COMPOSITIO MATHEMATICA

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Jean-Pierre Labesse and Joachim Schwermer

Compositio Math. **157** (2021), 1207–1210.

doi:10.1112/S0010437X21007181

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X21007181 Published online by Cambridge University Press

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Compositio Math. 157 (2021) 1207–1210
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Corrigendum

On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields

(Compositio Math. 102 (1996), 1–40)

Jean-Pierre Labesse and Joachim Schwermer

Abstract

The aim of this corrigendum is to correct an error in Corollary 10.7 to Theorem 10.6, one of the main results in the paper 'On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields'. This makes necessary a thorough investigation of the conditions under which a Cartan-type automorphism exists on $G_1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_0$, where G_0 is a connected semisimple algebraic group defined over $\mathbb{R}$.

Let G be a connected semisimple algebraic group defined over a number field k. Consider the Lie group $\mathbf{G} = G(k \otimes \mathbb{R})$ and let us denote by $\boldsymbol{\theta}$ a Cartan involution. A k-automorphism ϕ of G is said to be of Cartan type if the automorphism Φ induced by ϕ on $\mathbf{G}$ can be written as $\Phi = \mathrm{Int}(x) \circ \boldsymbol{\theta}$ where $\mathrm{Int}(x)$ is the inner automorphism defined by some $x \in \mathbf{G}$. Theorem 10.6 of [BLS96] establishes the following result regarding the existence of non-trivial cuspidal cohomology classes for S-arithmetic subgroups of G.

Theorem 1. *Let G be an absolutely almost simple algebraic group defined over k that admits a Cartan-type automorphism. When the coefficient system is trivial, the cuspidal cohomology of G over S does not vanish, that is, every S-arithmetic subgroup of G has a subgroup of finite index with non-zero cuspidal cohomology with respect to the trivial coefficient system.*

The following assertion appears as Corollary 10.7 in [BLS96]. *Assume that G is k-split and k totally real or $G = \mathrm{Res}_{k'/k} G'$ where k' is a CM-field. Then the cuspidal cohomology of G over S with respect to the trivial coefficient system does not vanish.*

The proof of Corollary 10.7 amounts to exhibiting, in each case, a Cartan-type automorphism. In the first case, dealing with split groups, the proof is correct. As regards the second case where $G = \mathrm{Res}_{k'/k} G'$ with k' a CM-field, it was observed by Rohlfs and Clozel independently that the assertion (and the proof) must be corrected since, to make sense, the argument implicitly uses strong extra assumptions. First of all, G' has to be defined over k so that the complex conjugation c induced by the non-trivial element σ in $\mathrm{Gal}(k'/k)$ acts as a k-rational automorphism of G. Observe that, strictly speaking, one has to extend the scalars from k to k' before applying the restriction functor. Further assumptions are necessary so that Corollary 10.7 in [BLS96] should be replaced by the following statement.

Received 11 December 2019, accepted in final form 12 January 2021, published online 26 May 2021.
2020 Mathematics Subject Classification 11F75 (primary), 11F70, 22E50, 22E55 (secondary).
Keywords: cohomology of arithmetic groups, cuspidal automorphic representations, Cartan-type automorphisms.
J.S. gratefully acknowledges the support of the Max-Planck-Institute for Mathematics, Bonn, in 2020.
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https://doi.org/10.1112/S0010437X21007181 Published online by Cambridge University Press

Theorem 2. *Let k be a totally real number field and G be an absolutely almost simple algebraic group defined over k. Consider the following cases:*

(1) *G is k-split.*
(2) *$G = \mathrm{Res}_{k'/k}G'$ where k' is a CM-field with G' defined over k totally real, satisfying one of the following hypotheses:*
   (2a) *$\mathbf{H} = G'(k \otimes \mathbb{R})$ has a compact Cartan subgroup;*
   (2b) *G' is split over k and simply connected.*

Then the cuspidal cohomology of G over S with respect to the trivial coefficient system does not vanish.

Proof. In case (1), when G is k-split, the proof is given in [BLS96]: it relies on the first case of [BLS96, Corollary 10.7] and Theorem 1. In case (2a) the result follows from Proposition 3 below and Theorem 1. In case (2b) the assertion is a particular case of [Lab99, Theorem 4.7.1], which in turn relies on case (1) above. □

Most of the following proposition is well known (see, in particular, [She79, Corollary 2.9], [Lan89, Lemma 3.1] and [Ada14, p. 2132]) but, not knowing of a convenient reference, we sketch a proof.

