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Pseudo-distributive near-rings

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In the study of the theory of rings, matrix rings, group rings, algebras, and so on, play a very important role. However, the analogous systems may not exist in the theory of near-rings. Recently Ligh obtained a necessary and sufficient condition for the set of $n \times n$ matrices with entries from a near-ring to be a near-ring. This opens the door for the study of other structures such as group near-rings, algebras, and so on. In this paper we initiate a study of the basic properties of pseudo-distributive near-rings, which is exactly the class of near-rings needed to carry out the construction of matrix near-rings, group near-rings, polynomials with near-ring coefficients, and so on.

1. Introduction

In the study of the theory of rings, matrix rings, group rings, algebras, and so on, play a very important role. However, the analogous systems may not exist in the theory of near-rings. In his dissertation [1] Beidleman considered the system $M_n(R)$ of all $n \times n$ matrices with entries from a near-ring R. He showed that if R is a near-ring with identity and $M_n(R)$, $n > 1$, is a near-ring, then R is a ring. Recently it was shown in [6] that $M_n(R)$ is a near-ring if and only if R is an n-distributive near-ring. This opens the door for the study of other structures such as group near-rings, algebras, and so on. The purpose of this paper is to initiate a study of the basic properties of pseudo-distributive near-rings, which is exactly the class of near-rings needed to carry out the construction of matrix near-rings, group near-rings,

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polynomials with near-ring coefficients, and so on.

2. Definitions and examples

DEFINITION 1. A near-ring R is called n-distributive, n a positive integer, if for each a, b, c, d, r, a_i, and b_i in R,

(i) $ab + cd = cd + ab$, and

(ii) $\left(\sum a_ib_i\right)r = \sum a_ib_ir$, $i = 1, 2, \ldots, n$.

DEFINITION 2. A near-ring R is called pseudo-distributive if it is n-distributive for each positive integer n.

It can easily be shown that a distributive near-ring is n-distributive for each n, hence, is a pseudo-distributive near-ring, while a distributively generated near-ring that is also pseudo-distributive must be distributive. We shall furnish some examples of pseudo-distributive near-rings which are not distributive to illustrate the importance and abundance of the class of pseudo-distributive near-rings.

EXAMPLE 1. The near-rings R given in [3, 2.1, #3, 2.2, #16] have the property that $(R, +)$ is abelian and yet R is not a ring. Recall that if R is distributively generated, then $(R, +)$ is not abelian unless R is a ring.

EXAMPLE 2. The near-ring given in [3, 2.2, #22], it is noteworthy to remark, is 3-distributive, but not 2 or 4-distributive. However, $0x$ is not always zero.

REMARK. We shall assume throughout the rest of the paper that all near-rings R considered have the property that $0x = 0$ for each x in R. Thus if R is n-distributive, then R is m-distributive for each $m < n$.

EXAMPLE 3 [6]. $M_n(R)$ is a near-ring if and only if R is n-distributive.

EXAMPLE 4. Group near-ring. Let G be any group (written multiplicatively) and N a near-ring. Let NG be the set of all mappings from G into N which have finite support. Define addition pointwise and multiplication via $(t)\alpha * \beta = \sum (g)\alpha \cdot \left(g^{-1}t\right)\beta$, $g \in G$, $\alpha, \beta \in NG$, and

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$t \in G$. In general multiplication will not be well-defined, but will depend on the order of the elements in the sum. However, if N is a pseudo-distributive near-ring, then NG is also a pseudo-distributive near-ring. Conversely, if NG is a near-ring and the order of G is k , then N is k-distributive. Hence it follows that N is pseudo-distributive if G is an infinite group.

EXAMPLE 5. Formal power series and polynomials. For an arbitrary near-ring N define the *formal power series* $F\langle N\rangle$ over N in the usual fashion, that is, each element $\langle a_i\rangle$ can be considered as a mapping from the set of non-negative integers into N , $i \to a_i$. Define addition pointwise and multiplication as the usual "Cauchy product":

$\langle a_i\rangle\langle b_j\rangle = \langle c_n\rangle$, where $c_n = \sum a_ib_j$, $i + j = n$. Then $F\langle N\rangle$ is a near-ring if and only if N is pseudo-distributive. The subset of $F\langle N\rangle$ of all functions of finite support, the *polynomials* over N , is a subnear-ring of $F\langle N\rangle$, when $F\langle N\rangle$ is a near-ring.

EXAMPLE 6. Gaussian near-ring. For an arbitrary near-ring N define $N(i)$ to be the system composed of the group $(N, +) \oplus (N, +)$ together with "complex" or "gaussian" multiplication:

$$(a, b)(c, d) = (ac-bd, ad+bc) .$$

Then $N(i)$ is a near-ring if and only if N is 2-distributive.

