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ON THE ADJOINT HOMOLOGY OF 2-STEP NILPOTENT LIE ALGEBRAS

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We give a lower bound and an upper bound for the dimension of the homology of 2-step nilpotent Lie algebras with adjoint coefficients. We conjecture, that the upper bound and the actual dimension are asymptotically equivalent.

The homology and cohomology of nilpotent Lie algebras $\mathfrak{n}$ is not yet well understood and in general very little is known about how to compute it effectively or how to interpret it.

In the case of trivial coefficients many efforts have been made to bound or estimate Betti numbers or their sum. For non trivial coefficients almost nothing is known. Dixmier [3] showed that $H_*(\mathfrak{n}, M) = 0$ unless M contains a trivial subquotient. The adjoint module $M = \mathfrak{n}$ contains trivial subquotients and is an example for which $H_*(\mathfrak{n}, M) \neq 0$.

In this note we consider the homology of 2-step nilpotent Lie algebras with adjoint coefficients. We provide a lower bound and an upper bound for $\dim H_*(\mathfrak{n}, \mathfrak{n})$. These bounds together with existing results suggest that in general $\dim H_*(\mathfrak{n}, \mathfrak{n})$ is close to $\dim H_*(\mathfrak{n}) \times \dim(\mathfrak{n})$. We state a precise conjecture in this direction.

For simplicity, all Lie algebras and modules considered here are finite dimensional over the complex field.

1. A LOWER BOUND FOR $H_*(\mathfrak{n}, \mathfrak{n})$

In this section we generalises some results in [7].

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra, let $\mathfrak{z}$ be its centre and V any complement of $\mathfrak{z}$ in $\mathfrak{n}$, so that

$$\mathfrak{n} = V \oplus \mathfrak{z}. \tag{1.1}$$

This decomposition is a graded decomposition, since $[V, V] \subseteq \mathfrak{z}$. Decompose the complex $(\Lambda\mathfrak{n} \otimes \mathfrak{n}, \partial)$ which computes the adjoint homology of $\mathfrak{n}$, as a direct sum of an even and

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an odd subcomplexes. For this recall that

$$\partial(x_1 \wedge \ldots \wedge x_p \otimes w) = \sum_{i<j}(-1)^{i+j+1}[x_i, x_j] \wedge x_1 \ldots \wedge \widehat{x_i} \ldots \wedge \widehat{x_j} \ldots \wedge x_p \otimes w$$
$$+ \sum_{i=1}^{p}(-1)^{i+1} x_1 \ldots \wedge \widehat{x_i} \ldots \wedge x_p \otimes \mathrm{ad}(x_i)(w).$$

Let,

$$C_p^{\text{even}} = \Lambda^{2p} V \otimes \Lambda\mathfrak{z} \otimes \mathfrak{z} + \Lambda^{2p-1} V \otimes \Lambda\mathfrak{z} \otimes V;$$
$$C_q^{\text{odd}} = \Lambda^{2q+1} V \otimes \Lambda\mathfrak{z} \otimes \mathfrak{z} + \Lambda^{2q} V \otimes \Lambda\mathfrak{z} \otimes V.$$

It is clear that $\Lambda\mathfrak{n} \otimes \mathfrak{n} = C^{\text{even}} \oplus C^{\text{odd}}$, and it is straightforward to verify that $\partial : C_p^{\text{even}} \longrightarrow C_{p-1}^{\text{even}}$ and $\partial : C_q^{\text{odd}} \longrightarrow C_{q-1}^{\text{odd}}$.

Theorem 1.1. *Let $\mathfrak{n} = V \oplus \mathfrak{z}$ be a 2-step nilpotent Lie algebra with centre $\mathfrak{z}$. If $z = \dim \mathfrak{z}$ and $v = \dim V$, then*

$$\dim H_*(\mathfrak{n}, \mathfrak{n}) \geqslant 2^{z+[v/2]} \times \begin{cases} v + z, & \text{if } v \text{ is even;} \\ v + z + |v - z| & \text{if } v \text{ is odd.} \end{cases}$$

Proof: A lower bound for $\dim H_*(\mathfrak{n}, \mathfrak{n})$ is

$$\left|\sum_{p\geqslant 0}(-1)^p \dim C_p^{\text{even}}\right| + \left|\sum_{q\geqslant 0}(-1)^q \dim C_q^{\text{odd}}\right|.$$

Writing $P = \sum_{p\geqslant 0}(-1)^p \binom{v}{2p}$ and $Q = \sum_{q\geqslant 0}(-1)^q \binom{v}{2q+1}$, we get that

$$\dim H_*(\mathfrak{n}, \mathfrak{n}) \geqslant 2^z \left(|zP - vQ| + |zQ + vP|\right).$$

Since $P = \left((1+i)^v + (1-i)^v\right)/2$ and $Q = i^{-1}\left((1+i)^n - (1-i)^v\right)/2$ (where $i^2 = -1$) it turns out that $P = r2^{[v/2]}$ and $Q = s2^{[v/2]}$, where r and s are given in the following table according to the congruence of v mod 8.

