Canad. Math. Bull. Vol. **57** (3), 2014 pp. 562–572
http://dx.doi.org/10.4153/CMB-2013-039-6

Note on the Grothendieck Group of Subspaces of Rational Functions and Shokurov's Cartier *b*-divisors

Kiumars Kaveh and A. G. Khovanskii

Abstract. In a previous paper the authors developed an intersection theory for subspaces of rational functions on an algebraic variety X over $\mathbf{k} = \mathbb{C}$. In this short note, we first extend this intersection theory to an arbitrary algebraically closed ground field $\mathbf{k}$. Secondly we give an isomorphism between the group of Cartier b-divisors on the birational class of X and the Grothendieck group of the semigroup of subspaces of rational functions on X. The constructed isomorphism moreover preserves the intersection numbers. This provides an alternative point of view on Cartier b-divisors and their intersection theory.

Introduction

In [K-K10] the authors developed an intersection theory for subspaces of rational functions on an arbitrary variety over $\mathbf{k} = \mathbb{C}$. In this short note we first extend this intersection theory to an arbitrary algebraically closed field $\mathbf{k}$, and secondly we observe that there is a direct connection between this intersection theory and Shokurov's Cartier b-divisors. This approach provides an alternative way of introducing Cartier b-divisors and their intersection theory and, in our opinion, is suitable for several applications in intersection theory.

Let X be an irreducible variety of dimension n over an algebraically closed ground field $\mathbf{k}$. Consider the collection $\mathbf{K}(X)$ of all the finite dimensional $\mathbf{k}$-subspaces of rational functions on X. The set $\mathbf{K}(X)$ is equipped with a natural product: for two subspaces $L, M \in \mathbf{K}(X)$, the product LM is the subspace spanned by all the fg where $f \in L$ and $g \in M$. With this product $\mathbf{K}(X)$ is a commutative semigroup (which is not cancellative). Let $L_1, \ldots, L_n$ be subspaces in $\mathbf{K}(X)$. In [K-K10] we associated a non-negative integer $[L_1, \ldots, L_n]$ with the subspaces L_i and called it their *intersection index*. It is defined to be the number of solutions x of a system $f_1(x) = \cdots = f_n(x) = 0$, where $f_i \in L_i$ are general elements and x lies in a certain non-empty Zariski open subset U of X (depending on the L_i). In [K-K10] it was shown that when $\mathbf{k} = \mathbb{C}$, the intersection index is well-defined and moreover is multi-additive with respect to the product of subspaces. It follows that the intersection index extends to a multi-additive integer valued function on the Grothendieck group $\mathbf{G}(X)$ of the semigroup

Received by the editors February 21, 2013; revised September 22, 2013.
Published electronically December 4, 2013.
The first author is partially supported by a Simons Foundation Collaboration Grants for Mathematicians (Grant ID: 210099) and a National Science Foundation Grant (Grant ID: 1200581).
The second author is partially supported by the Canadian Grant No. 156833-12.
AMS subject classification: **14C20**, 14Exx.
Keywords: intersection number, Cartier divisor, Cartier b-divisor, Grothendieck group.

$\mathbf{K}(X)$ (see Section 1). We regard $\mathbf{G}(X)$, together with its intersection index, as an extension of the intersection theory of Cartier divisors on complete varieties. In this note we observe that the well-definedness and multi-additivity of the intersection index for an arbitrary algebraically closed field $\mathbf{k}$ follow from the usual intersection theory on a product of projective spaces (see Sections 2 and 6).

Consider the collection of all projective birational models of X, *i.e.*, all birational maps $\pi\colon X_\pi \dashrightarrow X$ where X_π is projective. A *Cartier b-divisor* on X is a direct limit of Cartier divisors (X_π, D_π) with respect to a natural partial order on birational models of X. One verifies that the intersection product of Cartier divisors induces an intersection product on Cartier b-divisors (see Section 4). The b-divisors (birational divisors) were introduced by Shokurov (see [Iskovskikh03, S03]) and play an important role in birational geometry.

The main observations of this note are that the Grothendieck group $\mathbf{G}(X)$ of $\mathbf{K}(X)$ can be identified with the group of Cartier b-divisors on X and that this identification preserves the intersection index (Theorem 5.2).

A b-divisor is represented by a divisor on a projective birational model of the variety X. The collection of birational models of a variety, the main object of study in birational geometry, is a complicated object and intrinsically related to the notion of resolution of singularities and Minimal Model Program. Moreover, proving statements about b-divisors and their intersection theory relies on the statements about usual divisors and their intersection theory, while the intersection theory of b-divisors is more stable in the sense that it is invariant under birational isomorphisms, and one may regard it as easier to treat.

