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AMENABILITY AND TRANSLATION EXPERIMENTS

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In [**11**] it is shown that the deficiency of a translation experiment with respect to another on a σ-finite, amenable, locally compact group can be calculated in terms of probability measures on the group. This interesting result, brought to the writer's notice by [**1**], does not seem to be as well-known in the theory of amenable groups as it should be. The present note presents a simple proof of the result, removing the σ-finiteness condition and repairing a gap in Torgersen's argument. The main novelty is the use of Wendel's multiplier theorem to replace Torgersen's approach which is based on disintegration of a bounded linear operator from $L_1(G)$ into $C(G)^*$ for G σ-finite (c.f. [**5**], VI.8.6). The writer claims no particular competence in mathematical statistics, but hopes that the discussion of the above result from the "harmonic analysis" perspective may prove illuminating.

We then investigate a similar issue for discrete semigroups. A set of transition operators, which reduce to multipliers in the group case, is introduced, and a semigroup version of Torgersen's theorem is established.

The author is greatly indebted to the referee for pointing out an error in the original version of Theorem 1, and for his suggestions which have clarified the exposition.

We now discuss some preliminaries. Let X be a Banach space and S be a semigroup. Let $B(X)$ be the algebra of bounded linear operators on X. Let $\Phi: S \to B(X)$ be a homomorphism (anti-homomorphism) with $\sup_{s\in S} \|\Phi(s)\| < \infty$. Then X is called a *left* (*right*) *Banach S-space* (with respect to Φ). We write $sx(xs)$ for $\Phi(s)(x)$ $(x \in X)$. If X is a left (right) Banach S-space, then the dual X' is a right (left) Banach S-space, where for $\alpha \in X'$, we have $\alpha s(x) = \alpha(sx)$ $(s\alpha(x) = \alpha(xs))$.

Now let G be a locally compact group. A left Haar measure on G will be denoted by λ, and $L_1(G)$ is the group algebra of G. The convolution algebra of bounded, complex, regular Borel measures on G is denoted by $M(G)$. Then $M(G)$ is a left (right) Banach G-space where $x\mu = \delta_x * \mu$ $(\mu x = \mu * \delta_x)$. We shall often write $x * \mu$ $(\mu * x)$ in place of $x\mu$ (μx). It is well known that $L_1(G)$ can be regarded as an ideal in $M(G)$. The semigroup of probability measures in $M(G)$ $(L_1(G))$ is denoted by $PM(G)$ $(P(G))$. The Banach space of essentially bounded, complex-valued, measurable functions on G is denoted by $L_\infty(G)$. The following

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closed subspaces of $L_\infty(G)$ will be relevant:

$$C(G) = \{\phi \in L_\infty(G) : \phi \text{ is continuous}\}$$
$$C(G_\infty) = \{\phi \in C(G) : \phi(x) \text{ tends to a limit as } x \to \infty\}$$
$$C_0(G) = \{\phi \in C(G) : \phi(x) \to 0 \text{ as } x \to \infty\}.$$

Note that $C(G_\infty)$ "is" the space of continuous, complex-valued functions on the one-point compactification G_∞ of G. We can make $L_\infty(G)$ and the three subspaces above into left (right) Banach G-spaces by defining

$$x\phi(y) = \phi(yx), \quad \phi x(y) = \phi(xy).$$

We can regard $M(G)$ as a subspace of the dual of each of $C(G)$, $C(G_\infty)$ and $C_0(G)$, and the following important equalities hold:

$$x * \mu(\phi) = \mu(\phi x), \quad \mu * x(\phi) = \mu(x\phi).$$

Similar spaces and properties apply in the case of a discrete semigroup S.