	0	1	2	3	4	5	6	7
r	1	1	0	-1	-1	-1	0	1
s	0	1	1	1	0	-1	-1	-1

Therefore, we have that

	0	1	2	3	4	5	6	7
$rz - sv$	z	$z - v$	$-v$	$-z - v$	$-z$	$-z + v$	v	$z + v$
$sz + rv$	v	$z + v$	z	$z - v$	$-v$	$-z - v$	$-z$	$-z + v$

and the theorem follows. ∎

REMARK 1.2. Theorem 1.1 remains valid, with the same proof, if the adjoint module is replace for a module W satisfying the following two conditions:

(1) There is a vector space decomposition $W = W_1 \oplus W_2$ such that $x.W_0 \subseteq W_1$ and $x.W_1 \subseteq W_0$, for all $x \in \mathfrak{n}$;

(2) $z.W = 0$ for all $z \in \mathfrak{z}$.

These more general class of modules include, for example, the tensor, symmetric and exterior powers of the adjoint module. For example, for $M = \Lambda\mathfrak{n}$, let $W_0 = \sum_{i=\text{even}} \Lambda^i V \otimes \Lambda\mathfrak{z}$ and $W_1 = \sum_{i=\text{odd}} \Lambda^i V \otimes \Lambda\mathfrak{z}$.

Also the coadjoint module $\mathfrak{n}^*$ and its powers (tensor, symmetrica and exterior) are included. Notice that $H_*(\mathfrak{n}, \mathfrak{n}^*) = H^*(\mathfrak{n}, \mathfrak{n})$.

2. AN UPPER BOUND FOR $H_*(\mathfrak{n}, \mathfrak{n})$

Through this section $\mathfrak{n}$ is any nilpotent Lie algebra, not necessarily 2-step nilpotent.

Let us fix a vector space W of dimension n. Let $\mathcal{L}$ be the algebraic variety of n-dimensional representations of $\mathfrak{n}$. That is linear maps $\rho : \mathfrak{n} \longrightarrow \mathfrak{gl}(W)$ satisfying the polynomial restrictions that make ρ a Lie algebra morphism.

The group $GL(W)$ acts on $\mathcal{L}$ by *change of basis*. Precisely, if ρ is a representation and $A \in GL(W)$, then $A.\rho(x) = A\rho(x)A^{-1}$ for all $x \in \mathfrak{n}$. The $GL(W)$-orbit of ρ consists of precisely all representations μ which are isomorphic to ρ.

A representation ρ is said to *degenerate* to a representation μ, $\rho \to \mu$, if μ is in the closure of the orbit of ρ.

PROPOSITION 2.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra and let ρ be any nilpotent representation of $\mathfrak{n}$ of dimension n. Then ρ degenerates to the n-dimensional trivial representation.*

PROOF: Let W be the representation space of ρ. Take a basis for W such that $\rho(x)$ is in lower triangular form for every $x \in \mathfrak{n}$ and let $M_i = \rho(x_i)$, where $\{x_1, \dots, x_n\}$ is a basis of $\mathfrak{n}$. If A_t is the diagonal matrix with entries $t, t^2, \dots, t^n$, then conjugating the matrices M_i by A_t has the effect of multiplying the first diagonal below the main one of M_i by t, the second one by t^2, et cetera. Therefore, $\lim_{t \to 0} A_t M_i A_t^{-1} = 0$, for all i. ∎

PROPOSITION 2.2. *Let ρ and μ be two representations of $\mathfrak{n}$ on the same representation space W. If ρ degenerates to μ, then*

$$\dim H_*(\mathfrak{n}, \mu) \geqslant \dim H_*(\mathfrak{n}, \rho).$$

Moreover, $\dim H_p(\mathfrak{n}, \mu) \geqslant \dim H_p(\mathfrak{n}, \rho)$ *for* $p = 0 \dots \dim \mathfrak{n}$.

PROOF: Let ∂_p^ρ and ∂_p^μ be the p-differentials of the complexes that compute the homology of $\mathfrak{n}$ with ρ and μ coefficients respectively. Both are linear transformations from $\Lambda^p\mathfrak{n}\otimes W$ to $\Lambda^{p-1}\mathfrak{n}\otimes W$. It is easy to see that ∂_p^ρ *degenerates* to ∂_p^μ as linear transformations from one space to another under change of basis of both spaces. Then $\dim\ker\partial_p^\rho \leqslant \dim\ker\partial_p^\mu$ and $\dim\operatorname{Im}\partial_{p+1}^\rho \geqslant \dim\operatorname{Im}\partial_{p+1}^\mu$, for every p. $\square$

The following theorem is now an immediate corollary of these propositions.