3. Basic properties

In this section we consider some elementary facts about pseudo-distributive near-rings. It is worthwhile to mention that the class of pseudo-distributive near-rings is closed under direct products, epimorphic images, and subnear-rings; hence it is an equationally definable class (a variety). In this sense the class of pseudo-distributive near-rings is a better generalization of rings or distributive near-rings than is the class of distributively generated near-rings, which is not a variety, since a subnear-ring of a distributively generated near-ring need not be distributively generated [4]. Unfortunately the variety of pseudo-distributive near-rings has many of the same limitations that the variety of distributive near-rings has as shown by the following result.

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THEOREM 1. *Let* V *be the variety of pseudo-distributive near-rings and let* $V(p)$ *be the class of all pseudo-distributive near-rings with property* p . *If* p *is any one of the following properties, then* $V(p)$ *is* $R(p)$, *the class of rings with property* p :

(i) *there exists a (left, right, two-sided) identity;*

(ii) *there exists a left cancellable element;*

(iii) *every element is regular;*

(iv) *every element is an idempotent;*

(v) *every element is the sum of products;*

(vi) *there are no non-zero nilpotent ideals.*

Proof. It is known that any distributive near-ring with any of the properties *(i)*-*(vi)* must be a ring. It is easy to see that a pseudo-distributive near-ring with any of the properties *(i)*-*(v)* is distributive. The next theorem shows that a pseudo-distributive near-ring with property *(vi)* is a ring as well as setting up machinery needed in the sequel.

THEOREM 2. *Let* R *be a pseudo-distributive near-ring. Then*

(i) *the set* $A = \{x \in R : Rx = 0\}$ *is an ideal of* R *and* R/A *is a ring.*

Thus the commutator subgroup R' *of* R *is a subset of* A .

(ii) *If* R *is not a ring, then* $A \neq 0$.

(iii) *If* R *has a right identity, then* R *is a ring.*

(iv) *The set* $B = \{r \in R : (x+y)r = xr+yr,\ x,\ y \in R\}$ *is a subnear-ring of* R *and* $A \subseteq B$. *Also* $(B, +)$ *is a normal subgroup of* $(R, +)$.

Proof. *(ii)* If $(R, +)$ is abelian, then there is an $x \neq 0$ such that x is not right distributive. Hence there are w, z in R such that $a = [(w+z)x-wx-zx] \neq 0$. It is easy to show that $a \in A$.

If $(R, +)$ is not abelian and R is distributive, then $R' \neq 0$ and $R' \subseteq A$. If R is not distributive, there is an $a \neq 0$ in R such that $a \in A$ by the above argument.

The other parts follow from the definition of a pseudo-distributive

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near-ring and straightforward calculations.

If R is a simple pseudo-distributive near-ring, then $A = R$ and $R^2 = (0)$ or $A = (0)$ and R is a ring. So simple pseudo-distributive near-rings are distributive and hence the direct sum or product of simple pseudo-distributive near-rings is a distributive near-ring with summands being either rings or zero multiplication near-rings.

The structure of a ring R in which $(R, +)$ is a simple group is well-known; namely, R is either the zero multiplication ring on Z_p or R is the finite field $\mathrm{GF}(p)$. Hence, if N is a pseudo-distributive near-ring and $(N, +)$ is simple, then either N is the zero multiplication near-ring on $(N, +)$ or N is isomorphic to $\mathrm{GF}(p)$ for some prime p. Recall from Example 1 that there are pseudo-distributive near-rings which are not rings nor zero multiplication near-rings yet have Z_n as an additive group.

Note that if R is a pseudo-distributive near-ring and $(R, +)$ is a perfect group (that is, $R' = R$), then $R' = A = R$ and R is a zero multiplication near-ring. In contrast, the near-ring generated by the inner automorphisms on a finite non-abelian simple group G is a non-trivial distributively generated near-ring whose additive group is the direct sum of copies of G (and hence is perfect).

It is well known that the only ring that can be defined on a torsion divisible group is the zero ring. Now we consider pseudo-distributive near-rings defined on $Z(p^\infty)$.

THEOREM 3. *Let* R *be a pseudo-distributive near-ring and* $(R, +) \cong Z(p^\infty)$. *Then* R *is the zero ring.*

Proof. If every element is right distributive then R is a ring and hence a zero ring. Suppose there is an element $w \neq 0$ that is not right distributive. As in the proof of *(ii)* of Theorem 2, there is an $a \neq 0$ in R such that $a \in A$. Since A is an ideal and R/A is a ring, $(R/A, +) \cong Z(p^\infty)$. Let $x \in R$ and $B = \{xy : y \in R\}$; B is a homomorphic image of R and hence divisible. But $B \subseteq A$ since $(x+A)(y+A) = A$. Since A is finite, it follows that $B = 0$. Thus R has the zero multiplication.

It is well known that if R is a ring with a finite number of proper

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subrings, then R is finite. The situation for near-rings is unsettled, though it has been shown [2] to be affirmative for certain classes of near-rings. Recently, the above ring problem was investigated [5] from a different angle, by determining the structure of rings all of whose proper subrings are finite. Now we consider the case where R is a pseudo-distributive near-ring.