As in [**9**], Section 2, it is convenient to use the framework of (complex) L-space theory for our discussion. The reason for this is that a number of naturally occurring spaces (e.g. $C(G)'$) are L-spaces but are not explicitly given as L_1-spaces. Complex L-spaces are defined in [**10**], p. 138, and it is noted that the obvious analogues of many standard, real L-space results are valid. In particular, every L-space $\mathfrak{A}$ can be identified with some $L_1(X, \mu)$, where μ is a positive, regular Borel measure on a locally compact, Hausdorff space X, the order structures on $\mathfrak{A}$ and $L_1(X, \mu)$ corresponding. Under this identification, we write $\int a$ for $\int_X a\, d\mu$. Recall ([**10**], p. 56) that a vector subspace A of $\mathfrak{A}$ is called an *ideal* if whenever $a \in A$, $x \in \mathfrak{A}$ and $|x| \leqq |a|$, then $x \in A$. A version of the Lebesgue decomposition theorem states ([**10**], Proposition (8.3), (iii)) that if A is a closed ideal in $\mathfrak{A}$ and ([**10**], p. 50)

$$A^\perp = \{x \in \mathfrak{A} : |x| \wedge |a| = 0 \text{ for all } a \in A\},$$

then $\mathfrak{A} = A \oplus A^\perp$. If Π is the associated projection from $\mathfrak{A}$ into A, then $\|\Pi\| = 1$ and $\Pi(x) \geqq 0$ if $x \geqq 0$.

We note that the dual of any complex M-space (e.g. $C(G)$) is an L-space ([**10**], p. 121).

The definitions of this paragraph are based on [**8**], [**9**]. An *experiment* $\mathscr{E}$ is a pair $\{\mathfrak{A}, \{P_\theta : \theta \in \Theta\}\}$ where $\mathfrak{A}$ is a complex L-space and P_θ is a norm one, positive element of $\mathfrak{A}$ for each θ in the index set Θ. The closed ideal generated by the set $\{P_\theta : \theta \in \Theta\}$ in $\mathfrak{A}$ is denoted by $\mathscr{L}(\mathscr{E})$. If $\mathfrak{B}$ and $\mathfrak{C}$ are L-spaces, then a *transition* from $\mathfrak{B}$ to $\mathfrak{C}$ is a positive linear map $T: \mathfrak{B} \to \mathfrak{C}$ such that $\|T\mu\| = \|\mu\|$ for all $\mu \in \mathfrak{B}$ with $\mu \geqq 0$. Every transition has norm one, and the set $\mathfrak{J}(\mathfrak{B}, \mathfrak{C})$ of transitions is obviously a

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convex subset of $B(\mathfrak{B}, \mathfrak{C})$, the Banach space of bounded linear operators from $\mathfrak{B}$ to $\mathfrak{C}$. If

$$\mathscr{E} = \{\mathfrak{A}, \{P_\theta : \theta \in \Theta\}\} \quad \text{and} \quad \mathscr{F} = \{\mathfrak{B}, \{Q_\theta : \theta \in \Theta\}\}$$

are experiments, then a measure of the amount of information yielded by $\mathscr{E}$ relative to $\mathscr{F}$ is given by the number $\delta(\mathscr{E}, \mathscr{F})$, where

$$(1) \quad \delta(\mathscr{E}, \mathscr{F}) = \inf \{\sup_\theta \|Q_\theta - T(P_\theta)\| : T \in \mathfrak{J}(\mathscr{L}(\mathscr{E}), \mathscr{L}(\mathscr{F}))\}.$$

In general, the computation of $\delta(\mathscr{E}, \mathscr{F})$ is difficult. However, the situation is better when we consider translation experiments. A *translation experiment* is one of the form $\mathscr{E}_P$, where $P \in P(G)$ and

$$\mathscr{E}_P = \{L_1(G), \{y * P : y \in G\}\}.$$

It is easy to see that $\mathscr{L}(\mathscr{E}_P) = L_1(G)$. (Indeed, it is obvious that $x * \mathscr{L}(\mathscr{E}_P) = \mathscr{L}(\mathscr{E}_P)$, so that $\mathscr{L}(\mathscr{E}_P)$ is a closed (algebraic) left ideal in $L_1(G)$. We can suppose that e, the identity of G, belongs to the support of P. If V is open and relatively compact in G with $e \in V$ and $Q = P_{|V}$, then $Q/\|Q\|_1 \in \mathscr{L}(\mathscr{E}_P)$. Thus we obtain an approximate identity for $L_1(G)$ in $\mathscr{L}(\mathscr{E}_P)$ ([**7**], (20.15)) so that $\mathscr{L}(\mathscr{E}_P) = L_1(G)$.)

