13
Matrix models

Matrix models first appeared in statistical mechanics and nuclear physics [Wig51, Dys62] and turned out to be very useful in the analysis of various physical systems where the energy levels of a complicated Hamiltonian can be approximated by the distribution of eigenvalues of a random matrix. The statistical averaging is then replaced by averaging over an appropriate ensemble of random matrices. This idea has been applied, in particular, in studying the low-energy chiral properties of QCD [SV93, VZ93].

Matrix models possess some features of multicolor QCD described in Chapter 11 but are simpler and can often be solved as $N \to \infty$ (i.e. in the planar limit) using the methods proposed for multicolor QCD. For the simplest case of the Hermitian one-matrix model, the genus expansion in $1/N$ can be constructed.

The Hermitian one-matrix model is related to the problem of enumeration of graphs. Its explicit solution at large N was first obtained by Brézin, Itzykson, Parisi and Zuber [BIP78] and inspired a lot of activity in this subject. Further results in this direction are linked to the method of orthogonal polynomials [Bes79, IZ80, BIZ80].

A very interesting application of the matrix models along this line is for the problem of discretization of random surfaces and two-dimensional quantum gravity [Kaz85, Dav85, ADF85, KKM85]. The continuum limits of these matrix models are associated with lower-dimensional conformal field theories and exhibit properties of integrable systems.

We shall begin this chapter by describing the original approach [BIP78] for solving the Hermitian one-matrix model at large N and then concentrate on a more general approach based on the loop equations. Our main goal is to illustrate the methods described in the two previous chapters.

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13.1 Hermitian one-matrix model

The unitary one-matrix model (12.135) is generically a matrix model solvable in the large-N limit. A simplest and historically the first example of this kind is the Hermitian one-matrix model, the large-N solution of which is obtained in [BIP78].

The Hermitian one-matrix model is defined by the partition function

$$Z_{1\mathrm{h}} = \int \mathrm{d}\varphi\, \mathrm{e}^{-N \operatorname{tr} V(\varphi)}, \tag{13.1}$$

where

$$\mathrm{d}\varphi = \prod_{i=1}^{N} \mathrm{d}\varphi_{ii} \prod_{j>i}^{N} \mathrm{d}\,\mathrm{Re}\,\varphi_{ij}\, \mathrm{d}\,\mathrm{Im}\,\varphi_{ij} \tag{13.2}$$

is the measure for integrating over Hermitian $N \times N$ matrices. It is invariant under the shift

$$\varphi_{ij} \to \varphi_{ij} + \epsilon_{ij} \tag{13.3}$$

by an arbitrary $N \times N$ Hermitian matrix ϵ_{ij}.

We consider the most general potential

$$V(\varphi) = \sum_k t_k\, \varphi^k\,, \tag{13.4}$$

where t_k are coupling constants. We shall also use another normalization

$$t_k = \frac{g_k}{k} \qquad \text{for} \quad k \geq 1\,, \tag{13.5}$$

which respects the cyclic symmetry of the trace. The simplest Gaussian case is associated with $g_2 = 1$ and $g_k = 0$ for $k \neq 2$.

The averages in the Hermitian one-matrix model are defined by

$$\langle F[\varphi] \rangle_{1\mathrm{h}} = Z_{1\mathrm{h}}^{-1} \int \mathrm{d}\varphi\, \mathrm{e}^{-N \operatorname{tr} V(\varphi)} F[\varphi]\,. \tag{13.6}$$

Performing the Gaussian integral, it is easy to calculate the propagator

$$\langle \varphi_{ij}\varphi_{kl} \rangle_{\mathrm{Gauss}} \stackrel{\mathrm{def}}{=} \frac{\displaystyle\int \mathrm{d}\varphi\, \mathrm{e}^{-\frac{N}{2}\operatorname{tr}\varphi^2} \varphi_{ij}\varphi_{kl}}{\displaystyle\int \mathrm{d}\varphi\, \mathrm{e}^{-\frac{N}{2}\operatorname{tr}\varphi^2}} = \frac{1}{N}\delta_{il}\delta_{kj}\,. \tag{13.7}$$

Equation (13.7) can be obtained alternatively from the Schwinger–Dyson equation

$$\left\langle \frac{\partial \operatorname{tr} V(\varphi)}{\partial \varphi_{ji}} F[\varphi] \right\rangle_{1\mathrm{h}} = \left\langle \frac{1}{N} \frac{\partial F[\varphi]}{\partial \varphi_{ji}} \right\rangle_{1\mathrm{h}} \tag{13.8}$$

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which results from the invariance of the measure under the infinitesimal shift (13.3). It is enough to choose $F[\varphi] = \varphi_{kl}$ and to calculate the derivatives of the Gaussian potential on the LHS and of φ_{kl} on the RHS by the use of

$$\frac{\partial \varphi_{kl}}{\partial \varphi_{ji}} = \delta_{il}\delta_{kj} \,. \tag{13.9}$$

Problem 13.1 Derive Eq. (13.7) by calculating the Gaussian integral.

Solution Let us substitute $\varphi_{ij} = (X_{ij} + \mathrm{i}Y_{ij})/\sqrt{2}$ with real symmetric $X_{ij} = X_{ji}$ and antisymmetric $Y_{ij} = -Y_{ji}$. The number of independent components is $N(N+1)/2$ for X and $N(N-1)/2$ for Y, i.e. N^2 in total as it should be.
We then obtain

$$\begin{aligned} \langle \varphi_{ij}\varphi_{kl}\rangle_{\text{Gauss}} &= \frac{1}{2}\langle X_{ij}X_{kl}\rangle_{\text{Gauss}} - \frac{1}{2}\langle Y_{ij}Y_{kl}\rangle_{\text{Gauss}} \\ &= \frac{1}{2N}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{2N}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &= \frac{1}{N}\delta_{il}\delta_{jk} \end{aligned} \tag{13.10}$$

as in Eq. (13.7).

The Feynman graphs of the Hermitian one-matrix model can be represented by the double index lines quite similarly to Sect. 11.1. Now there are no commutators so all vertices are symmetric in the indices.

Generically, the Hermitian one-matrix model generates graphs of a zero-dimensional field theory. Since there is no momentum variable and each propagator is $1/N$, the contribution of each graph is simply $1/N^{2\,\text{genus}}$ times a symmetry factor. Hence solving the Hermitian one-matrix model is equivalent to calculating the number of graphs with a given genus.

A very important property of the model is that $\operatorname{tr} V(\varphi)$ depends only on the eigenvalues of the matrix φ. Similarly, representing φ in a canonical form

$$\varphi = VPV^\dagger \tag{13.11}$$

with unitary $N \times N$ matrix V and diagonal

$$P = \operatorname{diag}\{p_1, \ldots, p_N\}, \tag{13.12}$$

the measure (13.2) can be written in a standard Weyl form

$$\mathrm{d}\varphi = \mathrm{d}V \prod_{i=1}^{N} \mathrm{d}p_i \, \Delta^2(P)\,, \tag{13.13}$$

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where

$$\Delta(P) = \prod_{i<j} (p_i - p_j) \tag{13.14}$$

is the Vandermonde determinant.

We see that the contribution from angular degrees of freedom residing in V factorizes, so the partition function (13.1) becomes

$$Z_{1\mathrm{h}} = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \mathrm{d}p_i \prod_{i<j} (p_i - p_j)^2 \exp\left[-N \sum_{i=1}^{N} V(p_i)\right]. \tag{13.15}$$

Problem 13.2 Derive Eq. (13.13).

Solution The representation (13.11) of φ in the canonical form reminds one of fixing a gauge where V are matrices of a gauge transformation. The measure $\mathrm{d}\varphi$ can then be represented as

$$\mathrm{d}\varphi = \mathrm{d}V \prod_{i=1}^{N} \mathrm{d}p_i \, J(P)\,, \tag{13.16}$$

where the Jacobian $J(P)$ depends only on the eigenvalues of φ since $\mathrm{d}\varphi$ is invariant under

$$\varphi \to \Omega\, \varphi\, \Omega^\dagger\,. \tag{13.17}$$

To calculate the Jacobian, it is convenient [BIZ80] to apply the Faddeev–Popov method inserting

$$1 = \Delta^2(\varphi) \int \mathrm{d}\Omega \prod_{i<j} \delta^{(2)}\big([\Omega\varphi\Omega^\dagger]_{ij}\big) \tag{13.18}$$

in the measure $\mathrm{d}\varphi$. Here $\mathrm{d}\Omega$ is the Haar measure for $U(N)$ and the $N^2 - N$ distributions are only present for off-diagonal components. It is easy to see that $\Delta^2(\varphi)$ depends solely on eigenvalues of φ since the measure $\mathrm{d}\Omega$ is invariant under multiplication by a unitary matrix.