On the other hand, the Grothedieck group construction in [K-K10] suggests a different way that does not involve completions/birational models of X, and the invariance under a birational isomorphism is evident from the definition. This description of Cartier b-divisors and their intersection theory is suitable for several applications. We mention a few here: (1) It provides a framework to extend the celebrated Bernstein–Kushnirenko theorem (from toric geometry) on the number of solutions of a system of Laurent polynomial equations in the algebraic torus $(\mathbf{k}^*)^n$, to arbitrary varieties and arbitrary systems of equations (see [K-K10]). (2) Fix a valuation v on the field of rational functions $\mathbf{k}(X)$ and with values in $\mathbb{Z}^n$. One can then associate certain convex bodies (Newton–Okounkov bodies) with subspaces of rational functions, which using this approach can be identified with Cartier b-divisors such that their Euclidean volumes give the intersection numbers of the corresponding Cartier b-divisors. This way one obtains transparent proofs of the Hodge inequality, and its generalizations, for intersection numbers [K-K12].

It was pointed out to us by V. Shokurov that the group of Cartier b-divisors modulo the linear equivalence has another interpretation as the Picard group of the so-called *bubble space of* X. Thus, by the observation in this note, this Picard group can be interpreted as the Grothendieck group of subspaces modulo the linear equivalence of subspaces. Two subspaces $L, M \in \mathbf{K}(X)$ are called *linearly equivalent* if there exists a non-zero $h \in \mathbf{k}(X)$ with $hL = M$ where $hL = \{hf \mid f \in L\}$. Roughly speaking, the bubble space of X is the (huge) space obtained by blowing up X everywhere and continuing this process again and again. More precisely, it is the union of all birational models $\pi\colon X' \dashrightarrow X$ where we identify two points x', x'' in birational models X', X''

respectively, if the canonical birational map between X' and X'' is an isomorphism of a neighborhood of x' and a neighborhood of x'' (see [M74, §35]).

We give a few words about the organization of this short note. Section 1 covers basic definitions about the semigroup $\mathbf{K}(X)$ of subspaces of rational functions. Sections 2 and 3 recall some material about intersection index from [K-K10]. Section 4 recalls basic definitions about Cartier b-divisors. In Section 5 we observe that the Grothendieck group $\mathbf{G}(X)$ can be identified with the group of Cartier b-divisors. The last section is devoted to (short) proofs of the well-definedness and multi-additivity of the intersection index for an arbitrary algebraically closed field $\mathbf{k}$.

Finally, we believe that the elementary nature of the notions needed to define the semigroup $\mathbf{K}(X)$ and its intersection index, *i.e.*, subspaces of rational functions and number of solutions of a system of equations, makes this approach accessible to a wide audience and, in particular, suitable for a first course in algebraic geometry.

1 Subspaces of Rational Functions and the Grothendieck Group

Throughout this note the ground field $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic. Let X be an irreducible algebraic variety over $\mathbf{k}$.

Definition 1.1 We denote the collection of all non-zero finite dimensional vector subspaces (over $\mathbf{k}$) of the field of rational functions $\mathbf{k}(X)$ by $\mathbf{K}(X)$. Given $L, M \in \mathbf{K}(X)$, let the product LM be the subspace spanned by all the products fg, $f \in L$, and $g \in M$. With this product of subspaces $\mathbf{K}(X)$ is a commutative semigroup.

Let K be a commutative semigroup (whose operation we denote by multiplication). K is said to have the *cancellation property* if for $x, y, z \in K$, the equality $xz = yz$ implies $x = y$. Any commutative semigroup K with the cancellation property can be extended to an abelian group $G(K)$ consisting of formal quotients x/y, $x, y \in K$. For $x, y, z, w \in K$ we identify the quotients x/y and w/z, if $xz = yw$.

Given a commutative semigroup K (not necessarily with the cancellation property), we can get a semigroup with the cancellation property by considering the equivalence classes of a relation $\sim$ on K: for $x, y \in K$ we say $x \sim y$ if there is $z \in K$ with $xz = yz$. The collection of equivalence classes $K/\sim$ naturally has structure the of a semigroup with cancellation property. Let us denote the group of formal quotients of $K/\sim$ again by $G(K)$. It is called the *Grothendieck group of the semigroup* K. The map that sends $x \in K$ to its equivalence class $[x] \in K/\sim$ gives a natural homomorphism $\phi\colon K \to G(K)$.

The Grothendieck group $G(K)$ together with the homomorphism $\phi\colon K \to G(K)$ satisfies the following universal property: for any other group G' and a homomorphism $\phi'\colon K \to G'$, there exists a unique homomorphism $\psi\colon G(K) \to G'$ such that $\phi' = \psi \circ \phi$.