In [**11**], Torgersen defines $\delta(\mathscr{E}_P, \mathscr{E}_Q)$ for P, Q in $P(G)$ as in (1) above except that $\mathfrak{J}(\mathscr{L}(\mathscr{E}_P), \mathscr{L}(\mathscr{E}_Q))$ $(= \mathfrak{J}(L_1(G), L_1(G)))$ is replaced by $\mathfrak{J}(L_1(G), C(G)')$. It follows from the following simple proposition that the above difference does not matter.

PROPOSITION 1. *Let* $\{\mathfrak{C}, \{P_\theta : \theta \in \Theta\}\}$ *and* $\{\mathfrak{D}, \{Q_\theta : \theta \in \Theta\}\}$ *be experiments and let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be closed ideals of L-spaces* $\mathfrak{A}$ *and* $\mathfrak{B}$ *respectively. Then*

$$(2) \quad \inf \{\sup_\theta \|Q_\theta - T(P_\theta)\| : T \in \mathfrak{J}(\mathfrak{A}, \mathfrak{B})\} = \inf \{\sup_\theta \|Q_\theta - T(P_\theta)\| : T \in \mathfrak{J}(\mathfrak{C}, \mathfrak{D})\}.$$

Proof. Write $\mathfrak{A} = \mathfrak{C} \oplus \mathfrak{C}^\perp$ and $\mathfrak{B} = \mathfrak{D} \oplus \mathfrak{D}^\perp$, and let P_1 and P_2 be the canonical projections from $\mathfrak{A}$ onto $\mathfrak{C}$ and from $\mathfrak{B}$ onto $\mathfrak{D}$. Pick $\mu \in \mathfrak{B}$ with $\mu \geqq 0$, $\|\mu\| = 1$. Let $T \in \mathfrak{J}(\mathfrak{C}, \mathfrak{D})$. We can define $S \in \mathfrak{J}(\mathfrak{A}, \mathfrak{B})$ by setting

$$S(a) = T(P_1(a)) + (\textstyle\int (a - P_1(a)))\mu.$$

Then $\|Q_\theta - S(P_\theta)\| = \|Q_\theta - T(P_\theta)\|$ and LHS $\leqq$ RHS in (2).

We now establish the reverse inequality. Pick $\nu \in \mathfrak{D}$ with $\nu \geqq 0$, $\|\nu\| = 1$. Let $S' \in \mathfrak{J}(\mathfrak{A}, \mathfrak{B})$. Define $T' \in \mathfrak{J}(\mathfrak{C}, \mathfrak{D})$ by:

$$T'(c) = P_2(S'(c)) + (\textstyle\int (S'(c) - P_2(S'(c))))\nu.$$

Then $\|Q_\theta - S'(P_\theta)\| = \|Q_\theta - T'(P_\theta)\|$, and the reverse inequality is established.

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COROLLARY. *If $\mathfrak{B}$ is an L-space with $L_1(G)$ as a closed ideal, then*

$$\delta(\mathscr{E}_P, \mathscr{E}_Q) = \inf \{\sup_x \|x * Q - T(x * P)\| : T \in \mathfrak{J}(L_1(G), \mathfrak{B})\}.$$

The above corollary applies when $\mathfrak{B}$ is $M(G)$, $C(G)'$, $L_\infty(G)'$ or $C(G_\infty)'$. For our purposes, it will be convenient to use $\mathfrak{B} = C(G_\infty)'$.