We can insert the unity (13.18) into the integral of a function $f(\varphi)$ which is invariant under (13.17) and hence depends only on the eigenvalues of φ:

$$\begin{aligned}
\int \mathrm{d}\varphi f(\varphi) &= \int \mathrm{d}\varphi\, \Delta^2(\varphi) \int \mathrm{d}\Omega \prod_{i<j} \delta^{(2)}\big([\Omega\varphi\Omega^\dagger]_{ij}\big)\, f(\varphi) \\
&= \int \mathrm{d}\Omega \int \prod_{i<j} \mathrm{d}\varphi_{ij}\, \delta^{(2)}(\varphi_{ij}) \prod_{i=1}^{N} \mathrm{d}p_i\, \Delta^2(P)\, f(P) \\
&= \prod_{i=1}^{N} \mathrm{d}p_i\, \Delta^2(P)\, f(P)\,.
\end{aligned} \tag{13.19}$$

Comparing with Eq. (13.16), we conclude that $J(P) = \Delta^2(P)$.

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Let us now find $\Delta^2(P)$ by evaluating the integral over Ω in Eq. (13.18). We first reduce φ to the diagonal form (13.12) by the transformation (13.17). Then Ωs which are essential in the integral of the delta-function are close to a diagonal unitary matrix Ω_0. The integral can be calculated by substituting $\Omega = (1 + \mathrm{i}h)\Omega_0$ with an infinitesimal off-diagonal Hermitian matrix h. Since $\left[\Omega P \Omega^\dagger\right]_{ij} = \mathrm{i}h_{ij}\,(p_i - p_j)$ for $i \neq j$, we obtain

$$\begin{aligned}\Delta^{-2}(P) &= \int \mathrm{d}\Omega_0 \int \prod_{i<j} \mathrm{d}h_{ij}\, \delta^{(2)}(h_{ij}\,(p_i - p_j)) \\ &= \prod_{i<j} (p_i - p_j)^{-2}\,. \end{aligned} \tag{13.20}$$

This reproduces the Weyl measure (13.13).

We have therefore rewritten the Hermitian one-matrix model via N degrees of freedom in the spirit of Sect. 11.7. The integral on the RHS of Eq. (13.15) can be calculated as $N \to \infty$ using the saddle-point method.

To write down the saddle-point equation, let us introduce the spectral density

$$\rho(p) = \frac{1}{N} \sum_{i=1}^{N} \delta^{(1)}(p - p_i) \tag{13.21}$$

which becomes a continuous function of p as $N \to \infty$. It describes the distribution of eigenvalues of the matrix φ.

The spectral density (13.21) obeys

$$\rho(p) \geq 0\,, \tag{13.22}$$

$$\int \mathrm{d}p\, \rho(p) = 1 \tag{13.23}$$

as it follows from the definition (13.21).

Given the spectral density, we have

$$\frac{1}{N}\,\mathrm{tr}\,\varphi^k = \int \mathrm{d}p\, \rho(p)\, p^k \tag{13.24}$$

and, in particular,

$$\frac{1}{N}\,\mathrm{tr}\,V(\varphi) = \int \mathrm{d}p\, \rho(p)\, V(p)\,. \tag{13.25}$$

In the large-N limit where the integral over p_i is dominated by a saddle-point configuration, we obtain

$$W_k \stackrel{\mathrm{def}}{=} \left\langle \frac{1}{N}\,\mathrm{tr}\,\varphi^k \right\rangle_{1\mathrm{h}} \stackrel{N=\infty}{=} \int \mathrm{d}p\, \rho_{\mathrm{sp}}(p)\, p^k\,, \tag{13.26}$$

where $\rho_{\mathrm{sp}}(p)$ describes the distribution of eigenvalues at the saddle point.

Extrema of the integrand on the RHS of Eq. (13.15) are reached when

$$V'(p_i) \;=\; \frac{2}{N}\sum_{j\neq i}\frac{1}{p_i-p_j}\,, \tag{13.27}$$

where $V'(p) = \mathrm{d}V(p)/\mathrm{d}p$.

This determines the large-N saddle-point equation to be [BIP78]

$$V'(p) \;=\; 2\int \mathrm{d}\lambda \frac{\rho(\lambda)}{p-\lambda} \qquad \boxed{p \in \text{support of } \rho}\,, \tag{13.28}$$

where the RHS involves the principal part of the integral. Equation (13.28) holds only when p belongs to the support of ρ as is clear from the derivation.

Before solving the saddle-point equation (13.28), let us mention that the support of ρ must be finite for a general potential, say, the support is to be included in an interval $[a,b]$. Otherwise, the saddle-point equation would be inconsistent as $p\to\infty$ except for $V(p)$ which behaves asymptotically as $2\ln|p|$.

Remark on discretization of random surfaces

Matrix models are associated generically [Kaz85, Dav85, ADF85, KKM85] with discretization of random surfaces. The simplest Hermitian one-matrix model corresponds to a zero-dimensional embedding space, i.e. to two-dimensional Euclidean quantum gravity described by the partition function

$$\begin{aligned} Z_{\mathrm{2DG}} \;&=\; \int \mathcal{D}g\, \mathrm{e}^{-\int \mathrm{d}^2x\sqrt{g}(\Lambda - R/4\pi\mathcal{G})} \\ &=\; \int \mathcal{D}g\, \mathrm{e}^{-\int \mathrm{d}^2x\sqrt{g}\Lambda + \chi/\mathcal{G}}\,. \end{aligned} \tag{13.29}$$

Here Λ denotes the cosmological constant, R is the scalar curvature, and χ is the Euler characteristic of the two-dimensional world, while the coupling $\mathcal{G}$ weights topologies. The path integral in Eq. (13.29) is over all metrics $g_{\mu\nu}(x)$.

The idea of dynamical triangulation of random surfaces is to approximate a surface by a set of equilateral triangles. The coordination number (the number of triangles meeting at a vertex) does not necessarily equal six, which represents internal curvature of the surface.

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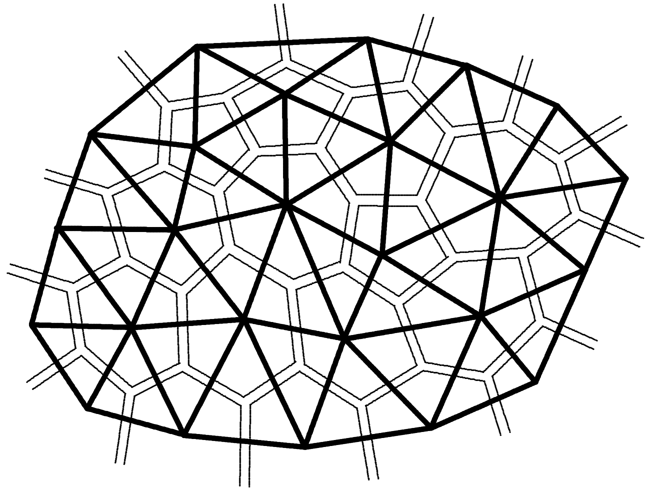

Fig. 13.1. Generic graph constructed from equilateral triangles (depicted by bold lines) and associated with dynamical triangulation of random surfaces. Its dual graph (depicted by double lines) coincides with that in the Hermitian one-matrix model with a cubic interaction potential.

The partition function (13.29) is approximated by

$$Z_{\mathrm{DT}} \;=\; \sum_{h} \mathrm{e}^{2(1-h)/\mathcal{G}} \sum_{T_h} \mathrm{e}^{-\sigma n_{\mathrm{t}}}\,, \tag{13.30}$$

where we split the sum over triangles into the sum over genus h and the sum over all possible triangulations T_h at fixed h. In (13.30) n_{t} denotes the number of triangles which is not fixed at the outset, but rather is a dynamical variable similar to that in Problem 1.12 on p. 27 for random paths.

The partition function (13.30) can be represented as a matrix model. A graph dual to a generic set of equilateral triangles coincides with a graph in the Hermitian one-matrix model with a cubic interaction as is depicted in Fig. 13.1. The precise statement is that Z_{DT} equals the (logarithm of the) partition function (13.1) with $N = \exp(1/\mathcal{G})$ and the cubic coupling constant $g_3 = \exp(-\sigma)$. This can be easily shown by comparing the graphs. The logarithm is needed to pick up connected graphs in the matrix model.

Analogously, the interaction $\operatorname{tr}\varphi^k$ in the matrix model is associated with discretization of random surfaces by regular k-gons, the area of which is $k-2$ times the area of the equilateral triangle.

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13.2 Hermitian one-matrix model (solution at $N = \infty$)

The saddle-point equation (13.28) can be solved by the Riemann–Hilbert method introducing an analytic function

$$W(\omega) = \int_a^b \mathrm{d}\lambda \frac{\rho(\lambda)}{\omega - \lambda}. \tag{13.31}$$

It has cuts (or simply one cut) in the complex ω-plane at the real axis where ρ has support. These cuts are included in the interval $[a, b]$.

Asymptotically, we have

$$W(\omega) \to \frac{1}{\omega} \tag{13.32}$$

as $\omega \to \infty$, as a consequence of the normalization (13.23) of ρ.

The idea is now to have $\operatorname{Re} W = V'/2$ at the support of ρ, i.e. where $\operatorname{Im} W \neq 0$, to satisfy Eq. (13.28). This is equivalent to the equation

$$\operatorname{Im} \left(V'W - W^2\right) = \left(V' - 2 \operatorname{Re} W\right) \operatorname{Im} W = 0 \tag{13.33}$$

which holds for the whole real axis: at the support owing to Eq. (13.28) and outside of the support since there $\operatorname{Im} W = 0$.