THEOREM 4. *Let R be an infinite pseudo-distributive near-ring in which each proper subnear-ring is finite. Then R is one of the following:*

(i) $R^2 = 0$ *and* $(R, +)$ *is non-abelian;*

(ii) $R^2 = 0$ *and* $R = Z(p^\infty)$ *for some prime* p *;*

(iii) $R = \bigcup_{n=0}^{\infty} \mathrm{GF}\left(p^{q^n}\right)$ *for some primes* p *and* q *.*

Proof. Case 1. Suppose $R^2 = 0$. If $(R, +)$ is not abelian, then this is the well-known unsolved problem in group theory. If $(R, +)$ is abelian, then each proper subgroup is a subnear-ring, hence finite, and it is well known that $(R, +) \cong Z(p^\infty)$ for some prime p .

Case 2. Suppose $R^2 \neq 0$. Thus there is an $x \neq 0$ in R such that $xR \neq 0$. It is easy to see that both xR and $A(x) = \{r \in R : xr = 0\}$ are subnear-rings of R . If xR is finite, then $A(x)$ is infinite and hence $A(x) = R$, a contradiction. Thus xR is infinite and $xR = R$. Now every element is right distributive and thus R is a distributive near-ring. Also it can easily be shown that $Rx = R$. Suppose $R' \neq 0$ and let $r' \in R'$. There is a y in R such that $xy = r'$. Let $e \in R$ such that $ex = x$. But $xy = r'$ implies $exy = er' = 0$ and $xy = r' = 0$, a contradiction. Thus $R' = 0$ and R is a ring and now the conclusion follows from the result in ring theory [5].

Right and anti-right distributive elements play an important role in the study of near-rings. (An element x is anti-right distributive if and only if $(a+b)x = bx + ax$ for each a, b .) In general, they do not form a subnear-ring. The following result gives a necessary and sufficient condition for them to be a subnear-ring. We shall omit the proof.

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THEOREM 5. *Let R be a near-ring. Then the set T of right and anti-right distributive elements forms a subnear-ring if and only if $ar_1 + br_2 = br_2 + ar_1$ for all $r_1, r_2 \in T$ and $a, b \in R$.*

Observe that in a pseudo-distributive near-ring T is indeed a subnear-ring.

The next result shows how one can construct a pseudo-distributive near-ring from any given near-ring. The proof follows from direct calculation.

THEOREM 6. *Let R be a near-ring and let*

$$S(R) = \left\{x \in R : ax+bc = bc+ax, \left(\sum a_ib_i\right)x = \sum a_ib_ix, \ i = 1, 2, \ldots, n, \ a, b, c, a_i, b_i \in R\right\} .$$

Then $S(R)$ is a pseudo-distributive subnear-ring of R.

Now we can examine $S(R)$ for certain classes of R.

THEOREM 7. *Let R be a near-field. Then $S(R)$ is a division ring.*

Proof. Since the set W of right distributive elements of R is a division ring, we wish to show $W = S(R)$. Clearly $W \subseteq S(R)$. Since $S(R)$ is pseudo-distributive, hence $S(R)$ is 2-distributive. Let $x \neq 0$ be in $S(R)$ and a, b be arbitrary elements of R. Then $x(a+b)x = (xa+xb)x = xax + xbx = x(ax+bx)$. Since x has a multiplicative inverse, it follows that $(a+b)x = ax + bx$. Thus $S(R) \subseteq W$ and the proof is complete.

Let G be a group. We adopt the following notations for our next result:

$Z(G)$ center of G;

end G set of endomorphisms of G;

$E(G)$ the distributively generated near-ring generated by end G;

$T(G)$ the near-ring of all mappings from G to G;

$U(G) = \{\phi \in \text{end } G : \text{Im } \phi \subseteq Z(G)\}$.

THEOREM 8. *Let G be any group. Then*

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(i) $S(T(G)) = U(G)$ *and* $U(G)$ *is a ring,*

(ii) $U(G) \subseteq S(E(G))$ *and* $S(E(G))$ *is a ring.*

Proof. To obtain *(i)*, let $\phi \in S(T(G))$. Since $T(G)$ has an identity I, for each $\alpha, \beta \in T(G)$,

$$(\alpha+\beta)\phi = (\alpha I+\beta I)\phi = \alpha I\phi + \beta I\phi = \alpha\phi + \beta\phi .$$

Hence ϕ is right distributive and it follows that $\phi \in \text{end } G$. Suppose $x, y \in G$. Then there are $\alpha, \beta \in T(G)$ and $g \in G$ such that $g\alpha = x$ and $g\beta = y$. Hence $g(\alpha\phi+\beta I) = g(\beta I+\alpha\phi)$. It follows that $x\phi + y = y + x\phi$ and $\phi \in U(G)$. Similarly $U(G) \subseteq S(T(G))$. The fact that $U(G)$ and $S(E(G))$ are rings follows from the fact that both $T(G)$ and $E(G)$ have an identity.

Finally we remark that many of the results in this paper remain valid if one assumes R only to be 2-distributive instead of pseudo-distributive.

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