Definition 1.2 For two subspaces $L, M \in \mathbf{K}(X)$, we write $L \sim M$ if L and M are equivalent as elements of the multiplicative semigroup $\mathbf{K}(X)$, that is, if there is $N \in \mathbf{K}(X)$ with $LN = MN$.

Let $L \in \mathbf{K}(X)$. A rational function f is said to be *integral over* L if it satisfies an equation $f^m + \sum_{i=0}^{m-1} g_i f^i = 0$, where $g_i \in L^{m-i}$, $i = 1, \dots, m-1$. The *completion* $\overline{L}$ is the collection of all rational functions that are integral over L. The following result describes the completion $\overline{L}$ of a subspace L as the largest subspace equivalent to L (see [S-Z60, Appendix 4] for a proof; also see [K-K10]).

Theorem 1.3 (i) *The completion $\overline{L}$ is finite dimensional.*
(ii) *The completion $\overline{L}$ is the largest subspace which is equivalent to L. That is,*

(a) *$\overline{L} \sim L$, and*
(b) *if for $M \in \mathbf{K}(X)$ we have $M \sim L$, then $M \subset \overline{L}$.*

Remark 1.4 Let us call a subspace L *complete* if $\overline{L} = L$. If L and M are complete subspaces, then LM is not necessarily complete. For two complete subspaces $L, M \in \mathbf{K}$, define

$$L * M = \overline{LM}.$$

The collection of complete subspaces together with $*$ is a semigroup with the cancellation property. Theorem 1.3 in fact shows that $L \mapsto \overline{L}$ gives an isomorphism between the quotient semigroup $\mathbf{K}/\sim$ and the semigroup of complete subspaces (with $*$).

We denote the Grothendieck group of the semigroup $\mathbf{K}(X)$ by $\mathbf{G}(X)$.

2 Intersection Index of Subspaces of Rational Functions

In this section we define the intersection index of finite dimensional subspaces of rational function. Let $\mathbf{L} = (L_1, \dots, L_n)$ be an n-tuple of non-zero finite dimensional subspaces of rational functions. Let $Z \subset X$ be a closed subvariety of X containing the poles of all rational functions from the L_i, as well as all the points x at which all functions from some subspace L_i vanish.

Theorem 2.1 (Intersection index is well-defined) *There exists a non-empty Zariski open subset $\mathbf{U} \subset L_1 \times \cdots \times L_n$ such that for any $(f_1, \dots, f_n) \in \mathbf{U}$ the number of solutions*

$$\{x \in X \setminus Z \mid f_1(x) = \cdots = f_n(x) = 0\}$$

is finite and is independent of the choice of Z and $(f_1, \dots, f_n) \in \mathbf{U}$.

We denote the number of solutions $\{x \in X \setminus Z \mid f_1(x) = \cdots = f_n(x) = 0\}$ in Theorem 2.1 by $[L_1, \dots, L_n]$ and call it the *intersection index* of the subspaces L_i.

The following are immediate corollaries of the definition of the intersection index: (a) $[L_1, \dots, L_n]$ is a symmetric function of $L_1, \dots, L_n \in \mathbf{K}(X)$, (b) the intersection index is monotone (*i.e.*, if $L_1' \subseteq L_1, \dots, L_n' \subseteq L_n$, then $[L_1', \dots, L_n'] \leq [L_1, \dots, L_n]$), and (c) the intersection index is non-negative.

Theorem 2.2 (Multi-additivity of intersection index) *Let $L_1', L_1'', L_2, \dots, L_n \in \mathbf{K}(X)$ and put $L_1 = L_1' L_1''$. Then*

$$[L_1, \dots, L_n] = [L_1', L_2, \dots, L_n] + [L_1'', L_2, \dots, L_n].$$

https://doi.org/10.4153/CMB-2013-039-6 Published online by Cambridge University Press

We will prove Theorems 2.1 and 2.2 in Section 6. The proofs rely on the notion of intersection product in the Chow rings of products of projective spaces.

From multi-additivity of the intersection index it follows that the intersection index is invariant under the equivalence of subspaces, namely if

$$L_1, \dots, L_n \quad \text{and} \quad M_1, \dots, M_n \in \mathbf{K}$$

are n-tuples of subspaces and for each i, $L_i \sim M_i$,

$$[L_1, \dots, L_n] = [M_1, \dots, M_n].$$

Hence one can extend the intersection index to the Grothendieck group $\mathbf{G}(X)$ of $\mathbf{K}(X)$. In particular, Theorem 1.3 implies that all the intersection indices of a subspace $L \in \mathbf{K}$ and its completion $\overline{L}$ are the same.