Equation (13.33) tells us that

$$V'W - W^2 = Q, \tag{13.34}$$

where an analytic function $Q(\omega)$ should have no singularities at the real axis. For a polynomial $V(p)$ it must be a polynomial of the same degree as $V'(p)/p$ to satisfy asymptotically Eq. (13.32).

We therefore find

$$W = \frac{V'}{2} - \frac{1}{2}\sqrt{(V')^2 - 4Q}, \tag{13.35}$$

where the minus sign is chosen to again provide the asymptotic behavior (13.32). Then ρ is given by the discontinuity of this W at the cuts (cut):

$$W(p \pm i0) = \frac{V'(p)}{2} \mp \mathrm{i}\pi\rho(p). \tag{13.36}$$

The simplest example is the Hermitian one-matrix model with the Gaussian potential when $V'(p) = \mu p$ ($\mu \equiv g_2$). The asymptotic behavior of Eq. (13.34) fixes $Q(p) = \mu$. Then Eq. (13.35) simplifies to

$$W(\omega) = \frac{\mu\,\omega}{2} - \frac{\mu}{2}\sqrt{\omega^2 - \frac{4}{\mu}} \tag{13.37}$$

for which the discontinuity determines the spectral density

$$\rho(p) \;=\; \frac{\mu}{2\pi}\sqrt{\frac{4}{\mu} - p^2}\,. \tag{13.38}$$

Note that this spectral density is nonnegative and has support on a finite interval $[-2/\sqrt{\mu}, 2/\sqrt{\mu}]$. The spectral density (13.38) was first calculated by Wigner [Wig51] and is called Wigner's *semicircle law*.

Problem 13.3 Calculate the density of eigenvalues for the Gaussian Hermitian one-matrix model using the Schwinger–Dyson equations.

Solution The calculation is similar to that in Problem 12.13 on p. 285. The difference is that now φ_{ij} is a Hermitian matrix with eigenvalues p_i which can take on values along the whole real axis.

The Schwinger–Dyson equations for W_n, defined in the Hermitian one-matrix model by Eq. (13.26), can be obtained from Eq. (13.8) by choosing $F[\varphi] = (\varphi^n)_{ij}$. Proceeding as before and using the large-N factorization, we obtain the set of equations [Wad81]

$$\left.\begin{aligned} \mu\, W_{n+1} \;&=\; \sum_{k=0}^{n-1} W_k W_{n-k} \qquad \text{for } n \geq 0\,, \\ W_0 \;&=\; 1\,. \end{aligned}\right\} \tag{13.39}$$

Introducing the generating function

$$W(p) \;\equiv\; \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{p-\varphi}\right\rangle_{\mathrm{1h}} \;=\; \sum_{n=0}^{\infty} \frac{W_n}{p^{n+1}}\,, \tag{13.40}$$

we rewrite Eq. (13.39) as the quadratic equation

$$\mu\, p\, W(p) - \mu \;=\; W^2(p)\,, \tag{13.41}$$

the solution of which is given by Eq. (13.37) determining the spectral density (13.38) which has support on a finite interval $[-2/\sqrt{\mu}, 2/\sqrt{\mu}]$ in analogy with the unitary one-matrix model it the weak-coupling regime.

We have already met the semicircle distribution in Problem 12.13 for the spectral density of the unitary one-matrix model at small λ (see Eq. (12.158)). This is because we can always substitute $U = \exp(\mathrm{i}\varphi)$ where U is unitary and φ is Hermitian and expand for small λ in φ up to the quadratic term. We then obtain the Hermitian model (13.1) with $\mu = 1/\lambda$ from the unitary model (12.135).

For a general polynomial potential, we are looking for a one-cut solution at small couplings $g_3, g_4, \ldots$ bearing in mind that it should look similar to the Gaussian case which is perturbed by the interactions. The expression (13.35) then takes the form

$$W(\omega) \;=\; \frac{V'(\omega)}{2} - \frac{M(\omega)}{2}\sqrt{(\omega - a)\,(\omega - b)}\,, \tag{13.42}$$

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where a and b are the ends of the cut and $M(\omega)$ is a polynomial of degree $K-2$ if $V(\omega)$ is a polynomial of degree K.

The coefficients of M are determined together with a and b from the asymptotic condition

$$\frac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} - M(\omega) \;\to\; \frac{2}{\omega^2} \tag{13.43}$$

as $\omega \to \infty$. There are precisely K conditions in Eq. (13.43) to unambiguously determine these K numbers.

A solution is acceptable if $M(p)$ is not negative in the interval $[a,b]$. Then the spectral density equals

$$\rho(p) \;=\; \frac{M(p)}{2\pi}\sqrt{(p-a)(b-p)} \tag{13.44}$$

which solves the problem for a general polynomial potential. This solution was first obtained in [BIP78] for cubic and quartic potentials.

For small values of the couplings $g_3, g_4, \ldots$, the one-cut solution is always realized. With increasing coupling, a third-order phase transition of the Gross–Witten type (see Sect. 12.9) may occur after which a more complicated multicut solution is realized.

An example of when such a phase transition happens is the quartic potential

$$V(p) \;=\; \frac{\mu}{2}p^2 + \frac{g_4}{4}p^4 \tag{13.45}$$

when the one-cut solution exists only for $-g_4 \le \mu^2/12$.

Problem 13.4 Elaborate the solution (13.44) for the quartic potential (13.45).

Solution Substituting the quartic potential (13.45) into Eq. (13.42), we obtain

$$W(p) \;=\; \frac{\mu p + g_4 p^3}{2} - \left(\frac{\mu + g_4 p^2 + g_4 a^2/2}{2}\right)\sqrt{p^2 - a^2}\,, \tag{13.46}$$

where

$$a^2 \;=\; \frac{2\mu}{3g_4}\left(-1 + \sqrt{1 + \frac{12 g_4}{\mu^2}}\right) \tag{13.47}$$

reproducing $a^2 \to 4/\mu$ as $g_4 \to 0$.

The RHS of Eq. (13.47) is well-defined only for $-g_4 \le \mu^2/12$ which determines the critical value $(g_4)_* = -\mu^2/12$. At $-g_4 \to \mu^2/12$ from below, two zeros of $M(p)$ approach two ends of the cut so that

$$\rho(p) \;\to\; \frac{\mu^2}{24\pi}\left(\frac{8}{\mu} - p^2\right)^{3/2}. \tag{13.48}$$

The one-cut solution is no longer realized for $-g_4 > \mu^2/12$.

The reason why we are interested in the one-cut solution is simple. This solution sums planar graphs of the Hermitian one-matrix model.

Remember that an effective expansion parameter associated with each quartic vertex in graphs is $-g_4/\mu^2$. Therefore, g_4 must be negative for the weight of each graph to be positive. At the critical value $-\mu^2/(g_4)_* = 12$, the sum of the planar graphs diverges, which determines the constant in Eq. (11.23) for the number of planar graphs with quartic vertices. Analogously, for a cubic potential, the solution (13.42) gives $\mu^3(g_3)_*^{-2} = 12\sqrt{3}$, which results in Eq. (11.43) for the number of trivalent planar graphs.

13.3 The loop equation

In the previous section, we have solved the Hermitian one-matrix model at $N = \infty$ using the saddle-point equation for the spectral density. This method of solution was historically the first one but cannot be extended to higher orders in $1/N^2$. In this section we present a very closely related method of solving the matrix model using loop equations which allows us to find a solution systematically order by order in $1/N^2$.

Choosing $F[\varphi] = (p-\varphi)^{-1}_{ij}$ in Eq. (13.8), we obtain the Schwinger–Dyson equation

$$\left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{V'(\varphi)}{p-\varphi}\right\rangle_{1\mathrm{h}} = \left\langle \frac{1}{N^2}\,\mathrm{tr}\,\frac{1}{p-\varphi}\,\mathrm{tr}\,\frac{1}{p-\varphi}\right\rangle_{1\mathrm{h}}. \tag{13.49}$$

Equation (13.49) can be expressed entirely via the resolvent

$$W(p) = \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{p-\varphi}\right\rangle_{1\mathrm{h}} \tag{13.50}$$

which is a Laplace transform of the "Wilson loop":

$$W(p) = \int_0^\infty \mathrm{d}l\,\mathrm{e}^{-pl}\left\langle \frac{1}{N}\,\mathrm{tr}\,\mathrm{e}^{l\varphi}\right\rangle_{1\mathrm{h}}. \tag{13.51}$$

The resulting loop equation reads

$$\int_{C_1}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{V'(\omega)}{(p-\omega)}W(\omega) = W^2(p) + \frac{1}{N^2}\frac{\delta}{\delta V(p)}W(p)\,, \tag{13.52}$$

where the contour C_1 encloses counterclockwise singularities of $W(\omega)$ leaving outside the pole at $\omega = p$ as depicted in Fig. 13.2. The contour integral on the LHS simply acts as a projector picking up negative powers of p.

