Corollary 6.2.5 of [**3**]). However we give another direct proof of it in Section 4. Actually, by means of the Lemma A below (also proved in Section 4), we obtain a monotonically decreasing sequence of numbers starting with the quantity on the left in (7) having the expression on the right as the limit, by showing that the expression in (8′) below is non-negative.

For the statement of Lemma A we introduce the following notation. Set $E_A :=$ $\{B : (|A - B| + |B - A|) \leq 1\}$; $|E_A| = (k+1)$ for all A. Also, define inductively, $z_A^{(j+1)} =$ $(k+1)^{-1} \sum_{B \in E_A} z_B^{(j)}$ for integers $j \geq 0$, for a given set of numbers $z_A =: z_A^{(0)}$. Now we notice that $\sum_A z_A^{(j)}$ is independent of j. Also, define δ_{BA} as 1 if $A = B$ and as 0, otherwise.

Lemma A. *We have, for every set A,*

$$\lim_{j \to \infty} z_A^{(j)} = |V_k|^{-1} \sum_B z_B.$$

REMARK. By considering the definition of $\{z_A^{(j)}\}$ as a linear transformation on $\{z_A\}$, we note that Lemma A can be viewed as the statement:

$$\lim_{j\to\infty} M^j = M_0,$$

where $M=(m_{AB})$ is the matrix defined by $m_{AB}=(k+1)^{-1}$ if $B\in E_A$ and $=0$ if $B\notin E_A$ and every entry of the matrix M_0 equals 2^{-k}. Now, on defining a chain $\{A_0, A_1, \ldots, A_j\}$ joining A_0 and A_j as a sequence satisfying $A_i\in E_{A_{i-1}}$ $(i=1,\ldots,j)$ and introducing $W_j(A_0, A_j)$ as the number of such chains we obtain the following equivalent result.

LEMMA A$'$. *We have*

$$W_j(A,B)\sim(k+1)^j2^{-k} \quad as \quad j\to\infty,$$

for every pair (A, B).

Proof of (5). In the present proof, the phrase "by (π)" means the use of the fact that the cardinality of a collection of sets A, subjected to some conditions, is unaltered if A' replaces A in those restrictions. First, we show that both the sets in (5) can be assumed to be disjoint. For, in the notation $\hat{B}_{\sigma\tau}$ (*resp.* $\hat{A}_{\sigma\tau}$) for the set on the left (*resp.* right), we observe that (5) is equivalent to $|\hat{B}_{\sigma\tau}-\hat{A}_{\sigma\tau}|\le|\hat{A}_{\sigma\tau}-\hat{B}_{\sigma\tau}|$ and that the set $\hat{B}_{\sigma\tau}-\hat{A}_{\sigma\tau}$ is seen to be equal to $\hat{B}_{\lambda\tau}$ with respect to some λ while the set $\hat{A}_{\sigma\tau}-\hat{B}_{\sigma\tau}$ contains $\hat{A}_{\lambda\tau}$ and thus the two sets occurring in (5) with respect to $(\lambda\tau)$ (instead of $(\sigma\tau)$) do not intersect. Thus, in particular, every pair of members of σ can be assumed to have (i) non-empty intersection and, as will be seen shortly, also to have (ii) their union $\neq U_k$. By (π), we see that $|\hat{B}_{\sigma\tau}|=|\hat{B}_{\tau'\sigma'}|$ and $|\hat{A}_{\sigma\tau}|=|\hat{A}_{\tau'\sigma'}|$. So, by starting with $(\tau'\sigma')$ instead of $(\sigma\tau)$ and proceeding as above, we can assume (i) above with respect to members of τ', which means that the union of any two members of τ can be assumed $\neq U_k$, proving (ii) for σ. Similarly, by starting with $(\tau'\sigma')$ instead of $(\sigma\tau)$ and proceeding as above (since the inequality (5) is unchanged), τ' also can be assumed to fulfil (i) and (ii).

Next, for any collection ρ of subsets of U_k satisfying (i) and (ii), we observe that ρ' does likewise and that the function $\epsilon_\rho(A)$ can be defined through (a): $=1$, if $A\in C_\rho$; (*b*): $=-1$, if $A'\in C_\rho$ and, (c): $=0$, otherwise. Clearly $\epsilon_\rho(A)\ge\epsilon_\rho(B)$, whenever $A\subseteq B$, and $\sum_A \epsilon_\rho(A)=0$. So, by (π) and (6), in view of Lemma *, we obtain $0\le\sum_A \epsilon_{\sigma'}(A)\epsilon_\tau(A)=2[|\hat{A}_{\sigma\tau}|-|\hat{B}_{\sigma\tau}|]$, which proves (4). Thus the proof of Lemma 0 is completed.