Analogous to the Kodaira map of a very ample line bundle, one can assign to a subspace $L \in \mathbf{K}(X)$ its Kodaira map Φ_L, which is a rational map from X to $\mathbb{P}(L^*)$, the projectivization of the dual space L^*, as follows: let $x \in X$ be such that $f(x)$ is defined for all $f \in L$. Then $\Phi_L(x)$ is represented by the linear functional in L^* that sends f to $f(x)$.

Let Y_L denote the closure of the image of X in $\mathbb{P}(L^*)$. The next proposition relates the self-intersection index of a subspace with the degree of Y_L. It easily follows from the definition of the intersection index.

Proposition 2.3 (Self-intersection index and degree) *Let $L \in \mathbf{K}$ be a subspace and $\Phi_L \colon X \dashrightarrow Y_L \subset \mathbb{P}(L^*)$ its Kodaira map.*

(i) *If $\dim X = \dim Y_L$, then Φ_L has finite mapping degree d and $[L, \dots, L]$ is equal to the degree of the subvariety Y_L (in $\mathbb{P}(L^*)$) multiplied by d.*

(ii) *If $\dim X > \dim Y_L$, then $[L, \dots, L] = 0$.*

3 Cartier Divisor Associated with a Subspace of Rational Functions with a Regular Kodaira Map

The material in this section is taken from [K-K10, Section 6].

A *Cartier divisor* on an irreducible variety X is a divisor that can be represented locally as a divisor of a rational function. Any rational function f defines a *principal Cartier divisor* denoted by (f). The Cartier divisors are closed under the addition and form an abelian group that we will denote by $\mathrm{CDiv}(X)$. A dominant morphism $\Phi \colon X \to Y$ between varieties X and Y gives a pullback homomorphism $\Phi^* \colon \mathrm{CDiv}(Y) \to \mathrm{CDiv}(X)$. Two Cartier divisors are linearly equivalent if their difference is a principal divisor. The group of Cartier divisors modulo linear equivalence is called the *Picard group* of X and is denoted by $\mathrm{CDiv}(X)$. One has an intersection theory on $\mathrm{Pic}(X)$: for given Cartier divisors $D_1, \dots, D_n$ on an n-dimensional complete variety there is an intersection index $[D_1, \dots, D_n]$ that obeys the usual properties (see [Fulton98]).

Now let us return to the subspaces of rational functions. For a subspace $L \in \mathbf{K}(X)$, in general, the Kodaira map Φ_L is a rational map, possibly not defined everywhere on X.

We denote by $\mathbf{K}_{\mathrm{Cart}}(X)$ the collection of subspaces $L \in \mathbf{K}(X)$ for which the rational Kodaira map $\Phi_L \colon X \dashrightarrow \mathbb{P}(L^*)$ extends to a regular map defined everywhere on X. We call a subspace $L \in \mathbf{K}_{\mathrm{Cart}}(X)$ a *subspace with regular Kodaira map*. One verifies that the collection $\mathbf{K}_{\mathrm{Cart}}(X)$ is closed under the multiplication.

To a subspace $L \in \mathbf{K}_{\mathrm{Cart}}(X)$ there naturally corresponds a Cartier divisor $\mathcal{D}(L)$ as follows: each rational function $h \in L$ defines a hyperplane $H = \{h = 0\}$ in $\mathbb{P}(L^*)$. The divisor $\mathcal{D}(L)$ is the difference of the pullback divisor $\Phi_L^*(H)$ and the principal divisor (h).

One has the following theorem ([K-K10, Theorem 6.8]).

Theorem 3.1 *Let X be a projective variety.*

(i) *For any $L \in \mathbf{K}_{\mathrm{Cart}}(X)$ the divisor $\mathcal{D}(L)$ is well defined,* i.e., *is independent of the choice of a function $h \in L$. The map $L \mapsto \mathcal{D}(L)$ is a homomorphism from the semigroup $\mathbf{K}_{\mathrm{Cart}}(X)$ to the semigroup* $\mathrm{CDiv}(X)$.

(ii) *The map $L \mapsto \mathcal{D}(L)$ preserves the intersection index,* i.e., *for $L_1, \dots, L_n \in \mathbf{K}_{\mathrm{Cart}}(X)$ we have*

$$[L_1, \dots, L_n] = \big[\mathcal{D}(L_1), \dots, \mathcal{D}(L_n)\big],$$

where the right-hand side is the intersection index of Cartier divisors.

Conversely, with any Cartier divisor we can associate a subspace of rational functions. The subspace $\mathcal{L}(D)$ associated with a Cartier divisor D is the collection of all rational functions f such that the divisor $(f)+D$ is effective (by definition $0 \in \mathcal{L}(D)$).

The following well-known fact can be found in [Hartshorne77, Chap. 2, Theorem 5.19].