At $N = \infty$, when the second term on the RHS can be omitted, Eq. (13.52) coincides for polynomial V with Eq. (13.34) derived above by

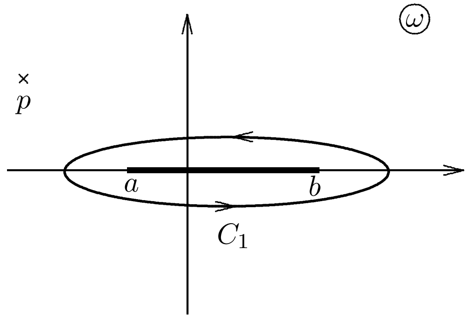

Fig. 13.2. Contour C_1 in the ω-plane for integration on the LHS of Eq. (13.52).

the other method. The polynomial Q can then be calculated by deforming the contour to infinity and taking the residue at $\omega = \infty$. The residue at $\omega = p$ simply yields $V'(p)W(p)$ which enters the LHS of Eq. (13.34).

The first term on the RHS of Eq. (13.52) is associated with the factorized part of the correlator, while the second term represents the connected part of the two-loop correlator which is $\sim 1/N^2$ as $N \to \infty$. It involves the variational derivative

$$\frac{\delta}{\delta V(p)} \quad = \quad -\sum_{k=0}^{\infty} p^{-k-1} \frac{\partial}{\partial t_k} \tag{13.53}$$

acting on $W(p)$. For this reason the operator (13.53) is often called the *loop insertion operator.*

Consequently, Eq. (13.52) is closed and determines $W(p)$ unambiguously, providing the boundary condition $W(p) \to 1/p$ is imposed as $p \to \infty$.

Note that we obtained a *single* (functional) equation for $W(p)$. This is due to the fact that $\operatorname{tr} V(\varphi)$ contains a complete set of traces $\operatorname{tr} \varphi^k$. They become independent as $N \to \infty$.

Problem 13.5 Obtain Eq. (13.52) from Eq. (13.49).

Solution The coupling t_k plays the role of a source for $\operatorname{tr} \varphi^k$:

$$\left\langle \frac{1}{N} \operatorname{tr} \varphi^k \right\rangle_{1\mathrm{h}} \quad = \quad -\frac{\partial}{\partial t_k} F \,, \tag{13.54}$$

where the free energy is

$$F \quad = \quad \frac{1}{N^2} \ln Z_{1\mathrm{h}} \,. \tag{13.55}$$

Analogously, using the definition (13.53) and Eq. (13.54), we find

$$\left\langle \frac{1}{N} \operatorname{tr} \frac{1}{p-\varphi} \right\rangle_{1\mathrm{h}} \quad = \quad \frac{\delta}{\delta V(p)} F \,. \tag{13.56}$$

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Applying the operator (13.53) one more time, we obtain [AM90] the connected correlator of two Wilson loops:

$$\frac{\delta}{\delta V(p_2)} W(p_1) \;=\; \left\langle \operatorname{tr} \frac{1}{(p_1-\varphi)} \operatorname{tr} \frac{1}{(p_2-\varphi)} \right\rangle_{1\mathrm{h}}^{\mathrm{conn}}, \tag{13.57}$$

which enters the RHS of Eq. (13.52). The higher-loop correlators can be obtained by further applying $\delta/\delta V(p_i)$.

Instead of introducing the sources t_k, we can consider $V(p)$ as a source for the Wilson loop $\operatorname{tr}[1/(p-\varphi)]$ from the very beginning by writing

$$\operatorname{tr} V(\varphi) \;=\; \int_{-\mathrm{i}\infty+0}^{+\mathrm{i}\infty+0} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} V(\omega) \operatorname{tr} \frac{1}{\omega-\varphi}. \tag{13.58}$$

According to this definition $\delta V(p)/\delta V(q) = 1/(p-q)$ which plays the role of a delta-function when integrated along the imaginary axis.

The LHS of Eq. (13.49) is transformed into the LHS of Eq. (13.52) using

$$\frac{1}{N} \operatorname{tr} \frac{V'(\varphi)}{p-\varphi} \;=\; \int_{-\mathrm{i}\infty+0}^{+\mathrm{i}\infty+0} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{V'(\omega)}{(p-\omega)} \frac{1}{N} \operatorname{tr} \frac{1}{\omega-\varphi}, \tag{13.59}$$

taking the average and deforming the contour to encircle singularities of $W(\omega)$.

Remark on the Virasoro constraints

The loop equation (13.52) can be represented as a set of Virasoro constraints imposed on the partition function.

We first rewrite Eq. (13.52) using the definitions (13.1) and (13.4) as

$$\frac{1}{Z_{1\mathrm{h}}} \sum_{n=-1}^{\infty} \frac{1}{p^{n+2}} L_n Z_{1\mathrm{h}} \;=\; 0, \tag{13.60}$$

where the operators

$$L_n \;=\; \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{N^2} \sum_{0\leq k\leq n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} \tag{13.61}$$

satisfy the Virasoro algebra

$$[L_n, L_m] \;=\; (n-m) L_{n+m}. \tag{13.62}$$

Equation (13.52) is therefore represented as the Virasoro constraints

$$L_n Z_{1\mathrm{h}} \;=\; 0 \qquad \text{for } \; n \geq -1\,. \tag{13.63}$$

It is enough to consider the constraints (13.63) only with $n = 2$ and $n = -1$. Then all the others are satisfied because of Eq. (13.62).

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13.4 Solution in $1/N$

Equation (13.52) can be solved order by order of the genus expansion in $1/N^2$.

The genus zero one-cut solution to Eq. (13.52) can be written as [Mig83]

$$W_0(p) = \int_{C_1} \frac{\mathrm{d}\omega}{4\pi\mathrm{i}} \frac{V'(\omega)}{(p-\omega)} \sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}\,, \tag{13.64}$$

where a and b are determined by

$$\int_{C_1} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} = 0\,, \qquad \int_{C_1} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\omega V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} = 2\,. \tag{13.65}$$

Performing the contour integral in (13.64) by taking the residues at $\omega = p$ and $\omega = \infty$, we reproduce Eq. (13.42) for polynomial V. However, Eq. (13.64) remains valid in the more general case of nonpolynomial V, e.g. having logarithmic singularities. The position of the cut is always such as to avoid these singularities of V.

Problem 13.6 Reproduce Eqs. (13.37) and (13.46) for the Gaussian and quartic potentials from the general one-cut solution (13.64) and (13.65).

Solution We substitute $a = -b$ for even potentials, then the first equation in (13.65) is always satisfied, while the second one yields

$$\sum_{j=1}^{J} g_{2j} \frac{(2j)!}{(j!)^2} \left(\frac{a}{2}\right)^{2j} = 2 \tag{13.66}$$

for an even polynomial potential of degree $K = 2J$. This equation is derived by expanding the square root in a^2/ω^2 and taking the residue at infinity. Analogously, Eq. (13.64) yields

$$M(p) = \sum_{j=1}^{J} p^{2j-2} \sum_{k=0}^{J-j} g_{2k+2j} \frac{(2k)!}{(k!)^2} \left(\frac{a}{2}\right)^{2k} \tag{13.67}$$

for the polynomial $M(p)$ in Eq. (13.42).

A solution to Eq. (13.66) reproduces the above explicit calculation for the Gaussian and quartic potentials. Analogously, Eqs. (13.37) and (13.46) are reproduced by substituting Eq. (13.67) into Eq. (13.42).

Problem 13.7 Elaborate the solution (13.64) and (13.65) for the Penner model where the potential is logarithmic:

$$V(\varphi) = \varphi - \lambda \ln \varphi\,. \tag{13.68}$$

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Solution The calculation is similar to the previous Problem, while now the residue is to be taken at $\omega = 0$ since

$$V'(\omega) = 1 - \frac{\lambda}{\omega} \tag{13.69}$$

has a pole there. For $\lambda > 0$ we find

$$a = 2 + \lambda - 2\sqrt{1+\lambda}\,, \qquad b = 2 + \lambda + 2\sqrt{1+\lambda} \tag{13.70}$$

and

$$\rho(p) = \frac{\sqrt{(b-p)(p-a)}}{2\pi p} \tag{13.71}$$

so that $W(p)$ is analytic at $p = 0$. The Gaussian formula (13.38) is reproduced as $\lambda \to \infty$ substituting $p \to \lambda + p$.

Note that both a and b are positive so the support is located for $\lambda > 0$ at the positive real axis where $\rho(p) > 0$. This is a manifestation of the general property already mentioned that the cut always avoids possible singularities of the potential. The location of the support of eigenvalues in the complex ω-plane for $\lambda < 0$ is studied in [AKM94].

The multiloop correlators in genus zero can be obtained from $W_0(p)$ given by Eq. (13.64) applying the loop insertion operator (13.53). For example, the two-loop correlator [AJM90]

$$W_0(p,q) = \frac{1}{4(p-q)^2}\left\{\frac{2pq-(p+q)(a+b)+2ab}{\sqrt{(p-a)(p-b)}\sqrt{(q-a)(q-b)}} - 2\right\} \tag{13.72}$$

depends on the potential V only via a and b but not explicitly. This property is called *universality.** It does not hold for higher multiloop correlators.

