COMPOSITIO MATHEMATICA

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Compositio Math. **149** (2013), 1871–1912.

doi:10.1112/S0010437X13007136

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Abstract

Let G be a simple algebraic group. Labelled trivalent graphs called webs can be used to produce invariants in tensor products of minuscule representations. For each web, we construct a configuration space of points in the affine Grassmannian. Via the geometric Satake correspondence, we relate these configuration spaces to the invariant vectors coming from webs. In the case of $G = \mathrm{SL}(3)$, non-elliptic webs yield a basis for the invariant spaces. The non-elliptic condition, which is equivalent to the condition that the dual diskoid of the web is CAT(0), is explained by the fact that affine buildings are CAT(0).

1. Introduction

1.1 Spiders

Let G be a simple, simply connected complex algebraic group. In previous work [Kup96], the third author defined a pivotal tensor category with generators and relations, called a 'spider', for G of rank 2. (The term 'spider' was originally intended to mean any pivotal category, but in common usage only categories as above are called spiders.) The Karoubi envelope of this category is equivalent to the category $\mathbf{rep}^u(G)$ of finite-dimensional representations of G with a modified pivotal structure. Actually, the spider comes with a parameter q, making it equivalent to the quantum deformation $\mathbf{rep}^u_q(G)$. These results in rank 2 are analogous to the influential result of Kauffman [Kau90] and Penrose [Pen71] that the Karoubi envelope of the Temperley–Lieb category (the category of planar matchings) is equivalent to $\mathbf{rep}^u_q(\mathrm{SL}(2))$. The Temperley–Lieb category can thus be called the SL(2) spider. Conjectural generalizations of spiders were proposed for SL(4) by Kim (see [Kim03]) and for SL(n) by Morrison (see [Mor07]).

In this article, for any G as above, we will define the free spider for G generated by the minuscule representations of G. A morphism in the free spider is given by a (linear combination) of labelled trivalent graphs called webs. For each web w with boundary edges labelled $\vec{\lambda}$, there is an invariant vector

$$\Psi(w) \in \mathrm{Inv}(V(\vec{\lambda})) = \mathrm{Inv}_G(V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_n)).$$

If G has rank 1 or 2, then the vectors $\Psi(w)$ coming from non-elliptic webs w (those whose faces have non-positive combinatorial curvature) form a basis for each invariant space $\mathrm{Inv}(V(\vec{\lambda}))$ of G, called a web basis. The web basis for SL(2) is well known as the basis of planar matchings, and it is known to be the same as Lusztig's dual canonical basis [FK95]. On the other hand, the SL(3) web bases are eventually not dual canonical [KK99], even though many basis vectors are dual canonical.

Received 24 June 2011, accepted 17 January 2012, received in final form 5 December 2012, published online 10 July 2013.
2010 Mathematics Subject Classification 22E47, 51E24, 57M27 (primary).
Keywords: webs, spiders, geometric Satake, buildings.
The first and second authors were supported in part by NSERC. The third author was partially supported by NSF grants DMS-0606795 and CCF-1013079.

1.2 Affine Grassmannians

The goal of this article is to introduce a new geometric interpretation of webs and spiders using the geometry of affine Grassmannians.

Let $\mathscr{O} = \mathbb{C}[[t]]$ and $\mathscr{K} = \mathbb{C}((t))$. In order to study the representation theory of G, we will consider the affine Grassmannian of its Langlands dual group,

$$\mathrm{Gr} = \mathrm{Gr}(G^\vee) = G^\vee(\mathscr{K})/G^\vee(\mathscr{O}).$$

The geometric Satake correspondence of [Lus83, Gin95, MV07a] will be our main tool in this article.

Theorem 1.1. *The category of equivariant perverse sheaves on the affine Grassmannian* Gr *is equivalent as a symmetric and pivotal tensor category to the tensor category* $\mathbf{rep}^u(G)$ *of representations of* G *with a modified pivotal and symmetric structure.*

As a consequence of this theorem, every invariant space $\mathrm{Inv}(V(\vec{\lambda}))$ for every G can be constructed from the geometry of Gr. Given a vector $\vec{\lambda}$ of dominant weights of G, there is a convolution morphism

$$m_{\vec{\lambda}} : \overline{\mathrm{Gr}(\vec{\lambda})} = \overline{\mathrm{Gr}(\lambda_1)} \mathbin{\tilde{\times}} \overline{\mathrm{Gr}(\lambda_2)} \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} \overline{\mathrm{Gr}(\lambda_n)} \longrightarrow \mathrm{Gr},$$

where each $\mathrm{Gr}(\lambda)$ is a sphere of radius λ (in the sense of weight-valued distances [KLM08]) in Gr. The fibre $F(\vec{\lambda}) = m_{\vec{\lambda}}^{-1}(t^0)$ is a projective variety that we call the Satake fibre. In particular, we will use the following corollary of the geometric Satake correspondence.

Theorem 1.2. *Every invariant space in* $\mathbf{rep}^u(G)$ *is canonically isomorphic to the top homology of the corresponding geometric Satake fibre with complex coefficients:*

$$\Phi : \mathrm{Inv}(V(\vec{\lambda})) \cong H_{\mathrm{top}}(F(\vec{\lambda}), \mathbb{C}).$$

Each top-dimensional component $Z \subseteq F(\vec{\lambda})$ *thus yields a vector* $[Z] \in \mathrm{Inv}(V(\vec{\lambda}))$. *These vectors form a basis, the Satake basis.*

A goal of this article is to understand how the invariant vectors coming from webs expand in this basis. (Throughout, we will assume complex coefficients for homology and cohomology.)

1.3 Diskoids

The orbits of $G(\mathscr{K})$ on the affine Grassmannian define a notion of distance on Gr with values in the set of dominant weights for G. Thus, we can interpret $F(\vec{\lambda})$ as the (contractive, based) configuration space in Gr of an abstract polygon $P(\vec{\lambda})$ whose side lengths are

$$\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

One of our ideas is to generalize this type of configuration space from polygons to diskoids. For us, a diskoid D is a contractible piecewise-linear region in the plane; in many cases it is a disk. (See § 3.2.) If D is tiled by polygons and its edges are labelled by dominant weights, then its vertices are a weight-valued metric space. We will define a (based) configuration space $Q(D)$ which consists of maps from the vertices of D to Gr that preserve the lengths of edges of D. We will also define a special subset $Q_g(D)$ that consists of maps that preserve all distances (globally isometric embeddings).

Assume that $\vec{\lambda}$ is a vector of minuscule highest weights. If w is a web with boundary $\vec{\lambda}$, then it has a dual diskoid $D = D(w)$ (or possibly a diskoid with bubbles). The boundary of this diskoid is a polygon $P(\vec{\lambda})$, and so we get a map of configuration spaces $\pi : Q(D) \to F(\vec{\lambda})$. Our first main result is that we can recover the vector $\Psi(w)$ using this geometry.

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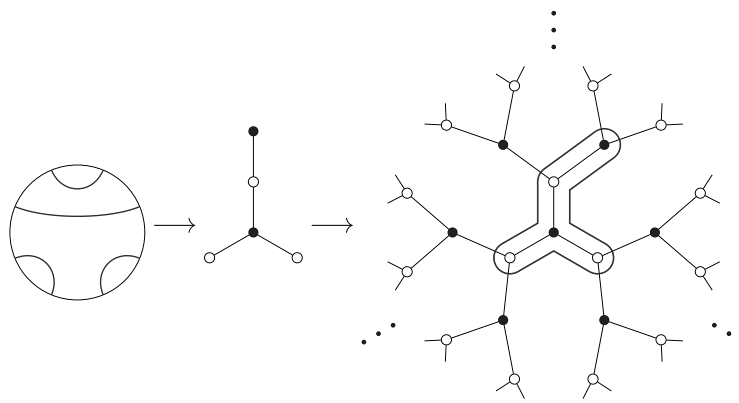


Figure 1. From a non-elliptic A_1 web, to a tree, to part of an affine A_1 building.

Theorem 1.3. *There exists a homology class* $c(w) \in H_*(Q(D))$ *such that* $\pi_*(c(w)) \in H_{\text{top}}(F(\vec{\lambda}))$ *corresponds to* $\Psi(w)$ *under the isomorphism from Theorem 1.2.*

We prove this theorem as an application of the geometric Satake correspondence. In many cases, the class $c(w)$ is the fundamental class of $Q(D)$, so that the coefficients of $\pi_*(c(w))$ (and hence $\Psi(w)$) in the Satake basis are just the degrees of the map π over the components of $F(\vec{\lambda})$.

1.4 Buildings

The affine Grassmannian Gr embeds isometrically into the affine building $\Delta = \Delta(G^\vee)$. We can use this perspective to gain greater insight into the variety $Q(D)$.

If $G = \mathrm{SL}(2)$, then a basis web is a planar matching (or cup diagram) and its dual diskoid D is a finite tree. The affine Grassmannian Gr is the set of vertices of the affine building Δ, which is an infinite tree with infinite valence. The configuration space $Q(D)$ is the space of colored, based simplicial maps $f : D \to \Delta$; see Figure 1. It is known that

$$Q(D) = \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \tilde{\times} \cdots \tilde{\times} \mathbb{P}^1$$

is a twisted product of copies of $\mathbb{P}^1$, and that these twisted products are the components of the Satake fibre $F(\vec{\lambda})$. Moreover, $Q_g(D)$ is the open dense subvariety of points in $Q(D)$ which are contained in no other component of $F(\vec{\lambda})$. Figure 1 is an illustration of the construction.

Our other main results are a generalization of this fact to $G = \mathrm{SL}(3)$. In this case, Gr is again the vertex set of Δ. If w is a non-elliptic web with boundary $\vec{\lambda}$, then $Q(D(w))$ is again the space of colored, based simplicial maps $f : D \to \Delta$, as in Figure 2. We then have the following result.

Theorem 1.4. *Let* $G = \mathrm{SL}(3) = A_2$ *and let* w *be a non-elliptic web with minuscule boundary* $\vec{\lambda}$ *and dual diskoid* D*. Then the global isometry configuration space* $Q_g(D)$ *is mapped isomorphically by* π *to a dense subset of a component of the Satake fibre* $F(\vec{\lambda})$*. This inclusion yields a bijection between non-elliptic webs and the components of* $F(\vec{\lambda})$*.*

Our construction can be viewed as an explanation of why basis webs are non-elliptic. A web is non-elliptic if and only if its diskoid is CAT(0), essentially by definition. It is well known that every affine building is a CAT(0) space [BT72]. Moreover, every convex subset of a CAT(0) space, such as a diskoid which is isometrically embedded in a building, is necessarily CAT(0). We will also show that the image of each diskoid embedding $f : D \to \Delta$ in $Q_g(D)$ has a least-

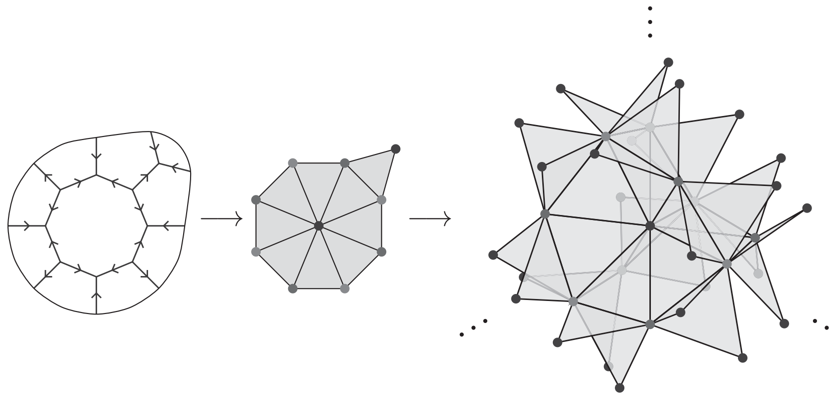


Figure 2. From a non-elliptic A_2 web, to a CAT(0) diskoid, to part of an affine A_2 building.

area property. Likewise, the elliptic relations of the A_2 spider can be viewed as area-decreasing transformations.

Meanwhile, if w is non-elliptic, then $Q(D)$ is sometimes the closure of $Q_g(D)$ and hence maps to a single component of $F(\vec{\lambda})$. Eventually, $Q(D)$ has other components and maps to more than one component of $F(\vec{\lambda})$. These other components seem related to the phenomenon that web bases are not dual canonical. However, we can get an upper triangularity result as follows. In § 5.3, we will define a partial order $\leqslant_S$ on the set of non-elliptic webs using pairwise distances between boundary vertices of their dual diskoids.