Theorem 3.2 *When X is projective, $\mathcal{L}(D)$ is finite dimensional.*

We record the following facts, which are direct corollaries of the definition.

Proposition 3.3 *Let $L \in \mathbf{K}_{\mathrm{Cart}}(X)$ and put $D = \mathcal{D}(L)$. Then $L \subset \mathcal{L}(D)$ and $\mathcal{L}(D) \in \mathbf{K}_{\mathrm{Cart}}(X)$.*

Proposition 3.4 *Let $\rho \colon X' \to X$ be a birational morphism. Let D be a Cartier divisor on X. Then*

$$\mathcal{L}\big(\rho^*(D)\big) = \rho^*\big(\mathcal{L}(D)\big).$$

One verifies that D is a very ample divisor if $\mathcal{D}(\mathcal{L}(D)) = D$. It is a well-known fact that the group of Cartier divisors is generated by the very ample divisors (see [L04, Example 1.2.6]).

Let $L \in \mathbf{K}_{\mathrm{Cart}}(X)$. The following describes the subspace $\mathcal{L}(\mathcal{D}(L))$. It can be found in slightly different forms in [Hartshorne77, Chap. 2, Proof of Theorem 5.19] and [S-Z60, Appendix 4].

Theorem 3.5 *Let X be an irreducible projective variety and let $L \in \mathbf{K}_{\mathrm{Cart}}(X)$ be such that the Kodaira map $\Phi_L \colon X \to \mathbb{P}(L^*)$ is an embedding.*

(i) *Every element of $\mathcal{L}(\mathcal{D}(L))$ is integral over L,* i.e., *$L \subset \mathcal{L}(\mathcal{D}(L)) \subset \overline{L}$, and hence $\mathcal{L}(\mathcal{D}(L)) \sim L$.*

(ii) *Moreover, if X is normal, then $\mathcal{L}(\mathcal{D}(L)) = \overline{L}$.*

https://doi.org/10.4153/CMB-2013-039-6 Published online by Cambridge University Press

4 Cartier b-divisors

Let X be an irreducible variety of dimension n defined over an algebraically closed field $\mathbf{k}$.

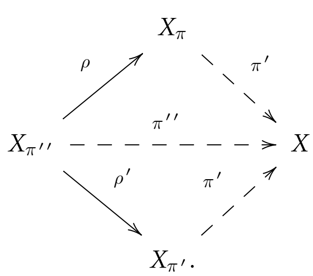
Definition 4.1 (Birational model) We call a proper birational map $\pi\colon X_\pi \dashrightarrow X$, where X_π is a projective variety, a *projective birational model* (or for short a birational model) of X. The collection of all models of X modulo isomorphism is partially ordered. We say $(X_{\pi'}, \pi')$ *dominates* (X_π, π) and write $\pi' \geq \pi$ if there is a morphism $\rho\colon X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \rho$.

Proposition 4.2 *The above partial order is a directed set,* i.e., *for any two models* (X_π, π) *and* $(X_{\pi'}, \pi')$ *there exists a third model* $(X_{\pi''}, \pi'')$ *that dominates both.*

Proof By Chow's lemma, without loss of generality we can assume that X is quasi-projective sitting in some projective space $\mathbb{P}^N$. Let $U \subset X$ be an open subset such that π and π' are isomorphisms restricted to $\pi^{-1}(U)$ and $\pi'^{-1}(U)$ respectively. Consider the set

$$\Gamma = \left\{ (x, \pi^{-1}(x), \pi'^{-1}(x)) \mid x \in U \right\} \subset X \times X_\pi \times X_{\pi'},$$

and let $X_{\pi''}$ be the Zariski closure of Γ in $\mathbb{P}^N \times X_\pi \times X_{\pi'}$. The morphisms to X_π and $X_{\pi'}$ as well as the rational map to X are given by the projections on the corresponding factors:



■

Definition 4.3 The *Riemann–Zariski space* $\mathfrak{X}$ of the birational class of X is defined as

$$\mathfrak{X} = \varprojlim_{\pi} X_\pi,$$

where the limit is taken over all the birational models of X.

Definition 4.4 (Cartier b-divisor) Following Shokurov one defines the group of *Cartier b-divisors* as

$$\mathrm{CDiv}(\mathfrak{X}) = \varinjlim_{\pi} \mathrm{CDiv}(X_\pi),$$

where $\mathrm{CDiv}(X_\pi)$ denotes the group of Cartier divisors on the variety X_π and the limit is taken with respect to the pullback maps $\mathrm{CDiv}(X_\pi) \to \mathrm{CDiv}(X_{\pi'})$, which are defined whenever $\pi' \geq \pi$.

https://doi.org/10.4153/CMB-2013-039-6 Published online by Cambridge University Press

Remark 4.5 Let $D_1, \ldots, D_n$ be Cartier divisors on a model X_π and suppose $\pi' \geq \pi$. One shows that the intersection number of the D_i on X_π is equal to the intersection number of the their pullbacks to $X_{\pi'}$. This shows that the intersection number of Cartier b-divisors is well defined.