To calculate the $1/N^2$ correction to the genus-zero result (13.64), we substitute

$$W_0(p,p) = \frac{(a-b)^2}{16(p-a)^2(p-b)^2} \tag{13.73}$$

extracted from Eq. (13.72) into the RHS of Eq. (13.52). We can now obtain $W_1(p)$ by solving a linear equation which, in turn, determines F_1.

An advantage of this method of solving the Hermitian one-matrix model using the loop equation over the orthogonal polynomial technique, used originally [Bes79, IZ80, BIZ80] in calculating the higher genera for polynomial potentials, is that now the free energy generates all multiloop correlators at a given genus.

* An analog of this correlator in condensed-matter physics is the correlator of two densities of energy eigenvalues which is universal [BZ93].

Remark on the iterative solution

The iterative procedure [ACK93] of solving the loop equation is based on the genus-zero solution (13.64). Inserting the genus expansion of $W(p)$ and F:

$$W(p) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W_h(p)\,, \qquad F = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} F_h \quad \text{with} \quad W_h(p) = \frac{\delta F_h}{\delta V(p)}\,, \tag{13.74}$$

into Eq. (13.52), we obtain the following equation for $W_h(p)$ at $h \geq 1$:

$$\begin{aligned} &\int\limits_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(p-\omega)} W_h(\omega) - 2\, W_0(p)\, W_h(p) \\ &\qquad = \sum_{h'=1}^{h-1} W_{h'}(p)\, W_{h-h'}(p) + \frac{\delta}{\delta V(p)} W_{h-1}(p)\,. \end{aligned} \tag{13.75}$$

It expresses $W_h(p)$ entirely in terms of $W_{h'}(p)$ with $h' < h$. This makes it possible to solve Eq. (13.75) iteratively genus by genus.

The iterative procedure simplifies if we introduce, instead of the coupling constants t_j, the moments M_k and J_k defined for $k \geq 1$ by

$$\left.\begin{aligned} M_k &= \int\limits_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(\omega-a)^{k+1/2}\,(\omega-b)^{1/2}}\,, \\ J_k &= \int\limits_{C_1} \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \frac{V'(\omega)}{(\omega-a)^{1/2}\,(\omega-b)^{k+1/2}}\,. \end{aligned}\right\} \tag{13.76}$$

These moments depend on the couplings t_j both explicitly and via a and b which are determined by Eq. (13.65). Note that M_k and J_k depend explicitly only on t_j with $j \geq k+1$.

Problem 13.8 Elaborate the moments (13.76) for the quartic potential (13.45).

Solution Given Eq. (13.47) for a, we simply calculate the moments (13.76) for the quartic potential (13.45), taking the residue at infinity. This results in an explicit representation of the moments via μ and g_4 which are given by the algebraic formulas

$$M_1 = J_1 = \mu + \frac{3}{2} g_4 a^2\,, \qquad M_2 = -J_2 = 2 g_4 a\,, \qquad M_3 = J_3 = g_4\,, \tag{13.77}$$

and $M_k = J_k = 0$ for $k \geq 4$.

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The main motivation for introducing the moments (13.76) is that $W_h(\lambda)$ depends only on $2 \times (3h-1)$ lower moments ($2 \times (3h-2)$ for F_h). This is in contrast to the t-dependence of W_h and F_h which always depend on the infinite set of t_j ($1 \leq j < \infty$).

To find F_h, we first solve Eq. (13.75) for $W_h(\lambda)$ and then use the last equation in (13.74). The result in genus one reads [ACM92]

$$F_1 \;=\; -\frac{1}{24}\ln\left(M_1 J_1\right) - \frac{1}{6}\ln\left(b-a\right). \tag{13.78}$$

The genus-two results are obtained in [ACK93]. More details on this subject can be found in Section 4.3 of the book [ADJ97].

The results for F_1 and F_2 for the quartic potential were first obtained in [Bes79] using the method of orthogonal polynomials.

13.5 Continuum limit

Continuum limits of the Hermitian one-matrix model are reached at the points of phase transitions. While no phase transition is possible at finite N since the system has a finite number of degrees of freedom, it may occur as $N \to \infty$ which plays the role of a statistical limit as has already been pointed out in the Remark in Sect. 11.8.

This third-order phase transition is of Gross–Witten type (see Sect. 12.9). It is associated with divergence of the sum over graphs at each fixed genus rather than with divergence of the sum over all graphs. The contribution of a graph with n_0 trivalent vertices is $\sim (-g_3)^{n_0}$ but an entropy (= the number) of such graphs at fixed genus is given by Eq. (11.23) so the sum can diverge at a certain critical value of g_3 calculated in Sect. 13.2.

This divergence has nothing to do with the divergence of the sum over all graphs which always occurs owing to a factorial growth of the total number of diagrams. The latter divergence is simply associated with the divergence of the integral over φ. For an even potential V, the couplings g_k are negative for $k > 2$ so the potential V is upside-down.

The phase structure of the Hermitian one-matrix model can be determined from the spectral density $\rho(p)$ given by (13.44) which vanishes under normal circumstances as a square root at both ends of its support. The critical behavior emerges when one or more roots of $M(p)$ approaches the end points a or b.

For example, the even potential

$$V(\varphi) \;=\; \frac{1}{\beta}\sum_{j=1}^{J}(-1)^{j-1}\frac{J!(j-1)!}{(J-j)!(2j)!}\left(\frac{\mu\varphi^2}{2}\right)^j \tag{13.79}$$

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becomes critical at $\beta = 1$ when $(J-1)$ zeros of $M(p)$ approach each of the two end points of the cut. They are determined by the equation

$$\beta \;=\; \sum_{j=1}^{J} (-1)^{j-1} \frac{J!}{(J-j)!j!} \left(\frac{\mu a^2}{8}\right)^j \;=\; 1 - \left(1 - \frac{\mu a^2}{8}\right)^J \tag{13.80}$$

which results from the substitution of Eq. (13.79) into Eq. (13.66).

The critical potential (13.79) with $\beta = 1$ is associated with the Jth *multicritical* point [Kaz89]. The case of $J = 2$ describes two-dimensional quantum gravity. The resulting continuum theory is unitary only at this critical point.

The continuum limit can be obtained near the critical point:

$$p^2 \;\to\; a_{\rm c}^2 + \epsilon\pi\,, \qquad a^2 \;\to\; a_{\rm c}^2 - \epsilon\sqrt{\Lambda}, \tag{13.81}$$

so that π plays the role of the continuum momentum and Λ is the cosmological constant.

The susceptibility near the critical point can be represented by the genus expansion

$$\begin{aligned} f(\beta) \;&\stackrel{\rm def}{=}\; \frac{1}{N^2}\left(\frac{d}{d\,1/\beta}\right)^2 \ln Z_{\rm 1h} \\ &=\; {\rm const} \;+ \sum_h N^{-2h}\,(1-\beta)^{-\gamma_h}\, f_h \end{aligned} \tag{13.82}$$

with the indices

$$\gamma_h \;=\; -\frac{1}{J} + \frac{2J+1}{J} h\,. \tag{13.83}$$

The genus-zero contribution to the susceptibility (13.82) does not diverge but rather exhibits a root singularity. This can be easily deduced by noting that f to genus zero is analytic in a^2 and contains a term $\sim \epsilon$, when the expansion (13.81) is substituted. According to Eq. (13.80), we have

$$\epsilon \;=\; \frac{a_{\rm c}^2}{\sqrt{\Lambda}}(1-\beta)^{1/J} \tag{13.84}$$

for the Jth multicritical point which explains Eqs. (13.82) and (13.83) to genus zero.

The dimensional cutoff ϵ should depend on N in such a way for the parameter $\mathcal{G} = N^{-2}\epsilon^{-2J-1}$ of the genus expansion to remain finite as $N \to \infty$. Then all terms of the genus expansion contribute in the continuum limit. This continuum limit was obtained in [BK90, DS90, GM90a] and is called the *double scaling limit*.

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The double scaling limit of the partition function (13.1) determines the genus expansion of the continuum partition function:

$$\begin{aligned}\ln Z_{1\mathrm{h}} \quad &\to \quad \text{const} \; + \sum_h N^{2-2h} \frac{(1-\beta)^{2-\gamma_h}}{(2-\gamma_h)(1-\gamma_h)} f_h \\ &= \quad \text{const} \; + \sum_h \left(\frac{\mathcal{G}}{\Lambda^{J+1/2}} \right)^{h-1} \frac{a_c^{2(2J+1)(h-1)}}{(2-\gamma_h)(1-\gamma_h)} f_h \,. \end{aligned} \tag{13.85}$$

At $J = 2$ this determines the partition function (13.29) of two-dimensional quantum gravity.

It is possible to construct explicitly a continuum theory which interpolates between multicritical points. We associate with Jth multicritical behavior a conformal operator of a certain scale dimension and introduce a proper source T_k.