Theorem 1.5. *The change of basis in* $\mathrm{Inv}(V(\vec{\lambda}))$ *from non-elliptic webs to the Satake basis is unitriangular, relative to the partial order* $\leqslant_S$.

We have learned from Sergei Ivanov (personal communication, 2011) that the partial order in Theorem 1.5 refines the partial order on webs given by the number of vertices.

Also, in § 5.4, we will show that the web basis, the Satake basis and the dual canonical basis for SL(3) are all eventually different.

Finally, in § 6, we will propose a different formulation of the geometric Satake correspondence based on convolution of constructible functions rather than convolution of homology classes. (In Theorem 4.5, we reinterpret geometric Satake in terms of convolution in homology.) We will prove this conjecture in the case of a tensor product of minuscule representations of SL(3).

1.5 Satake fibres and Springer fibres

Suppose $G = \mathrm{SL}(m)$ and that $\vec{\lambda} = (\omega_1, \ldots, \omega_1)$ is an n-tuple, with $n = mk$, consisting of ω_1, the highest weight of the standard representation; then $F(\vec{\lambda})$ is isomorphic to the $(k, k, \ldots, k)$ Springer fibre. In other words, $F(\vec{\lambda})$ is the variety of flags in $\mathbb{C}^n$ that are invariant under a nilpotent endomorphism with m Jordan blocks all of size $k \times k$. We have already mentioned the well-known description of the components of the Springer or Satake fibre in terms of planar matchings when $m = 2$. This Springer fibre formalism and the aforementioned description of it have been used as a model of Khovanov homology [Kho04a, Str09]. One motivation for the present work is to generalize this result to the $m = 3$ case and obtain a description of the components of the Springer or Satake fibre using non-elliptic webs. Theorem 1.4 accomplishes this task (see also the end of the introduction of [Tym12]).

https://doi.org/10.1112/S0010437X13007136 Published online by Cambridge University Press

2. Spiders

2.1 Pivotal and symmetric categories

The definitions used in this section are nicely summarized in a survey by Selinger [Sel11]; they are originally due to Freyd and Yetter [FY89] and Joyal and Street [JS93].

A *pivotal category* $\mathscr{C}$ is a (strict) monoidal tensor category such that each object A has a two-sided dual object A^*. This means that there is a contravariant functor $F(A) = A^*$ from $\mathscr{C}$ to itself which is also an order-reversing tensor functor, i.e.

$$(A \otimes B)^* = B^* \otimes A^*,$$

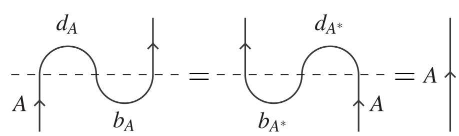
and which has the following extra properties: for each object A, there are 'cup' and 'cap' morphisms

$$b_A : I \longrightarrow A^* \otimes A, \quad d_A : A \otimes A^* \longrightarrow I$$

(where I denotes the unit object) such that

$$(1_A \otimes d_A)(b_A \otimes 1_A) = 1_A, \quad (d_{A_*} \otimes 1_{A_*})(1_{A_*} \otimes b_{A_*}) = 1_{A^*}.$$

In addition, $*$ is an anti-involution of the category $\mathscr{C}$. (We assume that $*$ is a strict involution of $\mathscr{C}$ that reverses both tensor products and compositions of morphisms.) The axiom can be summarized graphically as follows.



(1)

A *pivotal functor* is a tensor functor that preserves the above structure.

Every object A in a monoidal category has an *invariant space*

$$\mathrm{Inv}(A) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(I, A).$$

If the category is pivotal, then each invariant space has two other important properties. First, every space of morphisms is an invariant space by the relation

$$\mathrm{Hom}(A, B) \cong \mathrm{Inv}(A^* \otimes B).$$

Second, there is a cyclic action on the invariant spaces in tensor products,

$$R : \mathrm{Inv}(A \otimes B) \xrightarrow{\cong} \mathrm{Inv}(B \otimes A),$$

which we call a *rotation map*. It extends to a rotation of n tensor factors,

$$R : \mathrm{Inv}(A_1 \otimes \cdots \otimes A_n) \xrightarrow{\cong} \mathrm{Inv}(A_2 \otimes \cdots \otimes A_n \otimes A_1).$$

Another way to describe a pivotal category, already suggested in (1), is that it has the structure to evaluate a planar graph w drawn in a disk, if the edges of w are oriented and labelled by objects and the vertices are labelled by invariants. (In the literature, the words 'labelled' and 'colored' are used interchangeably in this context; Selinger calls an allowed set of colors a 'signature' [Sel11].) The value of such a graph w is another invariant, taking values in the invariant space of the boundary of w. The graph is considered up to isotopy rel boundary, and an edge labelled by A is equivalent to the opposite edge labelled by A^*. It is possible to write axioms for a pivotal category using invariants and planar graphs rather than morphisms. From this viewpoint, a word in a pivotal category is such a graph, and it can be called a *web*.

A web is a special case of a *ribbon graph* [RT90], the difference being that a ribbon graph can also have crossings. A *braided category* is a monoidal category with crossing isomorphisms

$$c_{A,B} : A \otimes B \to B \otimes A$$

that satisfy suitable axioms so that, among other things, a braid group acts on the invariant space of a tensor product. If the crossing isomorphisms are involutions, then the braid group action descends to a symmetric group action and the monoidal category is said to be *symmetric*. If a category is both symmetric and pivotal, then there is an important compatibility condition that makes it a *compact closed category*. We require that the two involutions on $\mathrm{Inv}(A \otimes A)$, one coming from the pivotal structure and the other from the symmetric structure, agree. Equivalently, we require that

$$c_{A^*,A}(b_A) = b_{A^*}.$$

In a compact closed symmetric category, abstract graphs w can be evaluated whether or not they are planar.

Two other intermediate types of categories between pivotal and compact closed are *ribbon categories* and *spherical categories*. A spherical category is a pivotal category with the extra property that left traces equal right traces, which allows the evaluation of a graph w embedded in the sphere rather than in the plane. A ribbon category is both pivotal and braided in a compatible way, and allows the evaluation of a framed graph w in $\mathbb{R}^3$. We will only need the pivotal category axioms in this article, but all categories considered are actually ribbon or compact closed.

2.2 Sign conventions

In many cases a pivotal category $\mathscr{C}$ which is linear over a field can be modified to a different category $\mathscr{C}'$. We will be interested in two modifications: sign changes to the pivotal structure of $\mathscr{C}$ that do not affect its tensor structure, and sign changes to the tensor structure of $\mathscr{C}$. We want to restrict attention to those sign changes that allow us to say that $\mathscr{C}$ and $\mathscr{C}'$ have the same algebraic information. For simplicity, when discussing signs, we assume that $\mathscr{C}$ is abelian-linear over an algebraically closed field k not of characteristic 2 and is semisimple with irreducible trivial object.

Another objective of this section is to correctly interpret labelled graphs w in a pivotal category with unoriented edges. Some edges in a labelled graph w for a pivotal or compact closed category can be unoriented. Suppose that $A \cong A^*$ is self-dual, and suppose further that the isomorphism $\phi_A \in \mathrm{Hom}(A, A^*)$ is cyclically invariant if interpreted as an element of $\mathrm{Inv}(A \otimes A)$. In this case we say that A is *symmetrically self-dual*. (This definition does not require any linearity assumption.) Then an unoriented edge can be defined by a replacement

(2)

where the dot on the right-hand side represents ϕ_A. Algebraically, if (and only if) A is symmetrically self-dual, then $\mathscr{C}$ is equivalent to a pivotal category in which $A = A^*$ outright, and $b_A = b_{A^*}$ and $d_A = d_{A^*}$. If every self-dual object in $\mathscr{C}$ is symmetrically self-dual, then $\mathscr{C}$ is said to be *unimodal* [Tur94]. If $A \cong A^*$ but A is not symmetrically self-dual, then only the right-hand side of (2) makes sense, and only if it is altered in some way to break symmetry; Morrison denotes such a morphism by a 'tag' [Mor07].

Suppose instead that $A \cong A^*$ but A is not symmetrically self-dual, and suppose that $\mathscr{C}$ is k-linear and semisimple and A is irreducible. Then, by Schur's lemma, $\mathrm{Hom}(A, A^*)$ is one-dimensional and rotation R is multiplication by -1. In this case, A is *anti-symmetrically self-dual*.

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Thus we can ask whether $\mathscr{C}$ could be made unimodal by changing signs. This is what happens in our case (see § 2.3), but there are also examples (namely, representation categories of finite groups) that are not unimodal for any pivotal structure.

To understand the allowed sign changes to the pivotal structure of $\mathscr{C}$, we first assume, by category equivalence, that A and A^* are different objects for every A. Then, by (1), we can negate b_A and d_A for some irreducible A, without changing b_{A^*} and d_{A^*}. This yields a new pivotal category $\mathscr{C}'$, provided that the sign change function $s(A)$ satisfies

$$s(A) = s(B)s(C)$$

whenever $\mathrm{Hom}(A, B \otimes C) \neq 0$. If A is self-dual and $s(A) = -1$, then this modification changes the sign of the self-duality of A. It also negates the dimension of A, which is by definition

$$\dim(A) = d_{A^*} \circ b_A.$$

Finally, since we are changing the pivotal structure by signs rather than by other phases, $\mathscr{C}$ is spherical if and only if $\mathscr{C}'$ is spherical.

We can change the sign of the tensor structure of $\mathscr{C}$ by a similar but more complicated construction. We can assume, after passing to an equivalent category, that the objects $\mathscr{C}$ are a free polynomial semiring over the irreducible objects of $\mathscr{C}$ with respect to the operations $\oplus$ and $\otimes$. If

$$A = A_1 \otimes A_2 \otimes \cdots \otimes A_a$$

is a tensor product of irreducibles, and likewise B, C and D are also tensor products of irreducibles, then we can change the sign of the tensor product map

$$\otimes : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(C, D) \to \mathrm{Hom}(A \otimes C, B \otimes D)$$

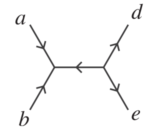
by some sign function $s(A, B; C, D) \in \{\pm 1\}$, defined when $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are both non-zero. In order for the result $\mathscr{C}'$ to be another pivotal category, we need to check that compositions and tensor products of morphisms are still both associative. In other words, we need to check the equations

$$s(A, C; D, F) = s(A, B; D, E)s(B, C; E, F),$$
$$s(A, B; C, D)s(A \otimes C, B \otimes D; E, F) = s(A, B; C \otimes E, D \otimes F)s(C, D; E, F)$$

when the right-hand sides are defined. It turns out that if $\mathscr{C}$ is pivotal or spherical, then the new tensor category $\mathscr{C}'$ can also be made pivotal or spherical.

2.3 Examples

A fundamental example of a pivotal category or, indeed, a compact closed category is the category $\mathbf{vect}(k)$ of finite-dimensional vector spaces over a field k. In this example, a web can be interpreted as the graph of a tensor calculus expression (or a 'spin network'). For example, if ϵ_{abc} is a trilinear determinant form on a three-dimensional vector space V, and ϵ^{abc} is the dual form on V^*, then the tensor $\epsilon_{abc}\epsilon^{cde}$ (with repeated indices summed) can be drawn as



with the convention in this case that the vertex labels can be inferred from context. If the characteristic of k is not 2, then another fundamental example is the category $\mathbf{svect}(k)$ of finite-dimensional *super vector spaces*, which are $\mathbb{Z}/2$-graded vector spaces with a non-trivial symmetric

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and pivotal structure. Specifically, if $v \in V$ and $w \in W$ are homogeneous elements of super vector spaces, then

$$c_{V,W}(v \otimes w) = (-1)^{(\deg v)(\deg w)} w \otimes v.$$

If $v \in V$ and $w \in V^*$ are homogeneous, then the cap d_V is likewise adjusted so that

$$d_V(v \otimes w) = (-1)^{(\deg v)} w(v).$$

If G is a group (or Lie group, Lie algebra or algebraic group), then $\mathbf{rep}(G, k)$, the category of finite-dimensional representations (or continuous or algebraic representations) over k is a pivotal category with a pivotal functor to $\mathbf{vect}(k)$. For the remainder of the article, we let G be a simple, simply connected algebraic group over $\mathbb{C}$ (and later we will specialize to $G = \mathrm{SL}(3)$). We will study the pivotal category $\mathbf{rep}(G) = \mathbf{rep}(G, \mathbb{C})$.