5 Main Statement

Let X be an irreducible variety over $\mathbf{k}$. In this section we prove the main statement of this note that the group of Cartier b-divisors is naturally isomorphic to the Grothendieck group of $\mathbf{K}(X)$ (Theorem 5.2). The proof is based on the following easy lemma.

Lemma 5.1 *Let $L \in \mathbf{K}(X)$ be a non-zero finite dimensional subspace with the Kodaira rational map $\Phi_L \colon X \dashrightarrow \mathbb{P}(L^*)$. Then there exists a normal projective birational model X_π of X such that $\Phi_L \circ \pi$ extends to a regular map on the whole X_π. In other words, the Kodaira map of the subspace $\pi^*(L)$ extends to a regular map on X_π.*

Proof Every irreducible variety is birationally isomorphic to a projective variety. So without loss of generality we assume that X is projective. Let $U \subset X$ be an open subset such that $\Phi_{L|U}$ is regular. Let $\Gamma \subset U \times \mathbb{P}(L^*)$ be the graph of $\Phi_{L|U}$ and let X_π be the closure of Γ in $X \times \mathbb{P}(L^*)$. Let $\pi \colon X_\pi \to X$ denote the projection onto the first factor. The map $\pi_{|\Gamma} \colon \Gamma \to U$ is an isomorphism with inverse $x \mapsto (x, \Phi_L(x))$. Thus X_π is birationally isomorphic to X. Also the Kodaira map Φ_L on X extends to $\Phi_{\pi*(L)}$, which is the projection on the second factor and hence defined on the whole X_π. If X_π is not normal, replace X_π with its normalization. ■

From Theorem 3.1 and Lemma 5.1 we get the following theorem.

Theorem 5.2 *The group of Cartier b-divisors* $\mathrm{CDiv}(\mathfrak{X})$ *is naturally isomorphic to the Grothendieck group* $\mathbf{G}(X)$ *of* $\mathbf{K}(X)$*. Moreover, the isomorphism preserves the intersection index.*

Remark 5.3 Sometimes it is customary to define Cartier b-divisors to be the direct limit of vector spaces of Cartier $\mathbb{Q}$-divisors on birational models of X. With this definition, the above result would assert that the $\mathbb{Q}$-vector space of Cartier b-divisors is isomorphic to the $\mathbb{Q}$-vector space $\mathbf{G}(X) \otimes \mathbb{Q}$.

Proof of Theorem 5.2 Define a map F from $\mathrm{CDiv}(\mathfrak{X})$ to the Grothendieck group of $\mathbf{K}(X)$ as follows. Let $\mathfrak{D}$ be a Cartier b-divisor represented by a Cartier divisor D_π on a birational model X_π. We know that D_π can be written as the difference of two very ample divisors on X_π. Thus to define F it is enough to define it on Cartier b-divisors that are represented by very ample divisors. So without loss of generality we assume that D_π is very ample. By Theorem 3.2 we know that the subspace $\mathcal{L}(D_\pi) \subset \mathbf{k}(X_\pi)$ associated with D_π is finite dimensional. Define

$$F(\mathfrak{D}) = \left(\pi^{-1}\right)^*(\mathcal{L}(D_\pi))\,.$$

https://doi.org/10.4153/CMB-2013-039-6 Published online by Cambridge University Press

Suppose $(X_{\pi'}, \pi')$ dominates (X_π, π) with the corresponding morphism $\rho\colon X_{\pi'} \to X_\pi$. Then by Proposition 3.4 we have

$$\mathcal{L}\big(\rho^*(D_\pi)\big) = \rho^*\big(\mathcal{L}(D_\pi)\big),$$

since ρ is a birational isomorphism. Thus F is well defined, *i.e.*, is independent of the choice of the representative (X_π, D_π) for $\mathfrak{D}$.

Now suppose $\mathfrak{D}$, $\mathfrak{D}'$ are two Cartier b-divisors represented by very ample divisors D_π, $D_{\pi'}$ on two birational models (X_π, π) and $(X_{\pi'}, \pi')$. By Proposition 4.2 we can find a third birational model $(X_{\pi''}, \pi'')$ dominating both. Now applying Theorem 3.1(2) to the pull-backs of D_π and $D_{\pi'}$ to $X_{\pi''}$, it follows that $F(\mathfrak{D} + \mathfrak{D}') = F(\mathfrak{D})F(\mathfrak{D}')$; that is, F is a homomorphism.