The relation between the set of sources t_{2k} for the Hermitian one-matrix model with an even potential when $t_{2k+1} = 0$ and their continuum counterparts T_k can be obtained* from the equation

$$W(p) - \frac{1}{2} V'(p) \quad = \quad \frac{1}{\epsilon\sqrt{\mathcal{G}}} \left[2W_{\mathrm{cont}}(\pi) - \mathcal{V}'(\pi) \right] \tag{13.86}$$

describing [Dav90, AM90, FKN91] a multiplicative renormalization of the Wilson loops.

Then a source for a continuum Wilson loop is

$$\mathcal{V}(\pi) \;\; = \;\; \sum_{n=0}^{\infty} T_n \pi^{n+1/2} \tag{13.87}$$

and

$$\frac{\delta}{\delta \mathcal{V}(\pi)} \;\; = \;\; -\sum_{n=0}^{\infty} \pi^{-n-3/2} \frac{\partial}{\partial T_n} \tag{13.88}$$

is the continuum loop insertion operator:

$$W_{\mathrm{cont}}(\pi_1, \ldots, \pi_m) \;\; = \;\; \mathcal{G} \frac{\delta}{\delta \mathcal{V}(\pi_1)} \cdots \frac{\delta}{\delta \mathcal{V}(\pi_m)} \ln Z_{\mathrm{cont}} \,, \tag{13.89}$$

where

$$Z_{\mathrm{cont}} \;\; \propto \;\; \sqrt{Z_{1\mathrm{h}}} \qquad\qquad \boxed{\text{even } V(\varphi)} \tag{13.90}$$

up to an infinite constant which is determined only by genus zero. The appearance of the square root is associated with a "doubling" of degrees of freedom for the even potential.

* These (linear algebraic) relations are obtained [MMM91] equating positive powers of p or π in Eq. (13.86) and using Eq. (13.81).

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The continuum loop equation can be obtained from Eq. (13.52) by substituting Eq. (13.86) and using Eqs. (13.87) and (13.88):

$$\int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \frac{\mathcal{V}'(\Omega)}{(\pi-\Omega)} W_{\mathrm{cont}}(\Omega) \;=\; W_{\mathrm{cont}}^2(\pi) + \mathcal{G}\frac{\delta W_{\mathrm{cont}}(\pi)}{\delta \mathcal{V}(\pi)} + \frac{\mathcal{G}}{16\pi^2} + \frac{T_0^2}{16\pi}\,. \tag{13.91}$$

This equation describes a model which interpolates between different multicritical points. The Jth multicritical point corresponds to $T_k = 0$ except for $k = 0$ and $k = J$ while

$$\sqrt{\Lambda} \;=\; \left(\frac{(-1)^{J-1}2^J J!}{(2J+1)!!}\frac{T_0}{T_J}\right)^{1/J}. \tag{13.92}$$

The continuum loop equation (13.91) can be solved order by order in $\mathcal{G}$ (genus expansion) analogously to that of Sect. 13.4. If $\mathcal{V}(\pi)$ is a polynomial ($T_k = 0$ for $k > K$), $K-1$ lower coefficients of the asymptotic expansion of $W_{\mathrm{cont}}(\pi)$ are not fixed and should be determined by requiring the one-cut analytic structure in π.

The continuum analog of Eq. (13.64) is given by

$$W_{\mathrm{cont}}^{(0)}(\pi) \;=\; \int_{C_1} \frac{\mathrm{d}\Omega}{4\pi \mathrm{i}} \frac{\mathcal{V}'(\Omega)}{(\pi-\Omega)} \frac{\sqrt{\pi-u}}{\sqrt{\Omega-u}}\,, \tag{13.93}$$

where u coincides to genus zero with $-\sqrt{\Lambda}$ at a given multicritical point. Then the vanishing of the $1/\sqrt{\pi}$ term is equivalent to Eq. (13.92). The cut of $W_{\mathrm{cont}}^{(0)}(\pi)$ is from u to ∞. This is because we are magnifying the region near the end a of the cut in the one-matrix model.

The function u versus T_k is determined to all genera from the asymptotic behavior. This dependence can be obtained by comparing $1/\pi$ terms in Eq. (13.91). Denoting the derivative with respect to $x = -T_0/2$ by D, this relation can be represented conveniently as

$$\int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \mathcal{V}'(\Omega) \left(D W_{\mathrm{cont}}(\Omega) + \frac{1}{2\sqrt{\Omega}} \right) \;=\; 0\,, \tag{13.94}$$

where

$$D W_{\mathrm{cont}}^{(0)}(\pi) + \frac{1}{2\sqrt{\pi}} \;=\; \frac{1}{2\sqrt{\pi-u}} \tag{13.95}$$

for the genus-zero solution (13.93).

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Remark on the KdV hierarchy

Equation (13.95) can be extended to all genera using the representation

$$\begin{aligned} DW_{\text{cont}}(\pi) + \frac{1}{2\sqrt{\pi}} &= \left\langle x \middle| \Big(-\mathcal{G}D^2 - u(x) + \pi \Big)^{-1} \middle| x \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{R_n[u]}{\pi^{n+1/2}} \equiv R(\pi)\,, \end{aligned} \tag{13.96}$$

where the diagonal resolvent of the Sturm–Liouville operator is expressed via the Gel'fand–Dikii differential polynomials [GD75]

$$R_n[u] = 2^{-n-1} \left(\frac{\mathcal{G}}{2} D^2 + u + D^{-1} u D \right)^n \cdot 1\,. \tag{13.97}$$

We have explicitly

$$R_0 = \frac{1}{2}, \qquad R_1 = \frac{u}{4}, \qquad R_2 = \frac{\mathcal{G}}{16} D^2 u + \frac{3}{16} u^2\,, \qquad \dots \tag{13.98}$$

for the lower polynomials. Equation (13.97) can be easily obtained from Eq. (1.127) derived in Problem 1.11 on p. 25.

Substituting the RHS of Eq. (13.96) into Eq. (13.94), we obtain the *string equation* [GM90b, BDS90]

$$\sum_{k=0}^{\infty} \left(k + \tfrac{1}{2}\right) T_k R_k[u] = 0 \tag{13.99}$$

which determines u versus T_k.

The meaning of u is clear from Eqs. (13.96) and (13.89):

$$u = 4R_1 = 2\,\mathcal{G} D^2 \ln Z_{\text{cont}}\,, \tag{13.100}$$

i.e. u is the continuum susceptibility. It is negative to genus zero because of the performed "renormalization".

Problem 13.9 Elaborate Eq. (13.99) for two-dimensional quantum gravity.

Solution Choosing $T_2 = 16/15$ and using Eq. (13.92), which gives $\Lambda = -T_0/2$, and Eq. (13.98), we represent Eq. (13.99) as the Painlevé equation

$$\Lambda = u^2 + \frac{\mathcal{G}}{3} D^2 u\,, \qquad D = \frac{\mathrm{d}}{\mathrm{d}\Lambda}\,, \tag{13.101}$$

the solution of which is given by a Painlevé transcendental. It can be found by solving Eq. (13.101) iteratively in $\mathcal{G}$:

$$u = \sqrt{\Lambda} \left[-1 + \sum_{h=1}^{\infty} \left(\frac{\mathcal{G}}{\Lambda^{5/2}} \right)^h \chi_h \right], \tag{13.102}$$

where the numerical coefficients $\chi_h > 0$ are determined by a recursion relation. This reproduces the indices (13.83) for $J = 2$. Substituting into Eq. (13.100) and integrating, one can obtain the genus expansion of the partition function of two-dimensional Euclidean quantum gravity introduced in Eq. (13.85). The series is asymptotic, since $\chi_h \sim (2h)!$ for large h.

These results were first obtained in [BK90, DS90, GM90a].

Problem 13.10 Show that the ansatz (13.96) satisfies Eq. (13.91).

Solution It is convenient to introduce

$$\widetilde{W}(\pi) = W_{\text{cont}}(\pi) - \frac{T_0}{4\sqrt{\pi}}\,, \tag{13.103}$$

$$\widetilde{\mathcal{V}}(\pi) = \mathcal{V}(\pi) - T_0\sqrt{\pi} = \sum_{n=1}^{\infty} T_n \pi^{n+1/2}. \tag{13.104}$$

In the new variables, the last term on the RHS of the loop equation (13.91) disappears and it can be written as

$$\int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \frac{\widetilde{\mathcal{V}}'(\Omega)}{(\pi-\Omega)} \widetilde{W}(\Omega) = \widetilde{W}^2(\pi) + \mathcal{G}\frac{\delta \widetilde{W}(\pi)}{\delta \mathcal{V}(\pi)} - \frac{3\mathcal{G}}{16\pi^2}\,. \tag{13.105}$$

We then apply the operator

$$\Delta_\pi = -\left(\frac{\mathcal{G}}{2}D^3 + uD + Du - 2\pi D\right) D\,, \tag{13.106}$$

which annihilates $\widetilde{W}(\pi)$ given by the Gel'fand–Dikii ansatz (13.96) (cf. Eq. (1.127)), to both sides of Eq. (13.105).