There is a deformation $\mathbf{rep}_q(G)$ of $\mathbf{rep}(G) = \mathbf{rep}_1(G)$ that consists of representations of the quantum group $U_q(\mathfrak{g})$, when the parameter q is not a root of unity. (The deformation also exists when q is a root of unity, but there is more than one standard choice for it.) This deformation is also a pivotal category, although it has no pivotal functor to $\mathbf{vect}$, because the cup and cap morphisms deform. Even though many ideas in this article are clearly related to quantum representations, we will concentrate on $\mathbf{rep}(G)$, except in § 6 where $\mathbf{rep}_{-1}(G)$ will also appear.

We are interested in two other variations of $\mathbf{rep}(G)$. First, we want to change its pivotal structure to make it unimodal. Recall that the irreducible representations $V(\lambda)$ of G are labelled by the set of dominant weights. For a dominant weight λ, we write λ^* for the dominant weight such that $V(\lambda)^* \cong V(\lambda^*)$. We also write ρ for the Weyl vector and $\rho^\vee$ for the dual Weyl vector. We make each $V(\lambda)$ a super vector space by giving it the grading $\langle 2\lambda, \rho^\vee \rangle \bmod 2$. In this way we realize $\mathbf{rep}(G)$ as a subcategory of $\mathbf{svect}(\mathbb{C})$ with a different pivotal and symmetric structure, and we call this version $\mathbf{rep}^u(G)$. Likewise, it has a unimodal pivotal deformation $\mathbf{rep}^u_q(G)$.

Following § 2.2, $\mathbf{rep}(G)$ and $\mathbf{rep}_{-1}(G)$ differ only by sign rules, and both of them have equivalent information. To obtain $\mathbf{rep}_{-1}(G)$ from $\mathbf{rep}(G)$ in this fashion, we use the abbreviations

$$\begin{aligned} \vec{\lambda} &= (\lambda_1, \lambda_2, \ldots, \lambda_n), \\ V(\vec{\lambda}) &= V(\lambda_1) \otimes \cdots \otimes V(\lambda_n), \\ \lambda &= \sum_i \lambda_i. \end{aligned}$$

Then we define the sign rule

$$s(V(\vec{\lambda}), V(\vec{\gamma}); V(\vec{\mu}), V(\vec{\nu})) = (-1)^{\langle 2\lambda, \rho^\vee \rangle \langle \mu - \nu, \rho^\vee \rangle}.$$

This sign rule takes $\mathbf{rep}(G)$ to $\mathbf{rep}_{-1}(G)$ and $\mathbf{rep}^u(G)$ to $\mathbf{rep}^u_{-1}(G)$.

The other variation is a restriction to minuscule representations. Recall that a dominant weight λ is said to be *minuscule* if $\langle \alpha^\vee, \lambda \rangle \leqslant 1$ for every positive coroot $\alpha^\vee$. If λ is a minuscule dominant weight, then $V(\lambda)$ is called a *minuscule representation*. These representations have the special property that all of their weights are in the Weyl orbit of the highest weight. We define $\mathbf{rep}(G)_{\min}$ to be the monoidal subcategory of $\mathbf{rep}(G)$ generated by minuscule representations. So the objects of $\mathbf{rep}(G)_{\min}$ are tensor products of minuscule representations. It is a symmetric category which is neither an additive nor an abelian category. If there exists a minuscule λ such that $\langle 2\lambda, \rho^\vee \rangle$ is odd, then $\mathbf{rep}(G)_{\min}$ is also not a pivotal category, because it is skeletal and yet has objects which are anti-symmetrically self-dual in $\mathbf{rep}(G)$. However, $\mathbf{rep}^u(G)_{\min}$ is a well-defined pivotal category in which $*$ is a strict involution and $V(\lambda^*) = V(\lambda)^*$.

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In the case where $G = \mathrm{SL}(n)$, as well as in some other cases, $\mathbf{rep}(G)$ can be recovered as the Karoubi envelope of $\mathbf{rep}(G)_{\min}$, although we will not use this construction in the present article.

The other main pivotal category which we will study in this paper is the category of $G^\vee(\mathscr{O})$-equivariant perverse sheaves $\mathbf{perv}(\mathrm{Gr})$ on Gr. This category has a relatively straightforward pivotal structure. It also has a more delicate symmetric structure which is called a 'commutativity constraint' or 'braiding', as defined by Ginzburg [Gin95] and by Mirković and Vilonen [MV07a] in two different ways. (See also [BD, § 5.3.8].) Theorem 1.1 states that $\mathbf{perv}(G)$ is equivalent to $\mathbf{rep}^u(G)$ both as a pivotal category and as a symmetric category; we shall be more interested in the pivotal structure. We will also be more interested in the minuscule analogue of $\mathbf{perv}(\mathrm{Gr})$, which we analyze in § 4.3.

2.4 Free spiders and presentations

Pivotal categories can also be presented by generators and relations. If the pivotal category is additive-linear over a ring or a field, then it can presented in the same sense, using linear combinations of words in the generators. In general, there are generating objects (or edges) and generating morphisms (or invariants or vertices), while the relations are all morphisms. Relations in a pivotal category are also known as *planar skein relations*.

We now define the free spider $\mathbf{fsp}(G)$ to be the free $\mathbb{C}$-linear pivotal category generated by an edge for each minuscule representation of G and a vertex for every triple (λ, μ, ν) of minuscule dominant weights such that

$$\mathrm{Inv}_G(V(\lambda, \mu, \nu)) \neq 0.$$

Note that the minuscule condition forces this vector space to be at most one-dimensional. In $\mathbf{fsp}(G)$, we also impose that the dual of the λ edge is λ^*. In [Mor07], $\mathbf{fsp}(\mathrm{SL}(n))$ was denoted by Sym_n.

A free spider has the same relationship to webs as a free group has to words in its generators, namely, two webs are equal in $\mathbf{fsp}(G)$ if and only if they are isotopic rel boundary. (Selinger also defines free categories of various kinds generated by signatures; see [Sel11].)

Let us fix $q \in \mathbb{C}$, non-zero and not a root of unity (but possibly equal to 1). There is a pivotal functor

$$\Psi : \mathbf{fsp}(G) \to \mathbf{rep}_q^u(G)_{\min},$$

which is defined by choosing a non-zero element in each invariant space

$$\mathrm{Inv}_{U_q(\mathfrak{g})}(V(\lambda, \mu, \nu)).$$

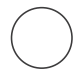
In particular, for each web w with boundary $\vec{\lambda}$, we obtain an element

$$\Psi(w) \in \mathrm{Inv}_{U_q(\mathfrak{g})}(V(\vec{\lambda})).$$

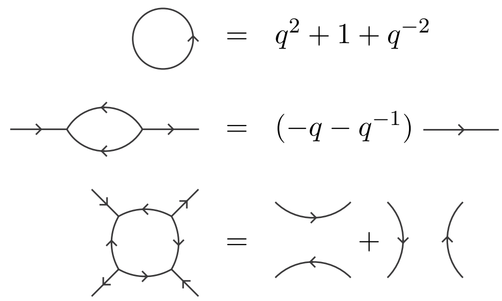
Actually, since webs are a notation for words in any pivotal category, we could also say that w 'is' $\Psi(w)$, or that its value is $\Psi(w)$. But the distinction between w and $\Psi(w)$ will be useful for us. The first result is that Ψ is surjective when $G = \mathrm{SL}(n)$ (see [Mor07, Proposition 3.5.8]). (This follows from Weyl's fundamental theorem of invariant theory.) Thus, the vectors $\Psi(w)$ of webs w span the invariant spaces.

It is an open problem to generate the kernel of Ψ with planar skein relations in $\mathbf{fsp}(G)$. This problem has been solved in the case where G has rank 1 or 2 by the third author [Kup96]. Kim has conjectured an answer for $\mathrm{SL}(4)$ in [Kim03], and Morrison has done so for $\mathrm{SL}(n)$ (see [Mor07]). Once these planar skein relations (which must depend on q) are determined, the resulting presented pivotal category can be called a spider and we denote it by $\mathbf{spd}_q(G)$.

We now review the known solutions for SL(2) and SL(3). The *Temperley–Lieb category* or A_1 *spider* $\mathbf{spd}_q(\mathrm{SL}(2))$ is the quotient of $\mathbf{fsp}(\mathrm{SL}(2))$ by the following single relation.


$$= -q - q^{-1} \tag{3}$$

(Since SL(2) has a single, self-dual minuscule representation, $\mathbf{fsp}(\mathrm{SL}(2))$ and $\mathbf{spd}_q(\mathrm{SL}(2))$ have unoriented edges with a single color or label.) The A_2 *spider* $\mathbf{spd}_q(\mathrm{SL}(3))$ is the quotient of $\mathbf{fsp}(\mathrm{SL}(3))$ by the following three relations.



(4)

(Since SL(3) has two minuscule representations which are dual to each other, $\mathbf{fsp}(\mathrm{SL}(3))$ and $\mathbf{spd}_q(\mathrm{SL}(3))$ have oriented edges with one label or color. By convention, the edge is labelled by the first fundamental representation ω_1 in the direction in which it is oriented.) The other two known spiders, $\mathbf{spd}_q(B_2)$ and $\mathbf{spd}_q(G_2)$, have similar but more complicated presentations.

Theorem 2.1 [Kau90]. *If q is not a root of unity, then $\mathbf{spd}_q(\mathrm{SL}(2))$ is equivalent to the pivotal category $\mathbf{rep}^u_q(\mathrm{SL}(2))_{\min}$ of minuscule representations.*

Theorem 2.2 [Kup96]. *If q is not a root of unity, then $\mathbf{spd}_q(\mathrm{SL}(3))$ is equivalent to the pivotal category*

$$\mathbf{rep}_q(\mathrm{SL}(3))_{\min} = \mathbf{rep}^u_q(\mathrm{SL}(3))_{\min}$$

of minuscule representations.

(In the case of SL(3), it turns out that $\mathbf{rep}_q(\mathrm{SL}(3))$ and $\mathbf{rep}^u_q(\mathrm{SL}(3))$ are the same; see § 2.3.)

A main property of the spider relations (4) is that they are confluent or of Gröbner type. In the free pivotal category generated by the generating edges and vertices, each web can be graded by the number of its faces. Then each relation has exactly one leading term, an elliptic face. (In the A_2 spider, a face is *elliptic* if it has fewer than six sides. In the other two rank-2 spiders, a face is elliptic if the total angle of the corresponding dual vertex is less than 2π, so that the vertex is CAT(0); see § 3.3.) A web having that face can be expressed, modulo the relation, as a linear combination of lower-degree webs. The Gröbner property, proved using a diamond lemma, is that any two sequences of simplifications of the same web lead to the same final expression. This means that the webs that cannot be simplified, i.e. the webs without elliptic faces or the non-elliptic webs, form a basis of each invariant space. There is an extended version of this result, but we will restrict our attention to the minuscule case, summarized in the following theorem.

Theorem 2.3 [Kup96]. *If $\vec{\lambda}$ is a sequence of dominant minuscule weights of* SL(3), *then the non-elliptic type-A_2 webs with boundary $\vec{\lambda}$ form a basis of* $\mathrm{Inv}(V(\vec{\lambda}))$.

Theorem 1.5 implies Theorem 2.3 as a corollary. However, this route is much more complicated than other proofs of Theorem 2.3 (cf. [KK99, Wes07]).

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3. Affine geometry

3.1 Weight-valued metrics and linkages

In the usual definition of a metric space, distances take values in the non-negative real numbers $\mathbb{R}_{\geqslant 0}$. However, Kapovich *et al.* [KLM08] have a theory of metric spaces in which distances take values in the dominant Weyl chamber of G. Two of the axioms of such a generalized metric space are easy to state:

$$d(x,x) = 0, \quad d(x,y) = d(y,x)^*.$$

The third axiom, the triangle inequality, is different. The main results of Kapovich *et al.* are generalized triangle inequalities that are satisfied in buildings and generalized symmetric spaces. On the one hand, the triangle inequalities in the A_1 case are the usual triangle inequality. On the other hand, the inequalities in higher-rank cases are decidedly non-trivial.