Next we define an inverse map G to F. Suppose $L \in \mathbf{K}(X)$. Then by Lemma 5.1 there exists a normal projective model X_π such that $\Phi_{\pi^*(L)}\colon X_\pi \dashrightarrow \mathbb{P}(\pi^*(L)^*)$ extends to a regular map on the whole X_π (which we again denote by $\Phi_{\pi^*(L)}$). Define $G(L)$ to be the element of $\mathrm{CDiv}(\mathfrak{X})$ represented by the divisor $\mathcal{D}(\pi^*(L))$ in the birational model X_π. Suppose $X_{\pi'}$ is another birational model such that $\Phi_{\pi'^*(L)}$ is regular. By Proposition 4.2 we can find a third model π'' that dominates both π and π'. Now $\rho^*(\mathcal{D}(\pi^*(L)) = \rho'^*(\mathcal{D}(\pi'^*(L)) = \mathcal{D}(\pi''^*(L))$. Hence the class in $\mathrm{CDiv}(\mathfrak{X})$ represented by $\mathcal{D}(L)$ is independent of the choice of the model X_π and the map G is well defined. Finally, if $\mathfrak{D}$ is represented by a very ample divisor, we know that $G(F(\mathfrak{D})) = \mathfrak{D}$, and also by Theorem 3.5, $L \subset F(G(L)) \subset \overline{L}$ and hence $F(G(L)) \sim L$. So F and G are inverses of each other, and the proposition is proved. ■

6 Intersection Index is Well Defined and Multi-additive

In this section we prove Theorems 2.1 and 2.2 using the intersection product in the Chow ring of a product of projective spaces. A standard reference for Chow rings and their intersection product is [Fulton98, Chapter 8].

Let X be an irreducible n-dimensional variety. Let $\mathbf{L} = (L_1, \dots, L_n)$ be an n-tuple of non-zero finite dimensional subspaces of rational functions on X. For each i let $\Phi_{L_i}\colon X \dashrightarrow \mathbb{P}(L_i^*)$ denote the corresponding Kodaira rational map. Suppose X is birationally embedded in some projective space $\mathbb{P}^N$ (Chow's lemma). Put

$$P = \mathbb{P}^N \times \mathbb{P}(L_1^*) \times \cdots \times \mathbb{P}(L_n^*),$$

and consider the rational map $\Phi_{\mathbf{L}}\colon X \dashrightarrow P$ given by

$$\Phi_{\mathbf{L}}\colon x \longmapsto \big(x, \Phi_{L_1}(x), \dots, \Phi_{L_n}(x)\big).$$

Let $Y_{\mathbf{L}}$ be the closure of the image of X under the rational map $\Phi_{\mathbf{L}}$. The map $\Phi_{\mathbf{L}}$ is a birational isomorphism between X and $Y_{\mathbf{L}}$.

Proof of Theorem 2.1 For each i, let H_i be a hyperplane in $\mathbb{P}(L_i^*)$ and let $\mathbf{H} = \mathbb{P}^N \times H_1 \times \cdots \times H_n$. Then H is a subvariety of P of codimension n. We note that for different choices of the hyperplanes H_i the cycles $[\mathbf{H}]$ are all rationally equivalent. From the definition of product in the Chow ring $A_*(P)$ we see that the intersection

https://doi.org/10.4153/CMB-2013-039-6 Published online by Cambridge University Press

index $[L_1, \dots, L_n]$ is equal to the intersection number of the product of cycles $[Y_{\mathbf{L}}]$ and $[\mathbf{H}]$, and hence well defined. ■

Proof of Theorem 2.2 There is a natural surjective map $L_1' \otimes L_1'' \to L_1'L_1''$. This induces an embedding $(L_1'L_1'')^* \hookrightarrow L_1' \otimes L_1''$ and thus an embedding $\mathbb{P}((L_1'L_1'')^*) \hookrightarrow \mathbb{P}((L_1' \otimes L_1'')^*)$. Consider the Segre map

$$\mathbb{P}(L_1'^*) \times \mathbb{P}(L_1'') \to \mathbb{P}(L_1'^* \otimes L_1''^*) \cong \mathbb{P}\big((L_1' \otimes L_1'')^*\big).$$