Consider a connected semisimple algebraic group G_0 defined over $\mathbb{R}$. We denote by $\mathbf{G} = G_0(\mathbb{C})$ the group of its complex points and by $\mathbf{c}$ the complex conjugation on $\mathbf{G}$. Then $\mathbf{H} = G_0(\mathbb{R})$ is the group of fixed points under $\mathbf{c}$. This anti-holomorphic involution is induced by an $\mathbb{R}$-automorphism c of $G_1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}G_0$. Let $\boldsymbol{\theta}$ be a Cartan involution of $\mathbf{G}$. The group $\mathbf{U}$ of fixed points of $\boldsymbol{\theta}$ in $\mathbf{G}$ is a compact real form: $\mathbf{U} = U(\mathbb{R})$ where U is a form of G_0 but not necessarily inner. Let G^{**} be the split outer form of G_0. Choose a splitting $(B^*, T^*, \{X_\alpha\}_{\alpha\in\Delta})$ for G^{**} over $\mathbb{R}$ where T^* is a torus in a Borel subgroup B^* and for each $\alpha \in \Delta$, the set of simple roots, X_α is a root vector. Let ψ^* be the automorphism of G^{**} that preserves the splitting and whose action defines G^*, the quasi-split inner form of G_0. Let w be the element of maximal length in the Weyl group for T^*.

Proposition 3. *The following assertions are equivalent.*

(i) *The automorphism c is of Cartan type.*
(ii) *The group $\mathbf{U}$ is an inner compact real form.*
(iii) *The group $\mathbf{H}$ has a compact Cartan subgroup.*
(iv) *The group $\mathbf{H}$ admits discrete series.*
(v) *The involution $w \circ \psi^*$ acts by -1 on the root system of T^*.*

Proof. The automorphism c is of Cartan type if, by definition, $\mathbf{c} = \mathrm{Int}(x) \circ \boldsymbol{\theta}$ for some $x \in \mathbf{G}$, that is, if and only if U is an inner form of G_0 or equivalently of G^*. This proves the equivalence of (i) and (ii). Assume now that U is an inner form of G_0. Up to conjugation under $\mathbf{G}$, we may assume that $\boldsymbol{\theta}$ is of the form $\boldsymbol{\theta} = \mathrm{Int}(x) \circ \mathbf{c}$ with x in the normalizer of a maximal torus T in G_0 defined over $\mathbb{R}$. In particular, x is semisimple and its centralizer $\mathbf{L}$ in $\mathbf{G}$ is a complex reductive subgroup of maximal rank. The cocycle relation $\mathrm{Int}(x\,\mathbf{c}(x)) = 1$ implies that $\mathbf{L}$ is stable under $\mathbf{c}$. Then $\mathbf{c}$ induces an anti-holomorphic involution on $\mathbf{L}$ whose fixed points $\mathbf{M} = \mathbf{L} \cap \mathbf{H} = \mathbf{U} \cap \mathbf{H}$ are a compact real form of $\mathbf{L}$. A Cartan subgroup $\mathbf{C}$ of $\mathbf{M}$ is a compact Cartan subgroup in $\mathbf{H}$. Hence (ii) implies (iii). Now, consider a torus T in G_0 such that $\mathbf{C} = T(\mathbb{R})$ is compact. Then the

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complex conjugation $\mathbf{c}$ acts by -1 on the weights of T. Hence there is $n \in \mathbf{G}$ which belongs to the normalizer of $T^* \subset G^*$ such that $\mathrm{Int}(n) \circ \psi^*$ acts as -1 on the root system of T^*. Now, since ψ^* preserves the set of positive roots, $w = \mathrm{Int}(n)|_{T^*}$ is the element of maximal length in the Weyl group. This shows that (iii) implies (v). The equivalence of (iii) and (iv) is a well-known theorem [Har66, Theorem 13] due to Harish-Chandra. Finally, Lemma 4 below shows that (v) implies (ii). $\square$

Lemma 4. *Assume $w \circ \psi^*$ acts by -1 on the root system. Then G^* has an inner form U such that $\mathbf{U} = U(\mathbb{R})$ is compact.*

Proof. Consider the complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Let Σ be the set of roots, Σ^+ the set of positive roots and $\mathfrak{g}_\alpha$ the vector space attached to $\alpha \in \Sigma$ with respect to the torus $T^*(\mathbb{C})$. Following Weyl [Wey26], Chevalley [Che55] and Tits [Tit66], one may choose elements $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Sigma$ such that, if we define $H_\alpha \in \mathrm{Lie}(T^*(\mathbb{C}))$ by $H_\alpha = [X_\alpha, X_{-\alpha}]$, we have

$$\alpha(H_\alpha) = 2 \quad \text{and} \quad [X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Sigma \quad \text{with } N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \mathbb{Z}.$$