Observe that $C(G_\infty)' = M(G_\infty)$, the space of bounded, complex, regular Borel measures on G_∞. We now make $X = B(L_1(G), M(G_\infty))$ into a left Banach G-space. We give $M(G_\infty)$ the dual Banach G-space structure induced by $C(G_\infty)$: so

$$x * \xi(\phi) = \xi(\phi x) \quad (x \in G, \phi \in C(G_\infty), \xi \in M(G_\infty)).$$

For $T \in X$, define $xT \in X$ by

$$xT(\mu) = x * (T(x^{-1} * \mu)) \quad (\mu \in L_1(G)).$$

One can check directly that X is a left Banach G-space. Let $\mathscr{S}$ be the topology on the unit ball X_1 of X regarded as a subset of $(M(G_\infty)_1)^{(L_1(G))_1}$, $(M(G_\infty))_1$ being given the relative weak* topology. So $T_\delta \rightarrow T$ in X_1 if and only if $T_\delta\mu \rightarrow T\mu$ weak* for all $\mu \in L_1(G)$. One readily checks that $(X_1, \mathscr{S})$ is compact Hausdorff. (A more instructive approach to the above is to make $L_1(G) \hat{\otimes} C(G_\infty)$ into a right Banach G-space by requiring $(\mu \otimes \phi)x = (x^{-1} * \mu) \otimes \phi x$, and then to identify X with $(L_1(G) \hat{\otimes} C(G_\infty))'$. Then X is given the dual G-space structure, and $\mathscr{S}$ is the restriction of the weak* topology to X_1.)

The following theorem in the σ-finite case is stated in [**11**]. In [**11**], Torgersen uses a set similar to K_ϵ below but in $B(L_1(G), C(G)')$ rather than $B(L_1(G), M(G_\infty))$. Unfortunately, his topology on K_ϵ is not Hausdorff, so that the fixed-point theorem (in [**3**]) he wishes to use does not apply. Further, as pointed out in [**4**], the fixed-point theorem for $C(G)'$ given in [**3**] is incorrect.

THEOREM 1. *Let G be amenable. Then if P, $Q \in P(G)$, we have*

$$(3) \quad \delta(\mathscr{E}_P, \mathscr{E}_Q) = \inf \{\|Q - P * \nu\| : \nu \in P(G)\}.$$

Proof. Let $\nu \in P(G)$ and define $T_\nu \in \mathfrak{J}(L_1(G), L_1(G))$ by:

$$T_\nu(\mu) = \mu * \nu.$$

Then

$$\delta(\mathscr{E}_P, \mathscr{E}_Q) \leqq \|Q - T_\nu(P)\| = \|Q - P * \nu\|$$

so that LHS $\leqq$ RHS in (3). It remains to prove the reverse inequality.

Let $\epsilon > 0$ and let

$$K_\epsilon = \{T \in \mathfrak{J}(L_1(G), M(G_\infty)): \|y * Q - T(y * P)\| \leqq \delta(\mathscr{E}_P, \mathscr{E}_Q) + \epsilon, y \in G\}.$$

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(In the obvious way, $y * Q$ is regarded as an element of $C(G_\infty)' = M(G_\infty)$.) If $T_\delta \to T$ in X_1 with $\{T_\delta\} \subset K_\epsilon$, then $T_\delta \mu \to T\mu$ weak* for each $\mu \in P(G)$, so that $T\mu \geqq 0$ and

$$T\mu(1) = \lim T_\delta\mu(1) = 1.$$

It follows that $T \in K_\epsilon$, so that K_ϵ is a compact subset of X_1. Clearly K_ϵ is also convex in X. Further, using Proposition 1, Corollary, K_ϵ is not empty.

As $\|y * Q - (xT)(y * P)\| = \|x^{-1}y * Q - T(x^{-1}y * P)\|$, we have $xK_\epsilon = K_\epsilon$ for all $x \in G$.