One has a commutative diagram:

$$\begin{array}{ccc} & & \mathbb{P}^N \times \mathbb{P}((L_1'L_1'')^*) \times \mathbb{P}(L_2^*) \times \cdots \times \mathbb{P}(L_n^*) \\ & \nearrow & \downarrow \\ X & \longrightarrow & \mathbb{P}^N \times \mathbb{P}((L_1' \otimes L_1'')^*) \times \mathbb{P}(L_2^*) \times \cdots \times \mathbb{P}(L_n^*) \\ & \searrow & \uparrow \\ & & \mathbb{P}^N \times \mathbb{P}((L_1'^*) \times \mathbb{P}(L_1''^*) \times \mathbb{P}(L_2^*) \times \cdots \times \mathbb{P}(L_n^*), \end{array}$$

where the top vertical map is given by $\mathbb{P}((L_1'L_1'')^*) \hookrightarrow \mathbb{P}((L_1' \otimes L_1'')^*)$ and the bottom vertical map is induced by the Segre map. Note that the image of X in $\mathbb{P}((L_1' \otimes L_1'')^*)$ lies in $\mathbb{P}((L_1'L_1'')^*)$. Take $f \in L_1'$, $g \in L_1''$. Then f, g define hyperplanes H_f, H_g in $\mathbb{P}(L_1'^*)$, $\mathbb{P}(L_1''^*)$ respectively. Moreover, $f \otimes g \in L_1' \otimes L_1'' \cong (L_1'^* \otimes L_1''^*)^*$ defines a hyperplane H in $\mathbb{P}((L_1' \otimes L_1'')^*)$ and hence in $\mathbb{P}((L_1'L_1'')^*)$. Also take hyperplanes $H_2, \dots, H_n$ in $\mathbb{P}(L_2^*), \dots, \mathbb{P}(L_n^*)$ respectively. We know from the proof of Theorem 2.1 that

$$[L_1', L_2, \dots, L_n] = [Y_{\mathbf{L}'}] \cdot [H_f \times H_2 \times \cdots \times H_n],$$
$$[L_1'', L_2, \dots, L_n] = [Y_{\mathbf{L}''}] \cdot [H_g \times H_2 \times \cdots \times H_n],$$
$$[L_1'L_1'', L_2, \dots, L_n] = [Y_{\mathbf{L}}] \cdot [H \times H_2 \times \cdots \times H_n],$$

where $\cdot$ denotes the product in the corresponding Chow rings, and

$$\mathbf{L}' = (L_1', L_2, \dots, L_n), \quad \mathbf{L}'' = (L_1'', L_2, \dots, L_n), \quad \mathbf{L} = (L_1'L_1'', L_2, \dots, L_n).$$

We note that the pullback, under the Segre map, of the hyperplane H to $\mathbb{P}(L_1'^*) \times \mathbb{P}(L_1''^*)$ coincides with $(H_f \times \mathbb{P}(L_1''^*)) + (\mathbb{P}(L_1'^*) \times H_g)$. This shows that

$$[H \times H_2 \times \cdots \times H_n] = [H_f \times H_2 \times \cdots \times H_n] + [H_g \times H_2 \times \cdots \times H_n].$$

This finishes the proof. ■

Acknowledgments We would like to thank A. Borisov and V. Shokurov for valuable suggestions and comments. In particular, V. Shokurov told us about the bubble space of X introduced in [M74].

References

[Fulton98] W. Fulton, *Intersection theory*. Second ed. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Folge, A Series of Modern Surveys in Mathematics, 2, Springer-Verlag, Berlin, 1998.

[Hartshorne77] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, 52, Springer-Verlag, New York-Heidelberg, 1977.

[Iskovskikh03] V. A. Iskovskikh, *b-divisors and Shokurov functional algebras*. (Russian) Tr. Mat. Inst. Steklova **240**(2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 8–20; translation in Proc. Steklov Inst. Math. **2003**, no. 1 (240), 4–15.

[K-K10] K. Kaveh and A. G. Khovanskii, *Mixed volume and an extension of intersection theory of divisors*. Mosc. Math. J. **10**(2010), no. 2, 343–375, 479.

[K-K12] ———, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*. Ann. of Math. **176**(2012), no. 2, 925–978. http://dx.doi.org/10.4007/annals.2012.176.2.5

[L04] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3, Folge, A Series of Modern Surveys in Mathematics, 48, Springer-Verlag, Berlin, 2004.

[M74] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*. North-Holland Mathematical Library, 4, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1974.

[S-Z60] P. Samuel and O. Zariski, *Commutative algebra*. Vol. II. Reprint of the 1960 ed., Graduate Texts in Mathematics, 29, Springer, New York, 1976.

[S03] V. V. Shokurov, *Prelimiting flips*. Tr. Mat. Inst. Steklova **240**(2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219; translation in Proc. Steklov Inst. Math. **2003**, no. 1(240), 75–213.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, USA
e-mail: kaveh@pitt.edu

Department of Mathematics, University of Toronto, Toronto, ON

and

Moscow Independent University, Institute for Systems Analysis, Russian Academy of Sciences
e-mail: askold@math.utoronto.ca