In this article, we will adopt the viewpoint of weight-valued metric spaces in order to discuss isometries and distance comparisons. We will not need the generalized triangle inequalities, but we will need isometries and distance comparisons. The definition of an isometry is straightforward. As for distance comparisons, we will say that $\mu \leqslant \lambda$ as a distance if and only if $\mu \leqslant \lambda$ in the usual partial order on dominant weights, namely that $\lambda - \mu$ is a non-negative integer combination of simple roots. Thus, a ball of radius λ is a finite union of spheres of radius $\mu \leqslant \lambda$. For one construction we will define distances that take values in the dominant Weyl chamber, instead of integral weights; we then say that $\mu \leqslant \lambda$ when $\lambda - \mu$ is a non-negative real combination of simple roots.

In addition to isometries, we will be interested in partial isometries in which only some distances are preserved. For this purpose, we define a *linkage* to be an oriented graph Γ whose edges are labelled by dominant weights. As with webs, an edge labelled by λ is equivalent to the opposite edge labelled by λ^*. Let $v(\Gamma)$ be the set of vertices of Γ. Then one may attempt to define a distance $d(p,q)$ between any two points $p, q \in v(\Gamma)$ by taking the shortest total distance of a connecting path. However, since weights are only partially ordered, this minimum may not be unique. We will say that Γ has *coherent geodesics* if the minimum distance $\min(d(p,q))$ between any two vertices p and q is unique, and if that minimum distance is the length of the edge (p,q) when Γ has that edge. In this case, Γ can be completed to another linkage Γ_g which is a complete graph, using all distances as weights.

3.2 Configuration spaces

Let X be a weight-valued metric space, and let Γ be a linkage as in § 3.1. Let $v(\Gamma)$ be the set of vertices of Γ. Then we define the *linkage configuration space* $Q(\Gamma, X)$ to be the set of maps

$$f : v(\Gamma) \to X$$

such that $d(f(p), f(q))$ equals the weight of the edge from p to q, when there is such an edge. If X and Γ both have a base point, then $Q(\Gamma, X)$ is instead the configuration space of based maps. Another possibility is that Γ has a base edge of length λ and X has two base points at distance λ; then $Q(\Gamma, X)$ is again the configuration space of based maps. We will be interested in four types of linkage Γ:

(i) a path or *polyline*;

(ii) a cycle or polygon;

(iii) the 1-skeleton $\Gamma(D)$ of a tiled diskoid D (see § 3.3) with edges labelled by weights;

(iv) the complete linkage $\Gamma_g(D)$, if $\Gamma(D)$ has coherent geodesics.

There is one final type of configuration space that is sometimes useful. If an edge (p,q) has weight λ, then we can ask that

$$d(f(p), f(q)) \leqslant \lambda$$

instead of

$$d(f(p), f(q)) = \lambda.$$

The result is the contractive configuration space $Q_c(\Gamma, X)$.

Suppose that $X = G/H$ for some group G with a subgroup H and that each sphere $X(\lambda)$ around the base point is a double coset of H. Let Γ be a linkage and let Γ_0 be the same linkage with a chosen base point 0. Then there is a fibration

$$Q(\Gamma_0, X) \longrightarrow Q(\Gamma, X) \longrightarrow X.$$

Similarly, if Γ_e denotes the same linkage with a base edge e of length λ incident to 0, then there is also a fibration

$$Q(\Gamma_e, X) \longrightarrow Q(\Gamma_0, X) \longrightarrow X(\lambda), \tag{5}$$

where $X(\lambda) = Q(\lambda, X)$ is the sphere of radius λ around the (first) base point of X, and the second base point is an arbitrary point in $X(\lambda)$.

If $f : \Gamma_2 \to \Gamma_1$ is a map between linkages, then there is a restriction map,

$$\pi^{\Gamma_1}_{\Gamma_2} : Q(\Gamma_1, X) \to Q(\Gamma_2, X), \tag{6}$$

between their configuration spaces. We will be particularly interested in this map when Γ_1 is a sublinkage of Γ_2 (for example its boundary).

Suppose now that $\Gamma = \Gamma_1 \cup \Gamma_2$ and that $\Gamma_1 \cap \Gamma_2$ is either an edge or a vertex. If we base Γ_2 (but not Γ_1) at this intersection, then the configuration space $Q(\Gamma, X)$ is a twisted product:

$$Q(\Gamma, X) = Q(\Gamma_1, X) \tilde{\times} Q(\Gamma_2, X).$$

Informally, Γ_2 is either an arm attached to Γ_1 at a point which can swing freely in any direction, or a flap attached to Γ_1 along a one-dimensional hinge which can swing freely in the remaining directions.

3.3 Diskoids

Recall that a *piecewise-linear diskoid* is a contractible, compact, piecewise-linear region in the plane. (We will not need diskoids that are not piecewise-linear, but if one were to consider such a diskoid, the most natural definition might be to make it a planar, cell-like continuum.) Any diskoid D has a polygonal boundary P with a boundary map $P \to D$, which, however, is not an inclusion unless D is either a point or a disk. Figure 3 shows an example of a diskoid D with its boundary P.

Note that since a diskoid comes with an embedding in the plane, its boundary P is implicitly oriented, so that the edges of P are cyclically ordered. We will assume a clockwise orientation in this article. Trees are diskoids, and Figure 1 has an example of the polygonal boundary of a tree; the polygon traverses each edge twice.

A diskoid D can be tiled by polygons. Formally, a *tiling* of D is a piecewise-linear CW complex structure on D with embedded 2-cells. If D is decorated in this way, then we define the graph $\Gamma(D)$ to be its 1-skeleton. Then, as above, $\Gamma(D)$ can be made into a linkage, which means, explicitly, that the edges of D are labelled by distances. In this article we will not need to label the faces (or 2-cells) of a tiled diskoid to define its configuration space, but only because the corresponding representation theory is multiplicity-free. In future work, the faces could also be

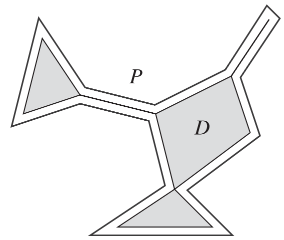

Figure 3. A diskoid D with boundary P.

labelled in order to define more restrictive configuration spaces. We will write $Q(D)$ for $Q(\Gamma(D))$ and $Q_g(D)$ for $Q(\Gamma_g(D))$.

In some cases, although not the most important cases, we will be interested in diskoids with bubbles. By definition, a *diskoid with bubbles* is, inductively, either a diskoid or a one-point union of a smaller diskoid with bubbles and either a line segment or a piecewise-linear 2-sphere. The extra line segments and 2-spheres are not embedded in the plane and do not affect the boundary of the diskoid, even if the attachment point is on the boundary. The discussion in the previous paragraph applies equally well to diskoids with bubbles.

Our interest in diskoids arises from the fact that they are geometrically dual to webs. As in the introduction, let w be a web in $\mathbf{fsp}(G)$ with boundary $\vec{\lambda}$. Then it has a dual diskoid $D = D(w)$, with bubbles if w has closed components, and with a natural base point. To be precise, D has a vertex for every internal or external face of w, two vertices are connected by an edge when the faces of w are adjacent, and there is a triangle glued to three edges whenever the dual edges of w meet at a vertex. We label the edges of D using the labels of the corresponding edges of w; also, if an edge of w is oriented, we transfer it to an orientation of the dual edge of D by rotating it counterclockwise. As a result, the boundary of the diskoid D is the polygon $P(\vec{\lambda})$. Figure 1 shows an example of an A_1 web and its dual diskoid, which in the A_1 case is always a tree. Figure 2 shows an example of an A_2 web and its dual diskoid, which happens to be a disk because the corresponding web is connected.

In this construction, D is always triangulated because w is always trivalent. The vertices of D are a weight-valued metric space, and by linear extension the whole of D is a Weyl-chamber-valued metric space. We can also simplify this metric to an ordinary metric space by taking the Euclidean length of the vector-valued distance. Finally, suppose that w is an A_2 web (or a B_2 or G_2 web). Then w is non-elliptic if and only if D, in its ordinary metric, is CAT(0) in the sense of Gromov [Gro87]. This follows from the fact that D is contractible and the condition that all complete angles in D are at least 2π.

3.4 Affine Grassmannians and buildings

As before, let G be a simple, simply connected complex algebraic group, and let $G^\vee$ be its Langlands dual group. Let $\mathscr{O} = \mathbb{C}[[t]]$ be the ring of formal power series over $\mathbb{C}$ and let $\mathscr{K} = \mathbb{C}((t))$ be its fraction field. Then

$$\mathrm{Gr} = \mathrm{Gr}(G^\vee) = G^\vee(\mathscr{K})/G^\vee(\mathscr{O})$$

is the *affine Grassmannian* for $G^\vee$ with residue field $\mathbb{C}$. It is an ind-variety over $\mathbb{C}$, meaning that it is a direct limit of algebraic varieties (of increasing dimension). The affine Grassmannian Gr is also a weight-valued metric space: the double cosets $G^\vee(\mathscr{O})\backslash G^\vee(\mathscr{K})/G^\vee(\mathscr{O})$ are bijective with the cone Λ_+ of dominant coweights of $G^\vee$, which is the same as the cone of dominant

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weights of G. More precisely, for each coweight μ of $G^\vee$, there is an associated point t^μ in the affine Grassmannian. If p and q are two arbitrary points of the affine Grassmannian, then we can find $g \in G^\vee(\mathscr{K})$ such that $gp = t^0$ and $gq = t^\mu$ for some unique dominant coweight μ. Under this circumstance, we write $d(p, q) = \mu$. So the action of $G^\vee(\mathscr{K})$ preserves distances and $d(t^0, t^\mu) = \mu$ for any dominant weight μ.

The affine Grassmannian Gr is also a subset of the vertices $\mathrm{Gr}' = v(\Delta)$ of an associated simplicial complex $\Delta = \Delta(G^\vee)$, called an *affine building* [Ron09], whose type is the extended Dynkin type of $G^\vee$. The simplices of this affine building are given by parahoric subgroups of the affine Kac–Moody group $\widehat{G^\vee}$. For a detailed description of affine buildings from this perspective, see [GL05].

An affine building Δ satisfies the following axioms.

(i) The building Δ is a non-disjoint union of *apartments*, each of which is a copy of the Weyl alcove simplicial complex of $G^\vee$.

(ii) Any two simplices of Δ of any dimension are both contained in at least one apartment Σ.

(iii) Given two apartments Σ and Σ' and two simplices $\alpha, \alpha' \in \Sigma \cap \Sigma'$, there is an isomorphism $f : \Sigma \to \Sigma'$ that fixes α and α' pointwise.

The axioms imply that the vertices of Δ, denoted by Gr', are canonically colored by the vertices of the extended Dynkin diagram $\hat{I} = I \sqcup \{0\}$ of $G^\vee$ or, equivalently, the vertices of the standard Weyl alcove δ of $G^\vee$. Moreover, every maximal simplex of Δ is a copy of δ; it has exactly one vertex of each color. The affine Grassmannian consists of those vertices colored by 0 and by minuscule nodes of the Dynkin diagram of $G^\vee$.

The axioms also imply that $v(\Delta)$ and, more generally, the realization $|\Delta|$ of Δ have a metric taking values in a Weyl chamber (but not necessarily integral weights as one sees in Gr). Specifically, if $p, q \in |\Delta|$, then $p, q \in |\Sigma|$ for an apartment Σ, and after a suitable automorphism $p = q + \lambda$ for some vector λ in the dominant Weyl chamber. We then define $d(p, q) = \lambda$. (The metric has coherent geodesics, and it extends the metric defined above for Gr.) We will need the following fact.

Lemma 3.1. *If $p, q \in |\Delta|$, then every geodesic path γ from p to q is contained in every apartment Σ such that $p, q \in |\Sigma|$.*

A subtle feature of the above affine building Δ is that it has two very different geometries. As an ordinary simplicial complex, its vertex set Gr' is discrete, and Gr' has a combinatorial, weight-valued metric. The vertex set Gr' is also naturally an algebraic ind-variety over $\mathbb{C}$, as is the set of vertices of any given color or the set of simplices of Δ of any given type. This second geometry endows Gr' with both a Zariski topology and an analytic topology. Among the relations between these two geometries, we will need the following fact.

Proposition 3.2. *The algebraic-geometric closure $\overline{\mathrm{Gr}'(\lambda)}$ of the sphere $\mathrm{Gr}'(\lambda)$ of radius λ is the set of all points in the metric ball of radius λ that have the same color as λ.*

An affine building Δ has a third geometry, which is related to the weight-valued metric but is not the same; namely, we can give the Weyl alcove δ its standard Euclidean structure and consider the induced metric on the realization $|\Delta|$ of Δ. This locally Euclidean metric can also be defined as $\|d(p, q)\|_2$, where $d(p, q)$ is the weight-valued metric on $|\Delta|$.