Now let $x_\delta \to x$ in G and $T_\delta \to T$ in K_ϵ. If $\mu \in L_1(G)$, $\phi \in C(G_\infty)$, then

$$\|x_\delta * \mu - x * \mu\|_1 \to 0$$

([**7**], (20.4)), and since ϕ is uniformly continuous, we also have

$$\|\phi x_\delta - \phi x\| \to 0.$$

So

$$(x_\delta T_\delta - xT)(\mu)(\phi) = (T_\delta(x_\delta^{-1} * \mu)(\phi x_\delta) - T(x^{-1} * \mu)(\phi x)) \to 0.$$

Hence the map $(x, T) \to xT$ is jointly continuous on K_ϵ, and since G is amenable, there exists ([**6**], Theorem 3.3.1) an element $T \in K_\epsilon$ such that $xT = T$ for all $x \in G$. Unravelling xT, we see that T is a multiplier in the sense that

$$T(x * \mu) = x * T\mu \quad (x \in G, \mu \in L_1(G)).$$

Obviously, $L_1(G)$ is a closed ideal of the L-space $M(G_\infty)$. The elements of $L_1(G)^\perp$ are of the form $\nu + k\delta_\infty$, where $\nu \in M(G)$ and the Haar measure λ are mutually singular, $k \in \mathbf{C}$ and ∞ is the "point at infinity". Observing that $x * \delta_\infty = \delta_\infty$ for all $x \in G$, we see that the canonical projection $\Pi : M(G_\infty) \to L_1(G)$ is such that $\Pi(x * \mu) = x * \Pi\mu$ for all μ. Hence $\Pi \circ T$ is a multiplier on $L_1(G)$, and so by Wendel's theorem ([**12**]), there exists $\nu \geqq 0$ in $M(G)$ such that $\Pi \circ T = T_\nu$. Find $\nu_1 \geqq 0$ in $L_1(G)$ such that $(\nu + \nu_1) \in PM(G)$ and let $T_1 = T_{(\nu+\nu_1)}$. Then if $\mu \in P(G)$, we have

$$\begin{aligned}\|\Pi \circ T(\mu)\| + \|\mu\nu_1\| = \|T_1\mu\| = 1 = \|T\mu\| \\ = \|\Pi \circ T(\mu)\| + \|(I - \Pi) \circ T(\mu)\|\end{aligned}$$

so that $\|\mu\nu_1\| = \|(I - \Pi) \circ T(\mu)\|$. Hence

$$\begin{aligned}\|y * Q - T_1(y * P)\| &\leqq \|y * Q - (\Pi \circ T)(y * P)\| + \|(y * P) * \nu_1\| \\ &= \|y * Q - (\Pi \circ T)(y * P)\| \\ &\qquad + \|(I - \Pi) \circ T(y * P)\| \\ &= \|y * Q - T(y * P)\|.\end{aligned}$$

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So $T_1 \in K_\epsilon$. Now replace $(\nu_1 + \nu)$ by $e_\delta * (\nu_1 + \nu)$, where $\{e_\delta\}$ is an approximate identity for $L_1(G)$ in $P(G)$, to obtain the required inequality.

Note. The present writer does not know if the converse is true, i.e., if every group G for which (3) holds, is amenable. The case of the free group on two generators is obviously worth investigating in this context. The main difficulty involved seems to be the computation of $\delta(\mathscr{E}_P, \mathscr{E}_Q)$ when G is not amenable. The present writer speculates that the converse is false.

We now investigate an analogous theorem for the case of a left amenable (discrete) semigroup S. The present writer is unsure of what significance (if any) the following results have for statistics.