The following terms emerge:

$$\begin{aligned}\Delta_\pi \int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \frac{\widetilde{\mathcal{V}}'(\Omega)}{(\pi-\Omega)} \widetilde{W}(\Omega) &= \int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \frac{\widetilde{\mathcal{V}}'(\Omega)}{(\pi-\Omega)} \Delta_\Omega \widetilde{W}(\Omega) \\ +2D\int_{C_1} \frac{\mathrm{d}\Omega}{2\pi \mathrm{i}} \widetilde{\mathcal{V}}'(\Omega) D\widetilde{W}(\Omega) &= 2D\sum_{k=1}^{\infty}(k+\tfrac{1}{2})T_k R_k[u] = 1\,,\end{aligned} \tag{13.107}$$

$$\begin{aligned}\Delta_\pi \widetilde{W}^2(\pi) &= 2\widetilde{W}\Delta\widetilde{W} - 4\mathcal{G}D\tilde{W}D^3\widetilde{W} - 3\mathcal{G}(D^2\widetilde{W})^2 + 4(\pi-u)(D\widetilde{W})^2 \\ &= -4\mathcal{G}RD^2R - 3\mathcal{G}(DR)^2 + 4(\pi-u)R^2,\end{aligned} \tag{13.108}$$

$$\begin{aligned}\Delta_\pi \frac{\delta}{\delta\mathcal{V}(\pi)}\widetilde{W}(\pi) &= \frac{\delta}{\delta\mathcal{V}(\pi)}\Delta_\pi\widetilde{W}(\pi) + 2\frac{\delta u}{\delta\mathcal{V}(\pi)}D^2\widetilde{W}(\pi) + \frac{\delta(Du)}{\delta\mathcal{V}(\pi)}D\widetilde{W}(\pi) \\ &= 4(DR)^2 + 2RD^2R\,.\end{aligned} \tag{13.109}$$

We have used Eq. (13.99) in deriving Eq. (13.107), the equation

$$\frac{\delta}{\delta\mathcal{V}(\pi)}u = 2DR(\pi)\,, \tag{13.110}$$

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which arises from acting by the loop insertion operator on Eq. (13.100) and the expansion of which in $1/\pi$ reproduces the Korteweg–de Vries (KdV) hierarchy,

$$-\frac{\partial u}{\partial T_n} \;=\; 2DR_{n+1}[u]\,, \tag{13.111}$$

in deriving Eq. (13.109), and the fact that $\Delta\widetilde{W} = 0$ for the Gel'fand–Dikii ansatz owing to Eq. (1.127).

Combining the RHSs of Eqs. (13.107), (13.108), and (13.109), we obtain

$$1 \;=\; -2\mathcal{G}RD^2R + \mathcal{G}(DR)^2 + 4(\pi - u)R^2 \tag{13.112}$$

which is the same as Eq. (1.134) satisfied by the Gel'fand–Dikii resolvent. Its solution is unambiguous (at least perturbatively in $\mathcal{G}$).

Thus, the Gel'fand–Dikii ansatz is obtained [DVV91] as a solution of the continuum loop equation (13.91).

Remark on the continuum Wilson loop

The continuum Wilson loops are related [Kaz89] to boundaries of surfaces in two-dimensional quantum gravity. Given the path integral (13.29) over surfaces with fixed boundary, we integrate over metrics (including the metric at the boundary). The result can depend solely on the length of the boundary which is the only invariant. Then $W_{\rm cont}(\pi)$ is simply the Laplace transform of this object.

Remark on the continuum Virasoro constraints

The continuum loop equation (13.91) can be represented as a set of the continuum Virasoro constraints [FKN91, DVV91]

$$\mathcal{L}_n^{\rm cont} Z_{\rm cont} \;=\; 0 \qquad \text{for} \;\; n \geq -1\,, \tag{13.113}$$

where

$$\mathcal{L}_n^{\rm cont} \;=\; \sum_{k=0}^{\infty} \left(k+\tfrac{1}{2}\right) T_k \frac{\partial}{\partial T_{k+n}} + \mathcal{G} \sum_{0\leq k\leq n-1} \frac{\partial^2}{\partial T_k \partial T_{n-k-1}} + \frac{\delta_{0,n}}{16} + \frac{\delta_{-1,n} T_0^2}{16\mathcal{G}} \tag{13.114}$$

obey the Virasoro algebra (13.62). This is a consequence of conformal invariance of the continuum theory.

The Virasoro constraints (13.113) and (13.114) can be obtained [MMM91] from their matrix-model counterparts (13.63), (13.61) by passing from the variables t_{2k} to the variables T_k.

Remark on the Kontsevich matrix model

The above continuum model interpolating between multicritical points can be formulated as a matrix model [Kon91]:

$$Z_{\text{Kont}}[M] \;=\; \frac{\displaystyle\int \mathrm{d}X\, \mathrm{e}^{\operatorname{tr}\left(\frac{\sqrt[4]{\mathcal{G}}}{6}X^3-\frac{1}{2}MX^2\right)}}{\displaystyle\int \mathrm{d}X\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}MX^2}}\,, \tag{13.115}$$

where the integral goes over the Hermitian $N\times N$ matrix X. The RHS of Eq. (13.115) is well-defined perturbatively in $\mathcal{G}$.

The couplings T_k are expressed via the positive-definite Hermitian matrix M by

$$T_k \;=\; \frac{\sqrt{\mathcal{G}}}{k+\frac{1}{2}}\operatorname{tr}\left(M^{-2k-1}\right)-\frac{2}{3}\delta_{1k}\,. \tag{13.116}$$

The identification (13.116) makes sense as $N\to\infty$ when all $\operatorname{tr}\left(M^{-2k-1}\right)$ become independent but M is chosen such that they are finite. Alternatively, the standard topological expansion of the Kontsevich model in $1/N^2$ is associated with $\mathcal{G}\sim 1/N^2$.

The partition function of continuum two-dimensional quantum gravity coincides with the partition function of the Kontsevich model:

$$Z_{\text{cont}}[T] \;=\; Z_{\text{Kont}}[M]\,. \tag{13.117}$$

This equality is valid in the sense of an asymptotic expansion at large M, each term of which is finite providing M is positive definite.

Remark on 2D topological gravity

Equation (13.117) represents the fact that quantum gravity is, in fact, a *topological* theory in two dimensions:

$$\boxed{\text{2D quantum gravity}} \;=\; \boxed{\text{2D topological gravity}}\,. \tag{13.118}$$

A crucial property of topological theories is that correlators of operators $\sigma_{n_i}(x_i)$ with definite (nonnegative integer) scale dimension n_i, located at the point x_i of a two-dimensional Riemann surface of genus h, depend only on the dimensions n_i and genus h but not on the metric on the surface and, therefore, not on the positions of the punctures x_i. The Kontsevich matrix model appeared [Kon91] as an explicit realization of the Witten geometric formulation [Wit90] of two-dimensional topological gravity.

13.6 Hermitian multimatrix models

An obvious extension of the Hermitian one-matrix model is the model of two Hermitian matrices φ_1 and φ_2. The partition function of the Hermitian two-matrix model is

$$Z_{2\mathrm{h}} \;=\; \int \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{e}^{N\,\mathrm{tr}\,[-V(\varphi_1)-V(\varphi_2)+\varphi_1\varphi_2]}\,, \tag{13.119}$$

where for simplicity we take the same potentials for self-interactions of each matrix.

The presence of two matrices adds matter to two-dimensional gravity. The Hermitian two-matrix model is precisely associated with the Ising model on a random two-dimensional lattice.

There is a vast literature on the Hermitian two-matrix model starting from the work by Itzykson and Zuber [IZ80], who showed how to reduce it to an eigenvalue problem. We shall rather briefly review the loop equations for the Hermitian two-matrix model.

Let us define the Wilson loop average and the one-link correlator in the Hermitian two-matrix model (13.119), respectively, by

$$W(\lambda) \;=\; \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{\lambda-\varphi_1}\right\rangle_{2\mathrm{h}}, \tag{13.120}$$

$$G(\nu,\lambda) \;=\; \left\langle \frac{1}{N}\,\mathrm{tr}\left(\frac{1}{(\nu-\varphi_1)}\frac{1}{(\lambda-\varphi_2)}\right)\right\rangle_{2\mathrm{h}}. \tag{13.121}$$

The definition of $W(\lambda)$ is similar to Eq. (13.50) while $G(\nu,\lambda)$, which is symmetric in ν and λ since the potentials of self-interaction are the same for both matrices, is absent in the one-matrix model. Expanding $G(\nu,\lambda)$ in $1/\nu$, we obtain

$$\left.\begin{aligned} G(\nu,\lambda) \;&=\; \frac{W(\lambda)}{\nu}+\sum_{n=1}^{\infty}\frac{G_n(\lambda)}{\nu^{n+1}}\,, \\ G_n(\lambda) \;&=\; \left\langle \frac{1}{N}\,\mathrm{tr}\left(\varphi_1^n\frac{1}{\lambda-\varphi_2}\right)\right\rangle_{2\mathrm{h}}. \end{aligned}\right\} \tag{13.122}$$

In the large-N limit, the correlator $G(\nu,\lambda)$ obeys the following loop equation:

$$\int\limits_{C_1}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{V'(\omega)}{(\nu-\omega)}\,G(\omega,\lambda) \;=\; W(\nu)\,G(\nu,\lambda)+\lambda G(\nu,\lambda)-W(\nu)\,, \tag{13.123}$$

where the contour C_1 encircles counterclockwise the cut (or cuts) of the function $G(\omega,\lambda)$ as depicted in Fig. 13.2.