Theorem 3.3 (Bruhat–Tits theorem [BT72]). *Every affine building is a* CAT(0) *space with respect to its locally Euclidean metric.*

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If $G=\mathrm{SL}(n)$ and hence $G^\vee=\mathrm{PGL}(n)$, then $\mathrm{Gr}=\mathrm{Gr}'$ and there is a simple description of Δ, namely, a finite set of vertices in Gr subtends a simplex if and only if the distances between them are all minuscule.

Finally, to close a circle, let $L(\vec{\lambda})$ be a polyline whose sides are labelled by

$$\vec{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n),$$

based at the beginning. Let $P(\vec{\lambda})$ be the corresponding polygon, based between λ_n and λ_1. Then the contractive polyline configuration space

$$\mathrm{Gr}(\vec{\lambda})=Q_c(L(\vec{\lambda}),\mathrm{Gr})$$

is the domain of the convolution morphism. The restriction map coming from the projection onto the boundary $L(\vec{\lambda})\to\mathrm{pt}$, or

$$\pi_{\mathrm{pt}}^{L(\vec{\lambda})}:Q_c(L(\vec{\lambda}),\mathrm{Gr})\to\mathrm{Gr},$$

is the convolution morphism. In keeping with the standard notation, we will denote it by

$$m_{\vec{\lambda}}=\pi_{\mathrm{pt}}^{L(\vec{\lambda})}.$$

Meanwhile, the contractive polygon configuration space

$$Q_c(P(\vec{\lambda}),\mathrm{Gr})=F(\vec{\lambda})=m_{\vec{\lambda}}^{-1}(t^0)$$

is the Satake fibre. As another bit of notation, if Γ is a linkage, we will elide the Gr and write $Q(\Gamma)$ for $Q(\Gamma,\mathrm{Gr})$ and so on.

4. Geometric Satake for tensor products of minuscule representations

4.1 Minuscule paths and components of Satake fibres

The full geometric Satake correspondence, Theorem 1.1, simplifies considerably when the weights are minuscule. In this special case, Haines showed in [Hai06, Theorem 3.1] that all components of $F(\vec{\lambda})$ are of maximal dimension. We can use his ideas to give an explicit description of these components using minuscule paths. In addition to previous notation, let W be the Weyl group of G.

Let λ be a minuscule dominant weight. Then there are no dominant weights less than λ, so the sphere of radius λ equals the ball of radius λ. Hence the sphere $\mathrm{Gr}(\lambda)$ is closed in the algebraic geometry of Gr by Proposition 3.2, and thus it is projective and smooth. In fact, $G^\vee$ acts transitively on $\mathrm{Gr}(\lambda)$. The stabilizer of t^λ is $M(\lambda)$, the opposite maximal proper parabolic subgroup corresponding to the minuscule weight λ. Thus $\mathrm{Gr}(\lambda)$ is isomorphic to the partial flag variety $G^\vee/M(\lambda)$.

More generally, if Γ is a *minuscule linkage*, meaning that all of its edges are minuscule, then

$$Q(\Gamma)=Q_c(\Gamma)=\overline{Q(\Gamma)}.$$

Let

$$\vec{\lambda}=(\lambda_1,\ldots,\lambda_n)$$

be a sequence of minuscule dominant weights. A *minuscule path* (ending at 0) of type $\vec{\lambda}$ is a sequence of dominant weights

$$\vec{\mu}=(\mu_0,\mu_1,\mu_2,\ldots,\mu_n)$$

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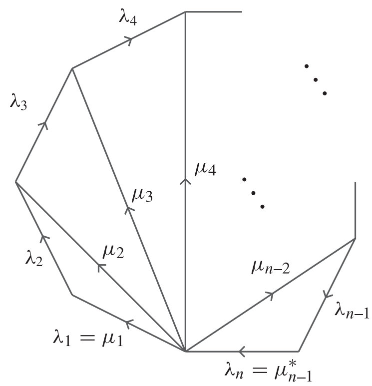


Figure 4. The fan diskoid $A(\vec{\lambda}, \vec{\mu})$.

such that $\mu_k - \mu_{k-1} \in W\lambda_k$ for every k, and such that

$$\mu_0 = \mu_n = 0.$$

In other words, the kth step of the path $\vec{\mu}$ is a weight of $V(\lambda_k)$, and the path is restricted to the dominant Weyl chamber Λ_+. Minuscule paths are a special case of Littelmann paths [Lit95], but it was much earlier folklore knowledge that the number of minuscule paths of type $\vec{\lambda}$ is the dimension of $\mathrm{Inv}(V(\vec{\lambda}))$. (See [Hum72, Exercise 24.9] and use induction.)

Given a minuscule path $\vec{\mu}$ of type $\vec{\lambda}$, we define a based diskoid $A(\vec{\lambda}, \vec{\mu})$ in the shape of a fan, whose boundary is the polygon $P(\vec{\lambda})$ and whose ribs are labelled by $\vec{\mu}$, as in Figure 4. Then there is a natural inclusion

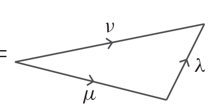
$$Q(A(\vec{\lambda}, \vec{\mu})) \subseteq F(\vec{\lambda}).$$

The following result is implicit in the work of Haines [Hai06].

Theorem 4.1. *For each minuscule path $\vec{\mu}$, the fan configuration space $Q(A(\vec{\lambda}, \vec{\mu}))$ is a dense subset of one component of $F(\vec{\lambda})$. The induced correspondence is a bijection between minuscule paths and components of $F(\vec{\lambda})$.*

The key to the proof of this theorem is the following lemma.

Lemma 4.2. *Let*

$$T_e(\mu, \lambda, \nu) =$$ 

be a triangle with a minuscule edge λ, based at the edge e of length μ. Then $Q(T_e(\mu, \lambda, \nu))$ is non-empty if and only if there exists $w \in W$ such that $\mu + w\lambda = \nu$. If it is non-empty, then it is smooth and has complex dimension $\langle \nu - \mu + \lambda, \rho^\vee \rangle$.

Proof. Let $W(\mu)$ denote the stabilizer of μ in the Weyl group. It is a parabolic subgroup of W.

Let us choose the base edge in Gr to be the edge connecting $t^{-\mu}$ and t^0. Then the edge based configuration space $Q(T_e(\mu, \lambda, \nu))$ is a subvariety of $\mathrm{Gr}(\lambda)$, since there is only one free vertex. In fact

$$Q(T_e(\mu, \lambda, \nu)) = \{p \in \mathrm{Gr}(\lambda) \mid d(t^{-\mu}, p) = \nu\}.$$

Let A denote the set $W/W(\lambda)$, which we regard as a poset using the opposite Bruhat order. With this order, A becomes the poset of B-orbits on $\mathrm{Gr}(\lambda) = G^\vee/M(\lambda)$, where B is the Borel subgroup

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of $G^\vee$. We will be interested in the action of $W(\mu)$ on A by left multiplication. The quotient $W(\mu)\backslash A$ is the set of $M_+(\mu)$ orbits on $\mathrm{Gr}(\lambda)$, where $M_+(\mu) = \mathrm{Stab}_{G^\vee}(t^{-\mu})$ is the parabolic subgroup corresponding to the minuscule weight μ.

Hence we can write any point p of $\mathrm{Gr}(\lambda)$ as $p = gt^{a\lambda}$ where $g \in M_+(\mu)$ and $a \in A$ is chosen to be a maximal-length representative for the orbit of $W(\mu)$. The action of $M_+(\mu)$ on Gr stabilizes $t^{-\mu}$, so

$$d(t^{-\mu}, gt^{a\lambda}) = d(t^{-\mu}, t^{a\lambda}) = d(t^0, t^{\mu+a\lambda}).$$

Now, we claim that $\mu + a\lambda$ is always dominant. Let us write $a = [w]$ for $w \in W$. We must check that

$$\langle \mu + w\lambda, \alpha_i^\vee \rangle = \langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w\alpha_i^\vee \rangle \geqslant 0$$

for all simple coroots $\alpha_i^\vee$. We break this calculation into two cases.

First, suppose that $s_i\mu = \mu$. Then $\langle \mu, \alpha_i^\vee \rangle = 0$. On the other hand, $s_iw > w$ (in the usual Bruhat order) by the maximality of a in the $W(\mu)$-orbit. This implies that $w\alpha_i^\vee$ is a positive coroot, which implies that $\langle \lambda, w\alpha_i^\vee \rangle$ is non-negative (since λ is dominant). Hence

$$\langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w\alpha_i^\vee \rangle \geqslant 0.$$

Next, suppose that $s_i\mu \neq \mu$. Then, since μ is dominant, $\langle \mu, \alpha_i^\vee \rangle \geqslant 1$. On the other hand, $|\langle \lambda, w\alpha_i^\vee \rangle| \leqslant 1$ since $w\alpha_i^\vee$ is a coroot and λ is minuscule. Hence

$$\langle \mu, \alpha_i^\vee \rangle + \langle \lambda, w\alpha_i^\vee \rangle \geqslant 0$$

in this case as well.

Since $\mu + a\lambda$ is always dominant, we conclude that

$$d(t^{-\mu}, gt^{a\lambda}) = \mu + a\lambda.$$

Hence, $Q(T_e(\mu, \lambda, \nu))$ is non-empty if and only if there exists $w \in W$ such that $\mu + w\lambda = \nu$. (The above argument shows that $[w]$ will necessarily be a maximal-length representative for the $W(\mu)$-action on A.) If such w exists, then the configuration space $Q(T_e(\mu, \lambda, \nu))$ is simply the $M(\mu)$-orbit through $t^{w\lambda}$. Hence it is smooth and its dimension is given by the length of $[w]$ in A because it is of the same dimension as the B-orbit through $t^{w\lambda}$. Since λ is minuscule, this equals $\langle w\lambda + \lambda, \rho^\vee \rangle$ as desired. □

Proof of Theorem 4.1. It is easy to show by induction that the fan configuration space

$$Q(A(\vec{\lambda}, \vec{\mu})) = Q(P_e(\mu_0, \lambda, \mu_1)) \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} Q(P_e(\mu_{n-1}, \lambda_n, \mu_n))$$

is an iterated twisted product of triangle configuration spaces. Since each factor has a minuscule edge, Lemma 4.2 tells us that $Q(A(\vec{\lambda}, \vec{\mu}))$ is also a smooth variety. Moreover, the dimensions add to tell us that

$$\dim_{\mathbb{C}} Q(A(\vec{\lambda}, \vec{\mu})) = \langle \lambda_1 + \cdots + \lambda_n, \rho^\vee \rangle = \dim_{\mathbb{C}} F(\vec{\lambda}).$$

On the other hand, $F(\vec{\lambda}) = Q(P(\vec{\lambda}))$ is partitioned as a set by the subvarieties $Q(A(\vec{\lambda}, \vec{\mu}))$, simply by taking the distances between the vertices of $P(\vec{\lambda})$ and the origin. If X is any algebraic variety with an equidimensional partition into smooth varieties $X_1, \ldots, X_N$, then X has pure dimension and its components are the closures of the parts X_k. In our case, $X = F(\vec{\lambda})$. □

It will be convenient, for later, to abbreviate the dimension of $F(\vec{\lambda})$ as

$$d(\vec{\lambda}) \stackrel{\text{def}}{=} \langle \lambda_1 + \cdots + \lambda_n, \rho^\vee \rangle = \dim_{\mathbb{C}} F(\vec{\lambda}).$$

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The same integers also arise in a different dimension formula,

$$\dim_{\mathbb{C}} \operatorname{Gr}(\vec{\lambda}) = 2d(\vec{\lambda}).$$

(Indeed, $\operatorname{Gr}(\vec{\lambda})$ is a top-dimensional component of $F(\vec{\lambda} \sqcup \vec{\lambda}^*)$, given by collapsing the polygon $P(\vec{\lambda} \sqcup \vec{\lambda}^*)$ onto the polyline $L(\vec{\lambda})$.)

Another important corollary of Lemma 4.2 is the following.

Theorem 4.3. *Suppose that D is a diskoid with boundary $\vec{\lambda}$ and with no internal vertices, and suppose that all edges of D (including the terms of $\vec{\lambda}$) are minuscule. Then $Q(D)$ is smooth and projective and is therefore a single component of $F(\vec{\lambda})$.*

Proof. Let $T_e(\mu, \lambda, \nu)$ be a triangle of D with three minuscule edges, and let the base edge e be any of the edges. Then, by Lemma 4.2, $Q(T_e(\mu, \lambda, \nu))$ is smooth. Likewise $T_p(\mu, \lambda, \nu)$, based at a point p instead, is smooth. By construction, $Q(D)$ is a twisted product of configuration spaces of this form, so it is smooth too. It is also projective since D is a minuscule linkage.