Let S be a semigroup and $P, Q \in P(S)$. (So P, Q are of the form $\sum_{s\in S} \alpha_s \delta_s$, where $\alpha_s \geqq 0$ for all s, and $\sum_{s\in S} \alpha_s = 1$.) We define

$$(4) \quad \delta(P, Q) = \inf \{\sup_{x\in S} \|Q - x * (T(P * x))\| : T \in \mathfrak{J}(l_1(S), l_1(S))\}.$$

If S is a group, then $\delta(P, Q) = \delta(\mathscr{E}_{P^*}, \mathscr{E}_Q)$, where $P^*(x) = P(x^{-1})$. (Indeed

$$\|Q - x * (T(P * x))\| = \|x^{-1} * Q - T^*(x^{-1} * P^*)\|$$

where $T^*(\mu) = T(\mu^*)$ for $\mu \in P(S)$.) It follows using Proposition 1, Corollary, that we can replace $\mathfrak{J}(l_1(S), l_1(S))$ in (4) by $\mathfrak{J}(l_1(S), l_\infty(S)')$.

The analogue of the multipliers arising in the proof of Theorem 1 is now introduced. We take X to be $B(l_1(S), l_\infty(S)')$. (Note that we cannot use the analogue $C(S_\infty)$ of $C(G_\infty)$ since ϕx need not belong to $C(S_\infty)$ if ϕ does.) In the obvious way, $l_\infty(S)'$ is a Banach S-space and we define a left Banach S-space structure on X by requiring

$$(xT)(\mu) = x * (T(\mu * x)).$$

The set of elements $T \in \mathfrak{J}(l_1(S), l_\infty(S)')$ for which $xT = T$ for all $x \in S$ is denoted by $\mathfrak{J}_m(S)$.

LEMMA 1. *If S is left amenable and $P, Q \in P(S)$, then*

$$\delta(P, Q) = \inf \{\|Q - T(P)\| : T \in \mathfrak{J}_m(S)\}.$$

Proof. We modify the proof of Theorem 1 in the obvious way. Define

$$K_\epsilon = \{T \in \mathfrak{J}(l_1(S), l_\infty(S)') : \|Q - (xT)(P)\| \leqq \delta(P, Q) + \epsilon, x \in S\}.$$

The topology $\mathscr{S}$ is defined in the obvious way on X_1, and K_ϵ is a non-void, weak* compact, convex subset of X_1. Further, $xT \in K_\epsilon$ if $T \in K_\epsilon$ and $x \in S$. By Day's fixed-point theorem [**3**], we can find $T \in \mathfrak{J}_m(S) \cap K_\epsilon$, and the desired result follows.

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With S identified as a subset of $l_\infty(S)'$, the *Stone-Čech compactification* W of S can be regarded as the weak* closure of S in $l_\infty(S)'$. With the relative weak* topology, W is a compact Hausdorff space. Further, $sw \in W$ if $w \in W$, and the map $w \to sw$ is continuous for each $s \in S$. Also $l_\infty(S)$ is canonically identified with $C(W)$, and hence $l_\infty(S)'$ with $M(W)$.

LEMMA 2. *If S is left amenable and $\bigcap\{tS : t \in S\} = \emptyset$, then $\delta(P, Q) = 2$ for all $P, Q \in P(S)$.*

Proof. Let W be the Stone-Čech compactification of S. Let $T \in \mathfrak{J}_m(S)$. Let $\mu \in P(S)$ and $S_{T\mu} \subset W$ be the support of $T\mu(\in M(W))$. Since $sT(\mu s) = T\mu$, we have $S_{T\mu} \subset sW$ for all $s \in S$. So $Y = \bigcap\{sW : s \in S\}$ is a non-void compact subset of W, and $S_{T\mu} \subset Y$. If $s \in S \cap Y$ and $t \in S$, then $t\alpha = s$ for some $\alpha \in W$, and by considering a net in S converging to α, we obtain $s \in tS$ since S is discrete. Consequently if S is left amenable and $\bigcap\{tS : t \in S\} = \emptyset$, then $S \cap Y = \emptyset$, $T\mu \in (l_1(S))^\perp$ and, using Lemma 1, $\delta(P, Q) = 2$ for all $P, Q \in P(S)$.