Theorem 2.3. *For any nilpotent Lie algebra* $\mathfrak{n}$,

$$\dim H_*(\mathfrak{n},\mathfrak{n}) \leqslant \dim H_*(\mathfrak{n}) \times \dim\mathfrak{n}.$$

3. Two examples and a conjecture

Example 1. (Heisenberg Lie algebras.)

Let $\mathfrak{h}(m)$ be the $(2m+1)$-dimensional Heisenberg Lie algebra. The Betti numbers of $\mathfrak{h}(m)$ are

$$\beta_i = \beta_{2m+1-i} = \binom{2m}{i} - \binom{2m}{i-2}, \qquad 0 \leqslant i \leqslant m,$$

(see [6] or [2, Corollary 4.4]) and hence the total homology is

$$TH(m) = \sum_{i=0}^{2m+1} \beta_i = 2\binom{2m+1}{m}.$$

The adjoint Betti numbers, $\beta_i^{\mathrm{ad}} = \dim H_i(\mathfrak{h}(m),\mathfrak{h}(m))$, have been computed in [5] (see also [2, Corollary 4.16]).

$$\begin{aligned}
\beta_0^{\mathrm{ad}} &= 1;\\
\beta_i^{\mathrm{ad}} &= (2m+1)\binom{2m+1}{i} - \binom{2m+1}{i+1} - 2m\binom{2m+1}{i-1}, \quad 1 \leqslant i \leqslant m;\\
\beta_{m+1}^{\mathrm{ad}} &= (2m+1)\left[\binom{2m}{m} - \binom{2m}{m-2}\right] - \binom{2m}{m-1} + \binom{2m}{m-3};\\
\beta_{2m+1-i}^{\mathrm{ad}} &= 2m\left[\binom{2m}{i} - \binom{2m}{i-2}\right] - \binom{2m}{i-1} + \binom{2m}{i-3}, \quad 0 \leqslant i \leqslant m-1.
\end{aligned}$$

It is straightforward to compute the total adjoint homology,

$$\begin{aligned}
TH^{\mathrm{ad}}(m) &= \sum_{i=0}^{2m+1} \beta_i^{\mathrm{ad}}\\
&= 2 - \binom{2m}{m+1} + 4m\binom{2m+1}{m} - \binom{2m}{m-1} - 2\binom{2m}{m-2}.
\end{aligned}$$

We have that

$$(2m+1) - \frac{TH^{\mathrm{ad}}(m)}{TH(m)} \longrightarrow 2, \quad \text{as } m \to \infty,$$

from which it follows that

$$\frac{TH^{\mathrm{ad}}(m)}{TH(m).\dim\mathfrak{h}(m)} \longrightarrow 1, \quad \text{as } m \to \infty.$$

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EXAMPLE 2. (Free 2-step nilpotent Lie algebras.)

$\mathfrak{L}(r) = V \oplus \Lambda^2 V$, where $\dim V = r$, is the free 2-step nilpotent Lie algebra of rank r. A closed formula for the total homology of $\mathfrak{L}(r)$, $TH(r)$, is stated in [4]. In [1], the adjoint homology of $\mathfrak{L}(r)$ was computed and for their total adjoint homology, $TH^{\mathrm{ad}}(r)$, it was found that

$$TH^{\mathrm{ad}}(r) = \frac{1}{2}\left(r(r-1) + \frac{[(r+1)/2]}{[(r+1)/2] - 1/2}\right)TH(r).$$

Since $\dim \mathfrak{L}(r) = r + \big(r(r-1)\big)/2$, it follows that

$$\frac{TH^{\mathrm{ad}}(r)}{TH(r).\dim \mathfrak{L}(r)} \longrightarrow 1, \quad \text{as } r \to \infty.$$

These two examples, Heisenberg Lie algebras and free 2-step Lie algebras are both extreme cases of 2-step nilpotent Lie algebras. Among non-abelian 2-step Lie algebras generated by a given (even) number of generators the Heisenberg Lie algebras are the smallest while the free ones are the largest.

The trivial homology of $\mathfrak{L}(r)$ is as small as possible for a (non-abelian) 2-step Lie algebra (see [4]) while we suspect the trivial homology of Heisenberg Lie algebras is as big as possible, for 2-step Lie algebras without Abelian factor.

We make the following conjecture.

CONJECTURE. There is a convergent sequence of numbers $a_n \longrightarrow 0$, such that for every 2-step nilpotent Lie algebra of dimension n,

$$\left|\frac{TH^{\mathrm{ad}}(\mathfrak{n})}{TH(\mathfrak{n}).\dim \mathfrak{n}} - 1\right| < a_n.$$

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