There is one delicate point in the inference that $Q(D)$ is a component of $F(\vec{\lambda})$: is the restriction map $Q(D) \to F(\vec{\lambda})$ injective? As in the proof of Lemma 4.2, the restriction map

$$\pi : Q(T_e(\mu, \lambda, \nu)) \to \operatorname{Gr}(\lambda)$$

is injective, and so is the restriction map

$$\pi : Q(T(\mu, \lambda, \nu)) \to \operatorname{Gr}(\mu, \lambda).$$

The diskoid D must have a triangle with at least two edges on the boundary, so by induction its restriction map to $F(\vec{\lambda})$ is also injective. □

4.2 A homological state model

This subsection discusses our motivation for the technical constructions in the remainder of § 4.

We would like to use Theorem 1.1 as a state model or counting model to evaluate webs in $\mathbf{rep}^u(G)$. If w is a web with dual diskoid D, then there is a map of linkages

$$P(\vec{\lambda}) = \partial D \longrightarrow \Gamma(D)$$

given by the inclusion of the boundary. This gives rise to a restriction map

$$\pi = \pi^{\Gamma(D)}_{P(\vec{\lambda})} : Q(D) \to F(\vec{\lambda}).$$

A point in $Q(D)$ is a 'state' of D in the sense of mathematical physics, such that each vertex of D (or each face of w) is assigned an element of Gr. We would like to count the number of states of D with some fixed boundary or, in other words, find the cardinality of a diskoid fibre $\pi^{-1}(f)$ for $f \in F(\vec{\lambda})$. If f is chosen generically in a top-dimensional component of $F(\vec{\lambda})$, then optimistically this cardinality will be the coefficient of $\Psi(w)$ in the Satake basis.

However, this sketch is naive. The diskoid fibre $\pi^{-1}(f)$ often has a complicated geometry for which it is hard to define 'counting'. The first and main solution for us is to replace counting by a homological intersection. (In § 6 we will propose a second solution, in which we count by taking the Euler characteristic of the fibre.) In particular, for each web w, we will define a homology class $c(w) \in H_{\text{top}}(Q(D))$ such that $\pi_*(c(w))$ equals $\Psi(w)$.

4.3 The homology convolution category

If M is an algebraic variety over $\mathbb{C}$, we will consider its intersection cohomology sheaf IC_M as a simple object in the category of perverse sheaves on M. If M is smooth, then IC_M is isomorphic

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to $\mathbb{C}_M[\dim_{\mathbb{C}} M]$, the constant sheaf shifted by the complex dimension of M. For brevity, we will write this perverse sheaf as $\mathbb{C}[M]$.

The geometric Satake correspondence is a tensor functor that takes the usual product on $\mathbf{rep}^u(G)$ to the convolution tensor product on $\mathbf{perv}(\mathrm{Gr})$. In particular, the tensor product $V(\vec{\lambda})$ of irreducible minuscule representations corresponds to the convolution tensor product of the simple perverse sheaves $\mathbb{C}[\mathrm{Gr}(\lambda_i)]$ on minuscule spheres, which are closed in the algebraic geometry. By definition, this convolution tensor product is given by the pushforward $(m_{\vec{\lambda}})_*(\mathbb{C}[\mathrm{Gr}(\vec{\lambda})])$ along the convolution morphism.

Let $\mathbf{perv}(\mathrm{Gr})_{\min}$ denote the subpivotal category of $\mathbf{perv}(\mathrm{Gr})$ consisting of such pushforwards. By construction, $\mathbf{perv}(\mathrm{Gr})_{\min}$ is equivalent to $\mathbf{rep}^u(G)_{\min}$. Our goal is to study $\mathbf{perv}(\mathrm{Gr})_{\min}$ using convolutions in homology, following ideas of Ginzburg. We begin by reviewing some generalities, following [CG97, § 2.7].

Let $\{M_i\}$ be a set of connected, smooth complex varieties and let M_0 be a possibly singular, stratified variety with strata $\{U_\alpha\}$. For each i, let $\pi_i : M_i \to M_0$ be a proper semismall map. In this context, the statement that π_i is semismall means that π_i restricts to a fibre bundle over each stratum U_α and that the dimensions of these fibres are given by

$$\dim_{\mathbb{C}} \pi_i^{-1}(u) = \frac{\dim_{\mathbb{C}} M_i - \dim_{\mathbb{C}} U_\alpha}{2}$$

for $u \in U_\alpha$ (note that we have equality above). Let $d_i = \dim_{\mathbb{C}} M_i$.

With this setup, let $Z_{ij} = M_i \times_{M_0} M_j$. The semismallness condition implies that $\dim_{\mathbb{C}} Z_{ij} = (d_i + d_j)/2$. Let

$$H_{\mathrm{top}}(Z_{ij}) = H_{d_i+d_j}(Z_{ij})$$

be the top homology of Z_{ij}. If the M_i are proper, which they will be in our situation, then we will obtain a valid definition of the convolution product using the ordinary singular homology of Z_{ij}. (Otherwise the correct type of homology would be Borel–Moore homology.)

Define a *homological convolution product*

$$* : H_{\mathrm{top}}(Z_{ij}) \otimes H_{\mathrm{top}}(Z_{jk}) \to H_{\mathrm{top}}(Z_{ik})$$

by the formula

$$c_1 * c_2 = (\pi_{ik})_*(\pi_{ij}^*(c_1) \cap \pi_{jk}^*(c_2)),$$

where $\cap$ here denotes the intersection product (with support), relative to the ambient smooth manifold $M_i \times M_j \times M_k$. This may be defined using the cup product in cohomology via Poincaré duality. For more details about this construction, see [CG97, § 2.6.15] or [Ful98, § 19.2]. Note that because

$$\dim_{\mathbb{C}} Z_{ij} = \frac{d_i + d_j}{2},$$

the correct homological degree is preserved by the convolution product.

This construction is relevant for us because of a theorem of Ginzburg that relates $H_{\mathrm{top}}(Z_{ij})$ to morphisms in the category $\mathbf{perv}(M_0)$ of perverse sheaves on M_0.

Theorem 4.4 [CG97, Theorem 8.6.7]. *With the above setup, there is an isomorphism*

$$H_{\mathrm{top}}(Z_{ij}) \cong \mathrm{Hom}_{\mathbf{perv}(M_0)}((\pi_i)_*\mathbb{C}[M_i], (\pi_j)_*\mathbb{C}[M_j]).$$

This isomorphism identifies convolution products on the left-hand side with compositions of morphisms on the right-hand side.

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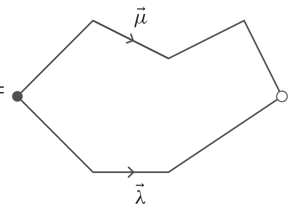
We will apply this setup by taking $M_0 = \mathrm{Gr}$ and letting each M_i be $\mathrm{Gr}(\vec{\lambda})$ for a sequence $\vec{\lambda}$ of dominant minuscule weights. The convolution morphism $m_{\vec{\lambda}} : \mathrm{Gr}(\vec{\lambda}) \to \mathrm{Gr}$ is semismall. (See [MV07a, Lemma 4.4]; it also follows from the proof of Theorem 4.1.) Then Z_{ij} becomes

$$Z(\vec{\lambda}, \vec{\mu}) = \mathrm{Gr}(\vec{\lambda}) \times_{\mathrm{Gr}} \mathrm{Gr}(\vec{\mu}) = Q(P(\vec{\lambda}^* \sqcup \vec{\mu})),$$

where $P(\vec{\lambda}^* \sqcup \vec{\mu})$ is the following polygon.

$$P(\vec{\lambda}^* \sqcup \vec{\mu}) =$$



Theorem 4.4 motivates the following construction of a category **hconv**(Gr). The objects in **hconv**(Gr) are the polyline varieties $\mathrm{Gr}(\vec{\lambda})$, where $\vec{\lambda}$ is a sequence of minuscule weights. The tensor product on objects is, by definition, given by convolution on objects, so

$$\mathrm{Gr}(\vec{\lambda}) \otimes \mathrm{Gr}(\vec{\mu}) \stackrel{\text{def}}{=} \mathrm{Gr}(\vec{\lambda} \sqcup \vec{\mu}),$$

where $\sqcup$ denotes concatenation of sequences. Therefore the identity object is the point $\mathrm{Gr}(\emptyset)$. Finally, the dual object $\mathrm{Gr}(\vec{\lambda})^* = \mathrm{Gr}(\vec{\lambda}^*)$ of $\mathrm{Gr}(\vec{\lambda})$ is given by reversing $\vec{\lambda}$ and taking the dual of each of its terms.

We define the morphism spaces of **hconv**(Gr) as

$$\mathrm{Hom}_{\mathbf{hconv}(\mathrm{Gr})}(\mathrm{Gr}(\vec{\lambda}), \mathrm{Gr}(\vec{\mu})) \stackrel{\text{def}}{=} H_{\text{top}}(Z(\vec{\lambda}, \vec{\mu})).$$

The composition of morphisms is given by the convolution product. Note that the identity morphism $1_{\vec{\lambda}} \in H_{\text{top}}(Z(\vec{\lambda}, \vec{\lambda}))$ is given by the class $[\mathrm{Gr}(\vec{\lambda})_\Delta]$ of the diagonal

$$\mathrm{Gr}(\vec{\lambda})_\Delta \subseteq Z(\vec{\lambda}, \vec{\lambda}) \subseteq \mathrm{Gr}(\vec{\lambda}) \times \mathrm{Gr}(\vec{\lambda}),$$

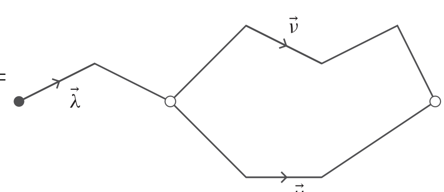
i.e. it is the configuration in which the polygon $P(\vec{\lambda}^* \sqcup \vec{\lambda})$ has collapsed onto the polyline $L(\vec{\lambda})$.

To describe the tensor structure on morphisms, it is enough to describe how to tensor with the identity morphism. So let $\vec{\lambda}, \vec{\mu}$ and $\vec{\nu}$ be three sequences of dominant minuscule weights and let $c \in H_{\text{top}}(Z(\vec{\mu}, \vec{\nu}))$. Our goal is to construct a class

$$1_{\vec{\lambda}} \otimes c \in H_{\text{top}}(Z(\vec{\lambda} \sqcup \vec{\mu}, \vec{\lambda} \sqcup \vec{\nu})).$$

For the moment, let Γ be a ρ-shaped graph with a tail of type $\vec{\lambda}$ and a loop of type $\vec{\mu}^* \sqcup \vec{\nu}$, based at the end of the tail, as shown below.

$$\Gamma =$$



Let $X = Q(\Gamma)$ be its based configuration space. We describe two fibration constructions related to X. First, there is a restriction map

$$\pi^{L(\vec{\lambda} \sqcup \vec{\mu})}_{L(\vec{\lambda}) \sqcup \mathrm{pt}} : \mathrm{Gr}(\vec{\lambda} \sqcup \vec{\mu}) \to \mathrm{Gr}(\vec{\lambda}) \times \mathrm{Gr}$$

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given by restricting to the polyline $L(\vec{\lambda})$ and the free endpoint of $L(\vec{\lambda} \sqcup \vec{\mu})$. Then X is the fibred product

$$X = \mathrm{Gr}(\vec{\lambda} \sqcup \vec{\mu}) \times_{\mathrm{Gr}(\vec{\lambda}) \times \mathrm{Gr}} \mathrm{Gr}(\vec{\lambda} \sqcup \vec{\nu}).$$

Second, there is a projection

$$\pi^{\Gamma}_{L(\vec{\lambda})} : X \to \mathrm{Gr}(\vec{\lambda})$$

given by restricting from Γ to $L(\vec{\lambda})$. The fibres of this projection are $Z(\vec{\mu}, \vec{\nu})$.