We now define

$$\mathfrak{J}_m'(S) = \{T \in \mathfrak{J}_m(S) : T(l_1(S)) \subset l_1(S)\}.$$

LEMMA 3. *Let S be right simple, $T \in \mathfrak{J}_m(S)$ and Π be the canonical projection map from $l_\infty(S)'$ onto $l_1(S)$ associated with the decomposition $l_\infty(S)' = l_1(S) \oplus l_1(S)^\perp$. Suppose that $\Pi \circ T \neq 0$. Then there exists $U \in \mathfrak{J}_m'(S)$ such that*

$$\|Q - U(P)\| \leqq \|Q - T(P)\| \quad (P, Q \in P(S)).$$

Proof. Let $T_1 = \Pi \circ T$. Then for $x \in S$, $\mu \in P(S)$, we have

$$(6) \quad xT_1(\mu x) = \Pi(xT_1(\mu x)) \leqq \Pi(xT(\mu x)) = T_1(\mu).$$

We show that $xT_1(\mu x) = T_1(\mu)$ for all $\mu \in P(S)$, $x \in S$. It is sufficient to show this when $\mu = y \quad (y \in S)$. Let $x \in S$. Since S is right simple, we have $yxS = S$, and we can find $a \in S$ with $yxa = y$. Then

$$xaT_1(y) = x[aT_1((yx)a)] \leqq xT_1(yx) \leqq T_1(y).$$

So $\|xT_1(yx)\| = \|T_1(y)\|$, and hence $xT_1(yx) = T_1(y)$. Since every $a \in S$ is of the form yz for some $z \in S$, we have

$$\|T_1(a)\| = \|T_1(y)\| = m.$$

Since $\Pi \circ T \neq 0$, we have $m > 0$. Let $U = m^{-1}T_1$. Then $U \in \mathfrak{J}_m'(S)$, and

$$\|Q - U(P)\| \leqq \|Q - T_1(P)\| + (1 - m) = \|Q - T(P)\|.$$

LEMMA 4. *Let S be right simple and left reversible and be such that*

$\mathfrak{J}_m'(S) \neq \emptyset$. Then S is a direct product $G \times E$ of a group G and a right zero semigroup E. For $\mu \in P(S)$ define $\mu^* \in P(S)$ by:

$$\mu^*(x, e) = \mu(x^{-1}, e).$$

Then the map α is a bijection from $P(S)$ onto $\mathfrak{J}_m'(S)$, where

(7) $\quad \alpha(\nu)(\mu) = \mu^* * \nu.$

Proof. Let $T \in \mathfrak{J}_m'(S)$. Since S is left reversible we can define a congruence $\sim$ on S by setting $a \sim b$ whenever $as = bs$ for some $s \in S$ ([**2**], (1.10)). The semigroup $S' = S/\sim$ is then right cancellative and right simple. Let $Q : S \to S'$ be the quotient map. If $as = bs$ in S, then

$$T(a) = sT(as) = sT(bs) = T(b).$$

So we can define $\tilde{T} \in \mathfrak{J}_m'(S')$ by setting

$$\tilde{T}(Q(x)) = \sum_{y \in S} \alpha_y Q(y)$$

where

$$T(x) = \sum_{y \in S} \alpha_y y \in P(S).$$

Let $u \in S'$, and find $v \in S'$ with $uv = u$. Then $u\tilde{T}(u) = \tilde{T}(u)$. Write $\tilde{T}(u) = \sum \alpha_w w$ and let

$$X = \{w : \alpha_w = \sup \{\alpha_z : z \in S'\}\}.$$

Then X is a non-void finite set, and $uX = X$. For some positive integer k, there exists $x_0 \in X$ with $a^k x_0 = x_0$. Then $(a^k)^2 x_0 = a^k x_0$, and cancelling x_0, we obtain an idempotent $\tilde{e}$ in S'. From [**2**], 1.27, S' is a right group, and since S' is right cancellative, S' is actually a group with identity $\tilde{e}$.