To analyze Eq. (13.123), let us consider the Hermitian two-matrix model with the general potential (13.4). The solution for $W(\lambda)$ versus $V(\lambda)$ is determined by the equation

$$\sum_{k\geq 1} g_k G_{k-1}(\lambda) \quad = \quad \lambda W(\lambda) - 1 \tag{13.124}$$

which is just the $1/\nu$ term of the expansion of Eq. (13.123) in $1/\nu$.

The functions $G_n(\lambda)$ are expressed via $W(\lambda)$ using the recurrence relation

$$\left.\begin{aligned} G_{n+1}(\lambda) &= \int_{C_1} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{V'(\omega)}{(\lambda-\omega)} G_n(\omega) - W(\lambda)\, G_n(\lambda)\,, \\ G_0(\lambda) &= W(\lambda) \end{aligned}\right\} \tag{13.125}$$

which is obtained by expanding Eq. (13.123) in $1/\lambda$. If $V(\lambda)$ is a polynomial of degree K, Eq. (13.124) contains $W(\lambda)$ up to degree K and the solution is algebraic [GN91, Alf93, Sta93].

For a cubic potential, this equation for $W(\lambda)$ is cubic and determines the critical index of the susceptibility $\gamma_0 = -1/3$. This is in contrast to the Hermitian one-matrix model where the loop equation is quadratic in $W(\lambda)$. We see that matter changes [Kaz86] the critical behavior of pure quantum gravity. The continuum theory associated with the $\gamma_0 = -1/3$ critical point of the Hermitian two-matrix model is unitary.

The correlator $G(\nu, \lambda)$ is symmetric in ν and λ for any solution of Eq. (13.124). This symmetry requirement can be used directly to determine $W(\lambda)$ alternatively to Eq. (13.124).

It is possible to further extend the Hermitian two-matrix model by considering a chain of matrices with the nearest-neighbor interaction:

$$Z_{q\mathrm{h}} \quad = \quad \prod_{i=1}^{q} \mathrm{d}\varphi_i \, \exp\left\{ N \,\mathrm{tr}\left[-\sum_{i=1}^{q} V(\varphi_i) + \sum_{i=1}^{q-1} \varphi_i \varphi_{i+1} \right] \right\}. \tag{13.126}$$

In the limit of $q \to \infty$, we obtain an infinite chain associated with discretization of a one-dimensional theory.

The Hermitian q-matrix model possesses unitary continuum limits with $\gamma_0 = -1/(q+1)$. In the $q \to \infty$ limit, this gives $\gamma_0 \to 0$.

Remark on the $d = 1$ barrier

The $d = 1$ barrier is associated with the formula [GN84, OW85, KPZ88]

$$\gamma_0 \quad = \quad \frac{d - 1 - \sqrt{(1-d)(25-d)}}{12} \tag{13.127}$$

for the critical index of string susceptibility of the bosonic string in a d-dimensional embedding space. Alternatively, it describes two-dimensional quantum gravity interacting with conformal matter of central charge $c = d$.

The RHS of Eq. (13.127) is well-defined for $d \leq 1$, where it is associated with topological theories of gravity (with matter). They can also be described by the Hermitian (multi)matrix models.

The RHS of Eq. (13.127) becomes complex for $d > 1$ which is physically unacceptable. This is termed the $d = 1$ *barrier*.

Remark on the Kazakov–Migdal model

A natural multidimensional extension of the matrix chain (13.126) is the Kazakov–Migdal model [KM92], which is defined by the partition function

$$Z_{\mathrm{KM}} = \int \prod_{x,\mu} \mathrm{d}U_\mu(x) \prod_x \mathrm{d}\varphi_x \, \mathrm{e}^{-S_{\mathrm{KM}}[U,\varphi]} \tag{13.128}$$

with the action

$$S_{\mathrm{KM}}[U,\varphi] = N \, \mathrm{tr}\Big[-\sum_{x,\mu} \varphi_{x+a\hat{\mu}} U_\mu(x) \varphi_x U_\mu^\dagger(x) + \sum_x V(\varphi_x)\Big]. \tag{13.129}$$

Here φ_x and $U_\mu(x)$ are $N \times N$ Hermitian and unitary matrices, respectively, with x labeling lattice sites on a d-dimensional hypercubic lattice. The integration over the gauge field $U_\mu(x)$ is over the Haar measure on $SU(N)$ at each link of the lattice.

The Kazakov–Migdal model is of the same type as Wilson's lattice gauge theory with adjoint matter but without the action for the gauge field, i.e. at $\beta = 0$ in front of the plaquette term. When integrated over φ_x, it induces an action for the gauge field $U_\mu(x)$ of the type discussed in Problem 8.6 on p. 155.

The model (13.128) obviously recovers the open matrix chain (13.126) if the lattice is just a one-dimensional sequence of points for which the gauge field can be absorbed by a unitary transformation of φ_x.

The Kazakov–Migdal model is described at $N = \infty$ by the loop equation which coincides with Eq. (13.123) for the two-matrix model with the potential [DMS93]

$$V'(\omega) \rightarrow \mathcal{V}'(\omega) \equiv V'(\omega) - (2d-1)F(\omega)\,. \tag{13.130}$$

The function

$$F(\omega) = \sum_n F_n \omega^n \tag{13.131}$$

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is defined by the pair correlator of the gauge fields

$$\frac{\displaystyle\int \mathrm{d}U\, \mathrm{e}^{N\,\mathrm{tr}\,(\Phi U \Psi U^\dagger)}\, \frac{1}{N}\,\mathrm{tr}\left(t^a U \Psi U^\dagger\right)}{\displaystyle\int \mathrm{d}U\, \mathrm{e}^{N\,\mathrm{tr}\,(\Phi U \Psi U^\dagger)}} = \sum_{n=1}^{\infty} F_n\, \frac{1}{N}\,\mathrm{tr}\,(t^a \Phi^n)\,, \tag{13.132}$$

where Φ and Ψ play the role of external fields and t^a $(a = 1, \ldots, N^2 - 1)$ denote the generators of $SU(N)$. Eq. (13.132) holds [Mig92] at $N = \infty$. The function $F(\omega)$ is determined by the loop equation itself.

The loop equation of the Hermitian two-matrix model emerges because the last term on the RHS of Eq. (13.130) disappears at $d = 1/2$, which is associated with the Hermitian two-matrix model, and we simply have $\mathcal{V}(\omega) = V(\omega)$.

An exact solution of the Kazakov–Migdal model was found for the quadratic potential [Gro92] and the logarithmic potential [Mak93].

Continuum limits of the Kazakov–Migdal model are associated again with lower-dimensional theories. It does not allow us to go beyond the $d = 1$ barrier.

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Bibliography to Part 3

Reference guide

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The large-N limit of the four-Fermi and φ^4 theories was obtained in the paper by Wilson [Wil73]. The renormalizability of the $1/N$-expansion of four-Fermi theory in $d < 4$ dimensions was demonstrated by Parisi [Par75]. The appearance of conformal invariance in the $1/N$-expansion of four-Fermi theory in three dimensions is discussed in [CMS93]. The scale and conformal symmetries are described in the lectures by Jackiw [Jac72].

The $1/N$-expansion of $SU(N)$ Yang–Mills theory and its relation to the topology of Riemann surfaces was introduced by 't Hooft [Hoo74a]. The incorporation of quarks into this picture was accomplished by Veneziano [Ven76]. The geometric growth of the number of planar graphs was demonstrated by Koplik, Neveu and Nussinov [KNN77]. The large-N factorization was observed by A.A. Migdal in the late 1970s (first published in [MM79]). Its consequences for the semiclassical nature of the large-N limit are discussed in the lectures by Witten [Wit79] and Coleman [Col79].

The loop equation of multicolor QCD was derived in [MM79]. The program of reformulating QCD entirely in loop space was realized in [MM81]. The renormalization of the Wilson loops was investigated in [GN80, Pol80, DV80, BNS81]. A solution of the loop equation in two dimensions was found by Kazakov and Kostov [KK80]. A string representation of large-N QCD_2 was constructed by Gross and Taylor [GT93].

For a canonical book on the matrix models see the one by Mehta [Meh67]. The solution of the Hermitian one-matrix model at large N

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was found by Brézin, Itzykson, Parisi and Zuber [BIP78]. The large-N phase transition in lattice QCD_2 was first observed by Gross and Witten [GW80]. The application of matrix models to discretization of random surfaces is described in the review by Di Francesco, Ginsparg, and Zinn-Justin [DGZ95] and in the book by Ambjørn, Durhuus, and Jonsson [ADJ97] which contain extensive references.

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