Since $\mathrm{Gr}(\vec{\lambda})$ is simply connected, we get an isomorphism

$$H_{\mathrm{top}}(X) \cong H_{\mathrm{top}}(\mathrm{Gr}(\vec{\lambda})) \otimes H_{\mathrm{top}}(Z(\vec{\mu}, \vec{\nu})),$$

and thus we obtain an isomorphism

$$H_{\mathrm{top}}(Z(\vec{\mu}, \vec{\nu})) \xrightarrow{\cong} H_{\mathrm{top}}(X)$$

given by $c \mapsto [\mathrm{Gr}(\vec{\lambda})] \otimes c$.

There is also an inclusion

$$i = \pi^{\Gamma}_{P(\vec{\lambda} \sqcup \vec{\mu} \sqcup \vec{\nu}^* \sqcup \vec{\lambda}^*)} : X \to Z(\vec{\lambda} \sqcup \vec{\mu}, \vec{\lambda} \sqcup \vec{\nu}),$$

using the polygon which travels twice along the tail of Γ and around the loop of Γ. Combining all this structure, we define

$$1_{\vec{\lambda}} \otimes c \stackrel{\mathrm{def}}{=} i_*([\mathrm{Gr}(\vec{\lambda})] \otimes c).$$

Tensoring by the identity morphism on the other side is similar, and we leave the construction to the reader.

Finally, to define the cap and cup morphisms for any $\vec{\lambda}$, we will define them for a single minuscule weight λ. Note that

$$Z(\lambda \sqcup \lambda^*, \emptyset) = Z(\emptyset, \lambda \sqcup \lambda^*) = F(\lambda, \lambda^*) \cong \mathrm{Gr}(\lambda).$$

We define the cup b_λ and the cap d_λ to each be the class $[\mathrm{Gr}(\lambda)]$ in their respective hom spaces.

Theorem 4.5. *There is an equivalence of pivotal categories*

$$\mathbf{hconv}(\mathrm{Gr}) \cong \mathbf{perv}(\mathrm{Gr})_{\min} \cong \mathbf{rep}^u(G)_{\min}.$$

Applying Theorem 4.5 to invariant spaces, we obtain an isomorphism

$$\mathrm{Inv}(V(\vec{\lambda})) \cong \mathrm{Hom}_{\mathbf{hconv}(\mathrm{Gr})}(\mathrm{Gr}(\emptyset), \mathrm{Gr}(\vec{\lambda})) = H_{\mathrm{top}}(Z(\emptyset, \vec{\lambda})) = H_{\mathrm{top}}(F(\vec{\lambda})),$$

which is Theorem 1.2.

Proof. The second equivalence is geometric Satake, so we will just prove the first equivalence. We begin by showing that it is an equivalence of monoidal categories.

By the definition, the objects in both categories are parameterized by sequences $\vec{\lambda}$, so the functor on objects is very simple. On morphisms, the functor is given by the isomorphisms from Theorem 4.4. By this theorem, the functor is fully faithful and is compatible with composition on both sides (i.e. it is a functor). To complete the proof this theorem, we need only show that the functor is compatible with the tensor product and with pivotal duality.

To see that it is compatible with the tensor product, we use the same notation as above. If

$$c \in \mathrm{Hom}((m_{\vec{\mu}})_* \mathbb{C}[\mathrm{Gr}(\vec{\mu})], (m_{\vec{\nu}})_* \mathbb{C}[\mathrm{Gr}(\vec{\nu})]),$$

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then with respect to the tensor structure in $\mathbf{perv}(\mathrm{Gr})$, $I_{(m_{\vec\lambda})_*\mathbb{C}[\mathrm{Gr}(\vec\lambda)]} \otimes c$ is given by the image of c under the map

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{perv}(\mathrm{Gr})}((m_{\vec\mu})_*\mathbb{C}[\mathrm{Gr}(\vec\mu)], (m_{\vec\nu})_*\mathbb{C}[\mathrm{Gr}(\vec\nu)])\\ &\quad\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{perv}(\mathrm{Gr}(\vec\lambda)\times \mathrm{Gr})}\Big(\big(\pi^{L(\vec\lambda\sqcup\vec\mu)}_{L(\vec\lambda)\sqcup \mathrm{pt}}\big)_*\mathbb{C}[\mathrm{Gr}(\vec\lambda\sqcup\vec\mu)], \big(\pi^{L(\vec\lambda\sqcup\vec\nu)}_{L(\vec\lambda)\sqcup \mathrm{pt}}\big)_*\mathbb{C}[\mathrm{Gr}(\vec\lambda\sqcup\vec\nu)]\Big)\\ &\quad\xrightarrow{p_*} \mathrm{Hom}_{\mathbf{perv}(\mathrm{Gr})}((\pi_{\vec\lambda\sqcup\vec\mu})_*\mathbb{C}[\mathrm{Gr}(\vec\lambda\sqcup\vec\mu)], (m_{\vec\lambda\sqcup\vec\nu})_*\mathbb{C}[\mathrm{Gr}(\vec\lambda\sqcup\vec\nu)]).\end{aligned}$$

Here $p : \mathrm{Gr}(\vec\lambda)\times \mathrm{Gr} \to \mathrm{Gr}$ is the projection onto the second factor. This is easily seen to match our above definition.

It remains to check that the pivotal structures match under this equivalence. Recall from § 2.2 that the pivotal structures on $\mathbf{rep}(G)$ are determined by the dimensions $\dim(V(\lambda))$, which are by definition the values of closed loops. (The discussion there was for pivotal structures that differ by a sign, but it is true in general.) Moreover, the discrepancy is multiplicative, so it only needs to be checked for minuscule λ.

Let λ be minuscule. In $\mathbf{hconv}(\mathrm{Gr})$, the value of a loop labelled λ, i.e. the composition

$$d_\lambda \circ b_\lambda \in \mathrm{Hom}(\mathrm{Gr}(\emptyset), \mathrm{Gr}(\emptyset)) = \mathbb{C},$$

is given by the self-intersection of $\mathrm{Gr}(\lambda) \cong F(\lambda, \lambda^*)$ with itself inside $\mathrm{Gr}(\lambda, \lambda^*)$.

There is a neighbourhood (defined using the pullback of the open big cell) of $F(\lambda, \lambda^*)$ in $\mathrm{Gr}(\lambda, \lambda^*)$ which is isomorphic to $T^*\mathrm{Gr}(\lambda)$, under an isomorphism which carries $F(\lambda, \lambda^*)$ to the zero section $\mathrm{Gr}(\lambda)$.

For any compact, complex d-manifold X, the self-intersection of X with itself inside T^*X is $(-1)^d\chi(X)$, where $\chi(X)$ is the Euler characteristic of X. (The self-intersection in TX is $\chi(X)$, and for a complex d-manifold the cotangent bundle T^*X has the opposite real orientation exactly when d is odd.) Applying this to $X = \mathrm{Gr}(\lambda)$, we conclude that

$$d_\lambda \circ b_\lambda = (-1)^d \chi(\mathrm{Gr}(\lambda)) = (-1)^{\langle 2\lambda, \rho^\vee\rangle} \dim V(\lambda).$$

This is the sign correction that is used to define the pivotal structure on $\mathbf{rep}^u(G)$, as desired. □

4.4 From the free spider to the convolution category

Section 2 describes a pivotal functor

$$\Psi : \mathbf{fsp}(G) \to \mathbf{rep}^u(G)_{\min}.$$

On the other hand, the geometric Satake correspondence and Theorem 4.5 yield equivalences

$$\mathbf{rep}^u(G)_{\min} \cong \mathbf{perv}(\mathrm{Gr})_{\min} \cong \mathbf{hconv}(\mathrm{Gr}).$$

The composition is a functor $\mathbf{fsp}(G) \to \mathbf{hconv}(\mathrm{Gr})$ which we will also denote by Ψ. Our goal now is to describe this functor and, in particular, its action on invariant vectors.

Let (λ, μ, ν) be a triple of dominant minuscule weights such that

$$\mathrm{Inv}_G(V(\lambda, \mu, \nu)) \neq 0.$$

There is a simple web $w \in \mathrm{Inv}_{\mathbf{fsp}(G)}(\lambda, \mu, \nu)$ which contains a single vertex. On the other hand,

$$\mathrm{Inv}_{\mathbf{hconv}(\mathrm{Gr})}(\lambda, \mu, \nu) \cong H_{\mathrm{top}}(F(\lambda, \mu, \nu))$$

is one-dimensional with canonical generator $[F(\lambda, \mu, \nu)]$. Recall from § 2 that in the construction of the functor $\mathbf{fsp}(G) \to \mathbf{rep}^u(G)_{\min}$, there was some freedom to choose the image of the simple

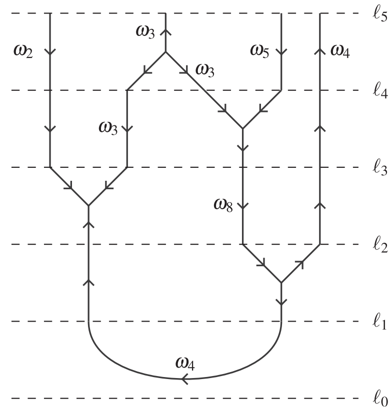

Figure 5. A web for SL(9) in Morse position.

web w (it was only defined up to a non-zero scalar). Now we fix this choice by setting

$$\Psi(w) \stackrel{\text{def}}{=} [F(\lambda, \mu, \nu)].$$

The functor Ψ is now determined by what it does on vertices and the fact that it preserves the pivotal structure on both sides.

We are now in a position to prove Theorem 1.3, which we will restate as follows. Recall that

$$d(\vec{\lambda}) = \dim_{\mathbb{C}} F(\vec{\lambda}).$$

Theorem 4.6. *Let w be a web with boundary $\vec{\lambda}$ and dual diskoid $D = D(w)$. Let*

$$\pi : Q(D) \to F(\vec{\lambda})$$

be the boundary restriction map. There exists a homology class $c(w) \in H_{2d(\vec{\lambda})}(Q(D))$ such that $\pi_*(c(w)) = \Psi(w)$. Moreover, when $Q(D)$ has dimension $d(\vec{\lambda})$ and is reduced as a scheme, then $c(w)$ is the fundamental class $[Q(D)]$.

Proof. We begin by picking a isotopy representative for w such that the height function is a Morse function and so that the boundary of w is at the top level. We assume a sequence of horizontal lines $\ell_0, \dots, \ell_m$ such that in between each pair, w has only a single cap, cup or vertex. We assume further that each vertex is either an ascending Y (i.e. it is in the shape of a Y) or a descending Y (an upside-down Y).

Let $\vec{\lambda}^{(k)}$ be the vector of labels of the edges cut by the horizontal line ℓ_k. Then $\vec{\lambda}^{(0)} = \emptyset$ and $\vec{\lambda}^{(m)} = \vec{\lambda}$. For example, Figure 5 shows an SL(9) web in Morse position, with edges labelled by its minuscule weights ω_k with $1 \leqslant k \leqslant 8$. In this example,

$$\vec{\lambda}^{(1)} = \{\omega_4, \omega_5\}, \quad \vec{\lambda}^{(3)} = \{\omega_7, \omega_6, \omega_1, \omega_4\}.$$

(Note that in SL(n) in general, $\omega_k^* = \omega_{n-k}$; if an edge points down as it crosses a line, then we must take the dual weight.)

Let

$$w_k \in \mathrm{Hom}_{\mathbf{fsp}(G)}(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$$

denote the web in the horizontal strip between the lines ℓ_{k-1} and ℓ_k. By examining the above definition, we see that for each $1 \leqslant k \leqslant m$, there exists a component $X_k \subset Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$ such

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that $\Psi(w_k) = [X_k]$. We would like to describe this component explicitly. For convenience, if

$$\vec{p} = (p_0, p_1, \ldots, p_m) \in \mathrm{Gr}^{m+1}$$

(with $p_0 = t^0$ for us), define $\sigma_i(\vec{p})$ by omitting the term p_i.