We assume the splitting compatible with this choice. Now let

$$Y_\alpha = X_\alpha - X_{-\alpha}, \quad Z_\alpha = i(X_\alpha + X_{-\alpha}), \quad W_\alpha = iH_\alpha.$$

The elements Y_α and Z_α for $\alpha \in \Sigma^+$ together with the W_α for $\alpha \in \Delta$ build a basis for a real Lie algebra $\mathfrak{u}$. As in the proof of [Hel62, Chapter III, Theorem 6.3], we see that the Killing form is negative definite on $\mathfrak{u}$ and hence the Lie subgroup $\mathbf{U} \subset \mathbf{G}$ with Lie algebra $\mathfrak{u}$ is compact. Since ψ^* preserves the splitting, $\psi^*(X_\alpha) = X_{\psi^*(\alpha)}$ for $\alpha \in \Delta$. Let w be the element of maximal length in the Weyl group for T^*. There is an $n^* \in \mathbf{G}$, uniquely determined modulo the center, such that the inner automorphism $w^* = \mathrm{Int}(n^*)$ acts as w on T^* and such that $w^*(X_\alpha) = -X_{w\alpha}$ for $\alpha \in \Delta$. This automorphism is of order 2 and commutes with ψ^*. Now let $\phi = w^* \circ \psi^*$. Since $w \circ \psi^*$ acts by -1 on Σ this implies $\phi(X_\alpha) = -X_{\phi(\alpha)} = -X_{-\alpha}$ for $\alpha \in \Delta$. It follows from the commutation relations and the relations $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ that $\phi(X_\alpha) = -X_{-\alpha}$ and $\phi(H_\alpha) = -H_\alpha$ for all $\alpha \in \Sigma$. Now ϕ, which acts as an automorphism of the real Lie algebras $\mathfrak{g}^{**}$ generated by the X_α for $\alpha \in \Sigma$, can be extended to an antilinear involution of $\mathfrak{g}^{**} \otimes \mathbb{C} = \mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$. This, in turn, induces a Cartan involution $\boldsymbol{\theta}$ on $\mathbf{G}$: its fixed point set is the compact group $\mathbf{U} = U(\mathbb{R})$ with Lie algebra $\mathfrak{u}$, and U is the inner form of G^* defined by the Galois cocycle $a_1 = 1$ and $a_\sigma = w^*$. $\square$

We observe that when, moreover, G^* is almost simple, which means that the root system of G^* is irreducible, the classification shows that condition (v) holds except when G^* is split of type A_n with $n \geqslant 2$, or D_n with $n \geqslant 3$ odd, or E_6 or when G^* is quasi-split but non-split of type D_n with $n \geq 4$ even.

Acknowledgements

We thank Laurent Clozel and Raphaël Beuzart-Plessis for very useful discussions as well as the referee for pertinent remarks.

https://doi.org/10.1112/S0010437X21007181 Published online by Cambridge University Press

References

Ada14 J. Adams, *The real Chevalley involution*, Compos. Math. **150** (2014), 2127–2142.

BLS96 A. Borel, J.-P. Labesse and J. Schwermer, *On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields*, Compos. Math. **102** (1996), 1–40.

Che55 C. Chevalley, *Sur certains groupes simples*, Tohoku Math. J. (2) **7** (1955), 14–66.

Har66 Harish-Chandra, *Discrete series for semisimple Lie groups. II. Explicit determination of the characters*, Acta Math. **116** (1966), 1–111.

Hel62 S. Helgason, *Differential geometry and symmetric spaces*, Pure and Applied Mathematics, vol. XII (Academic Press, New York, 1962).

Lab99 J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque **257** (1999).

Lan89 R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, in *Representation theory and harmonic analysis on semisimple Lie groups*, Mathematical Surveys and Monographs, vol. 31 (American Mathematical Society, Providence, RI, 1989), 101–170.

She79 D. Shelstad, *Characters and inner forms of a quasi-split group over* $\mathbb{R}$, Compos. Math. **39** (1979), 11–45.

Tit66 J. Tits, *Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples*, Publ. Math. Inst. Hautes Études Sci. **31** (1966), 21–58.

Wey26 H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. II*, Math. Z. **24** (1926), 328–376.

Jean-Pierre Labesse jean-pierre.labesse@univ-amu.fr
Institut de Mathématique de Marseille, UMR 7373, Aix-Marseille Université,
13288 Marseille, France

Joachim Schwermer joachim.schwermer@univie.ac.at
Faculty of Mathematics, University Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

Current address: Max-Planck-Institute for Mathematics, Vivatsgasse 7,
D-53111 Bonn, Germany

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