Let $E = Q^{-1}(\tilde{e})$. Then E is a subsemigroup of S. Since $e \sim f$ for all $e, f \in E$, we have $T(e) = T(f) = \mu$ for some $\mu \in P(S)$. For each $e \in E$, $e\mu = \mu$, and, arguing as above, we can find a non-void finite subset Y of S with $eY = Y$ for all $e \in E$. If Y contains n elements and $y \in Y$, then $e^n y = y$ $(e \in E)$. Find $x \in S$ with $Q(x) = Q(y)^{-1}$. Then $e^n(yx)^n = (yx)^n$, and putting $e = yx$, we obtain an idempotent in S. Again applying [**2**], 1.27, we have that S is a direct product $G \times E$ where G is a group and E is a right zero semigroup.

For each $e \in E$, define $T_e \in \mathfrak{J}_m'(G)$ by: $T_e(x) = Q_e T(x, e)$ where

$$Q_e\left(\sum_{s \in S} \alpha_s s\right) = \sum_{y \in G} \alpha_{(y,e)} y \quad \left(\sum_{s \in S} \alpha_s s \in l_1(S)\right).$$

One readily checks that there exists $\nu_e \geqq 0$ in $l_1(G)$ such that

$$T_e(x) = x^{-1}\nu_e \quad (x \in G).$$

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Let

$$\nu = \sum_{(y,e)\in S} \nu_e(y)(y, e).$$

Then $T(x, f) = (x^{-1}, f)\nu$ and $\nu \in P(S)$, and it follows that $T(\mu) = \mu^* * \nu$. The remainder of the proof is routine.

THEOREM 2. *Let S be a left amenable semigroup. If S has a kernel R which is a right group $G \times E$ and u is a left identity for R, then*

$$(8) \quad \delta(P, Q) = \inf \{\|Q - (Pu)^* * \nu\| : \nu \in P(G \times E)\}$$

for all $P, Q \in P(S)$. Otherwise, $\delta(P, Q) = 2$ for all $P, Q \in P(S)$.

Proof. Let Π be as in Lemma 3. If $\Pi \circ T_1 = 0$ for all $T_1 \in \mathfrak{J}_m(S)$, then $T_1(S) \subset l_1(S)^\perp$ for all $T_1 \in \mathfrak{J}_m(S)$, and so using Lemma 1, $\delta(P, Q) = 2$ for all $P, Q \in P(S)$. Suppose that for some $T_1 \in \mathfrak{J}_m(S)$, we have $\Pi \circ T_1 \neq 0$. Using Lemmas 3 and 1, we have $\mathfrak{J}_m'(S) \neq \emptyset$ and

$$\delta(P, Q) = \inf \{\|Q - T(P)\| : T \in \mathfrak{J}_m'(S)\} \quad (P, Q \in P(S)).$$

Let $T \in \mathfrak{J}_m'(S)$. As in Lemma 2, $T\mu$ is supported on $R = \bigcap_{t\in S} tS$ for all $\mu \in P(S)$. Let $T' = T_{|R}$. Then $T' \in \mathfrak{J}_m'(R)$. The semigroup R is obviously left reversible and right simple, and so, by Lemma 4, is a direct product $G \times E$, and there exists $\nu_T \in P(R)$ such that $T'\mu = \mu^* * \nu_T$ $(\mu \in l_1(\mathbf{R}))$. Now R is the kernel of S ([**2**], p. 84, 13(b)). Let u be a left identity for R; so $u = (1, e)$ where 1 is the identity of G and $e \in E$. Then for $\mu \in P(S)$,

$$T(\mu) = uT(\mu u) = T'(\mu u) = (\mu u)^* * \nu_T.$$

It is easy to check that the map $T \to \nu_T$ is a bijection from $\mathfrak{J}_m'(S)$ onto $P(R)$, and the equality (8) now follows.

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