4. Proofs of Lemmas A and *.

Proof of Lemma A. First we note that it suffices to prove the lemma for the choices $z_B=\delta_{BA}$ with respect to every A. To start with we shall obtain the result under the assumption that

$$z_A\ge z_B; \qquad A\subseteq B\subseteq U_k. \tag{6'}$$

Then we observe that $z_A^{(j)}$ also satisfy (6$'$) and $z_\phi^{(j)}\ge z_\phi^{(j+1)}$, for $j\ge0$. Put $z_0=\lim_{j\to\infty} z_\phi^{(j)}$, which exists because $z_A^{(j)}$ are bounded below by $\min_A z_A$. Now we can show, by induction on r, that there are (positive) constants $c_0, c_1, \ldots, c_k$ such that, for every $\epsilon>0$, the numbers $z_A^{(j)}$, with $|A|=r$, belong to the interval $[z_0-c_r\epsilon, z_0+c_r\epsilon]$ provided j is sufficiently large. This is true for $r=0$ with $c_0=1$ (say) by definition of z_0. The passage from r to $(r+1)$ is facilitated by the definition of $z_A^{(j+1)}$ and the fact that $z_A^{(j)}-z_B^{(j)}\geq 0$ whenever $B=A\cup\{e\}$ with $e\notin A$ and $|A|=r$. Thus $\lim_{j\to\infty} z_A^{(j)}=z_0$ for every A. Since $\sum_A z_A^{(j)}$ is independent of j the lemma is proved under (6′).

In particular, we can take $z_B=\delta_{B\phi}$. For any given A, we define the mapping $\psi=\psi_A$ by

$$\psi(B)=(B-A)\cup(A-B).$$

Now observe that $D\in E_B$ if and only if $\psi(D)\in E_{\psi(B)}$. Therefore we have the lemma for the choice $z_B=\delta_{\psi(B)\phi}=\delta_{BA}$. This completes the proof.

Proof of Lemma *. Given (6), we see that the two sets $\{x_A^{(j)}\}$, $\{y_A^{(j)}\}$ also satisfy (6), for every $j\geqslant 0$.

Now consider, for given A, the identity

$$2\left[(k+1)\sum_B{}^{(A)} x_B^{(j)}y_B^{(j)}-\left(\sum_B{}^{(A)} x_B^{(j)}\right)\left(\sum_B{}^{(A)} y_B^{(j)}\right)\right]=\sum_{B,C}{}^{(A)}(x_B^{(j)}-x_C^{(j)})(y_B^{(j)}-y_C^{(j)}),$$

where (A) denotes summation over E_A. On summing over A, the left side becomes

$$2(k+1)^2\left[\sum_A x_A^{(j)}y_A^{(j)}-\sum_A x_A^{(j+1)}y_A^{(j+1)}\right], \tag{8′}$$

while the right side gives

$$\sum_A\sum_{B,C}{}^{(A)}(x_B^{(j)}-x_C^{(j)})(y_B^{(j)}-y_C^{(j)}). \tag{8″}$$

Next, we show that the expression in (8″) (which is empty if $k=1$) is non-negative. To start with, observe that any pair (B,C) here satisfies $\big||B|-|C|\big|\leq 2$. Since, for (B,C) with $\big||B|-|C|\big|=1$, either $B\subseteq C$ or $C\subseteq B$, we have that the contribution to (8″) from such (B,C) is non-negative by our initial remark above regarding (6) and obviously there is no contribution from pairs (B,C) with $B=C$. The remaining part of (8″) can be rearranged to give

$$2\sum_R[(x_C^{(j)}-x_A^{(j)})(y_C^{(j)}-y_A^{(j)})+(x_D^{(j)}-x_B^{(j)})(y_D^{(j)}-y_B^{(j)})], \tag{9}$$

where $R=[A,B,C,D]$ denotes a (typical) configuration satisfying (i) $A\subset C$, $|C|=|A|+2$ and (ii) $|B|=|D|$, $B\cup D=C$, $B\cap D=A$. (For this we need only observe that each remaining pair (B,C) occurs twice in (8″) and appears in two configurations $[A',B',C',D']$ and $[A',D',C',B']$ (like R) as corresponding to the pair (A',C') or (C',A') or (B',D') or (D',B') and each of the last four pairs is among the pairs (B,C) remaining in (8″) and the summand in (9) corresponding to (C,A) (resp. to (D,B)) is unaltered if A and C (resp. B and D) are interchanged.) Now every summand in (9) is non-negative. For, with the notation $z_1=z_B^{(j)}-z_A^{(j)}$, $z_2=z_C^{(j)}-z_B^{(j)}$, $z_3=z_D^{(j)}-z_A^{(j)}$ and $z_4=z_C^{(j)}-z_D^{(j)}$, the summand becomes

$$(x_1+x_2)(y_1+y_2)+(x_3-x_1)(y_3-y_1),$$

which, on simplification by using $z_1+z_2=z_3+z_4$, becomes $x_1y_4+x_2y_2+x_3y_3+x_4y_1$, and

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this last expression is ≥ 0 since each of x_j, y_j $(j=1,2,3,4)$ is non-negative. Thus the expression in (8″) is non-negative and so is also the quantity in (8′). Now, by Lemma A, $x_A^{(j)}$'s (resp. $y_A^{(j)}$'s) tend, as $j\to\infty$, to the value $|V_k|^{-1}\sum_A x_A$ (resp. $|V_k|^{-1}\sum_A y_A$). So, (8′) gives (as $j\to\infty$)

$$\sum_A x_A y_A \geq |V_k|^{-1}\left(\sum_A x_A\right)\left(\sum_A y_A\right),$$

which is (7), thus completing the proof of Lemma *.

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