(i) If w_k is an ascending Y vertex that connects the ith point on ℓ_{k-1} to the ith and $(i+1)$st points on ℓ_k, then

$$X_k = \{(\vec{p}, \vec{p}\,') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) \mid \vec{p} = \sigma_i(\vec{p}\,')\}.$$

(ii) If w_k is a descending Y vertex that connects the ith and $(i+1)$st points on ℓ_{k-1} to the ith point on ℓ_k, then

$$X_k = \{(\vec{p}, \vec{p}\,') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) \mid \vec{p}\,' = \sigma_i(\vec{p})\}.$$

(iii) If w_k is a cup that connects the ith and $(i+1)$st points on ℓ_k, then

$$X_k = \{(\vec{p}, \vec{p}\,') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) \mid \vec{p} = \sigma_i(\sigma_i(\vec{p}\,'))\}.$$

(iv) If w_k is a cap that connects the ith and $(i+1)$st points on ℓ_{k-1}, then

$$X_k = \{(\vec{p}, \vec{p}\,') \in Z(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)}) \mid \vec{p}\,' = \sigma_i(\sigma_i(\vec{p}))\}.$$

Then $w = w_m \circ \cdots \circ w_1$. Since Ψ is a functor,

$$\Psi(w) = \Psi(w_m) * \cdots * \Psi(w_1) = [X_m] * \cdots * [X_1].$$

Now, compositions of convolutions can be computed as a single convolution as

$$[X_m] * \cdots * [X_1] = (\pi_{0,m})_*(\pi_{0,1}^*[X_1] \cdots \pi_{m-1,m}^*[X_m]),$$

where the intersection products take place in the ambient smooth manifold

$$X = \mathrm{Gr}(\vec{\lambda}^{(0)}) \times \cdots \times \mathrm{Gr}(\vec{\lambda}^{(m)}).$$

Here $\pi_{k-1,k}$ denotes the projection from X to $\mathrm{Gr}(\vec{\lambda}^{(k-1)}, \vec{\lambda}^{(k)})$.

From the definitions, we see that the diskoid configuration spaces $Q(D)$ can be obtained as

$$Q(D) = \pi_{0,1}^{-1}(X_1) \cap \cdots \cap \pi_{m-1,m}^{-1}(X_m).$$

Let

$$\begin{aligned} c(w) &= \pi_{0,1}^*[X_1] \cap \cdots \cap \pi_{m-1,m}^*[X_m] \\ &= [\pi_{0,1}^{-1}(X_1)] \cap \cdots \cap [\pi_{m-1,m}^{-1}(X_m)]. \end{aligned}$$

Because we are using the intersection product with support, $c(w)$ lives in $H_{d(\vec{\lambda})}(Q(D))$, the homology of the intersection. When $Q(D)$ is reduced of the expected dimension, the intersection product of the homology classes corresponds to the fundamental class of the intersection (see [Ful98, § 8.2]); so $c(w) = [Q(D)]$.

Finally, $\pi : Q(D) \to F(\vec{\lambda})$ is the restriction of $\pi_{0,m}$ to $Q(D)$. Hence we conclude that $\Psi(w) = \pi_*(c(w))$. □

Because $\pi_*(c(w))$ is supported on $\pi(Q(D))$, we immediately obtain the following result.

Corollary 4.7. *$\Psi(w)$ is a linear combination of the fundamental classes of the components of $F(\vec{\lambda})$ which are in the image of π.*

It may not seem clear that $c(w)$ depends only on the web w and not on the Morse position of w used to construct it. However, a posteriori, this must be verified by checking that it is invariant under basic isotopy moves (for example, straightening out a cup/cap pair).

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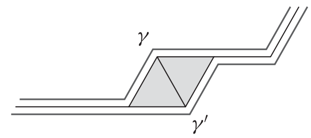

Figure 6. Two geodesics γ and γ' connected by a diamond move.

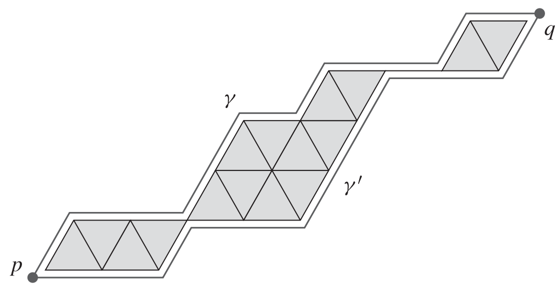

Figure 7. A skew partition bounded by extremal geodesics γ and γ'.

5. SL(3) results

In this section, we will prove Theorems 1.4 and 1.5. In preparation, we need to use and extend the geometry of non-elliptic webs. To review, if w is an A_2 web and $D = D(w)$ is its dual diskoid, then w is non-elliptic if and only if D is CAT(0).

5.1 Geodesics in CAT(0) diskoids

We will be interested in combinatorial (meaning edge-travelling) geodesics in a type-A_2 diskoid D. These are equivalent to 'minimal cut paths' of the dual web [Kup96] when the endpoints of the geodesic are boundary vertices D. Here we will consider geodesics between vertices that may be in the interior or on the boundary. If both vertices are on the boundary, then the geodesic is said to be *complete*.

Geodesics in an A_2 diskoid are often not unique. Define a *diamond move* of a geodesic to be a move in which the geodesic crosses two triangles, as in Figure 6. (This is equivalent to an 'H-move' on a cut path of a non-elliptic web.) We say that two geodesics are *isotopic* if they are equivalent with respect to diamond moves.

Theorem 5.1. *Let p and q be two vertices of a* CAT(0)*, type-A_2 diskoid D. Then the geodesics between p and q subtend a diskoid which is a skew Young diagram, with each square split into two triangles. In particular, all geodesics are isotopic, D is geodesically coherent, and all geodesics lie between two extremal geodesics. Both of the extremal geodesics are concave on the outside.*

Here a *skew Young diagram* is the same as the usual object in combinatorics with that name, namely the diskoid lying between two geodesic lattice paths in $\mathbb{Z}^2$. Figure 7 shows an example in which the squares have been split so that it becomes an A_2 diskoid.

Theorem 5.1 is proven in [Kup96] in the case where p and q are on the boundary. If they are not on the boundary, then we can reduce to the previous case by removing the simplices of D that do not lie between two geodesics. The final statement, that an extremal geodesic γ is concave outside of the skew Young diagram, is easy to check: if γ has an angle of $\pi/3$, then it is not a geodesic; if it has an angle of $2\pi/3$, then an isotopy is available and it is not extremal.

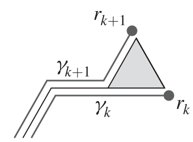

Figure 8. A triangle move connecting geodesics γ_k and γ_{k+1}.

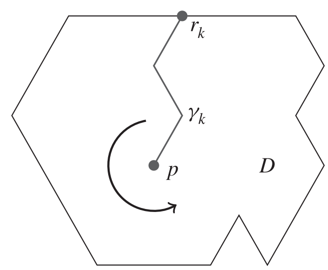

Figure 9. Making a sequence of geodesics that sweep out D.

Lemma 5.2. *If p and q are two vertices of a* CAT(0) *diskoid D, then every geodesic κ between them extends to a complete geodesic.*

Proof. The argument is based on a geodesic sweep-out construction. We claim that we can make a sequence of geodesics

$$\vec{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$$

from p to the boundary ∂D with certain additional properties. We require that each pair of consecutive geodesics γ_k and γ_{k+1} differ by either an elementary isotopy or an elementary boundary isotopy (for each $k \in \mathbb{Z}/m$). The latter consists of either appending an edge to γ_k or removing the last edge, or a *triangle move* as in Figure 8.

We require that the other endpoint r_k of γ_k travel all the way around ∂D in the counterclockwise direction, as in Figure 9.

If p is on the boundary, then $r_0 = p$, but this is okay. It is easy to see that if $\vec{\gamma}$ exists, then it uses every vertex in D. There is thus a geodesic γ from p to $r \in \partial D$ that contains q. We can then repeat the argument with r replacing p, to obtain a geodesic γ' from r to some $s \in \partial D$ that contains p. The geodesic γ' may not contain q, much less all of κ. However, because D is geodesically coherent, the path

$$\gamma'' = \gamma'(s, p) \sqcup \kappa \sqcup \gamma(q, r)$$

is a geodesic and satisfies the lemma, as in Figure 10.

To prove the claim, let γ_0 be the geodesic of length 0 if $p \in \partial D$; otherwise, let γ_0 be the geodesic from p to any $r_0 \in \partial D$ which is counterclockwise extremal. We construct $\vec{\gamma}$ iteratively. Given γ_k, we apply a diamond move to make γ_{k+1} if such a move is possible. If such a move is not possible, then let r_{k+1} be the next boundary vertex after r_k, and let γ_{k+1} be the clockwise-extremal geodesic from p to r_{k+1}, among geodesics that do not cross γ_k. (In other words, cut D along γ_k to make D', and then let γ_{k+1} be clockwise extremal in D'.) By geodesic coherence, the region between γ_k and γ_{k+1} is either empty or connected; otherwise we could splice γ_{k+1} with γ_k so that γ_{k+1} would not be clockwise extremal.

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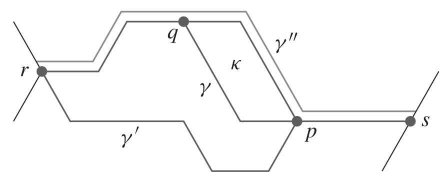

Figure 10. A geodesic replacement argument.

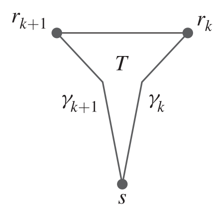

Figure 11. A topological triangle T made from geodesics.

If the region between γ_k and γ_{k+1} is empty, then either $\gamma_k \subseteq \gamma_{k+1}$ or $\gamma_{k+1} \subseteq \gamma_k$. If it is not empty, then there are two geodesic segments $\gamma_k(s, r_k)$ and $\gamma_{k+1}(s, r_{k+1})$ that together with the edge (r_k, r_{k+1}) make a topological triangle T, as shown in Figure 11.

We summarize the properties of the topological triangle T: it is CAT(0), all three sides are concave, and its angles at the corners are at least $\pi/3$. So T is flat, all three sides are flat (unlike in the figure), and all three angles equal $\pi/3$. Thus, T is a face of D and γ_k and γ_{k+1} differ by a triangle move.

As k increases, eventually $r_k = r_0$. Once the diamond moves are exhausted for this choice of r_k (there are none if p is on the boundary), the sequence of geodesics returns to the beginning. □

The sweep-out construction in the proof of Lemma 5.2 also yields this lemma.

Lemma 5.3. *Let D be a* CAT(0) *diskoid with a boundary vertex p. Then every edge of D either lies on a complete geodesic from p to some $q \in \partial D$, or lies in a diamond move or a triangle move between two geodesics from p.*

Finally, there is a relation between fans as described in § 4.1 and non-elliptic webs. Given a diskoid D with boundary $\vec{\lambda}$, let $\vec{\mu}(D)$ be the sequence of distances $d(p, q_k)$ where p is the base point of D and q_k is the sequence of boundary vertices of D. Then the following holds.

Theorem 5.4 [Kup96]. *Given a sequence of A_2 minuscule weights $\vec{\lambda}$, the map $D \mapsto \vec{\mu}(D)$ is a bijection between* CAT(0) *diskoids and minuscule paths of type $\vec{\lambda}$.*

So we can write $D(\vec{\lambda}, \vec{\mu})$ as the non-elliptic web with boundary $\vec{\lambda}$ and minuscule path $\vec{\mu}$.

5.2 Unitriangularity

We apply § 5.1 to prove the following result. It is a bridge result, based on the geometry of affine buildings, that we will use to relate web bases to the geometric Satake correspondence and, in particular, to prove Theorem 1.5.

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Theorem 5.5. *Let $\vec{\lambda}$ be a minuscule sequence of type A_2 and let $\vec{\mu}$ be a minuscule path of type $\vec{\lambda}$. If $f \in Q(A(\vec{\lambda}, \vec{\mu}))$ is a fan configuration, then it extends uniquely to a diskoid configuration $f \in Q(D(\vec{\lambda}, \vec{\mu}))$.*

Proof. The construction derives from the constraints that make the extension unique. Let p be the base vertex of D, so that $f(p) = 0 \in \mathrm{Gr}$. Suppose that q is the kth boundary vertex of D and that γ is a geodesic from p to q. Then $d(f(p), f(q)) = \mu_k$, and by definition μ_k is also the length of γ. If Σ is an apartment containing $f(p)$ and $f(q)$, then $f(q) = \mu_k$ in suitable coordinates in Σ. It follows that there is a unique geodesic in Σ with the same sequence of edge weights as γ and which connects $f(p)$ with $f(q)$. Thus f extends uniquely to γ.

We claim that this extension of f is consistent for vertices of D. First, every vertex of D is contained in some complete geodesic from p, since by Lemma 5.2 any geodesic from p to a vertex extends to a complete geodesic. Suppose that γ and γ' are two geodesics from p to $q \in \partial D$ and $q' \in \partial D$, respectively. Suppose further that $r \in \gamma \cap \gamma'$. Then every apartment that contains p and r contains both geodesics $\gamma(p, r)$ and $\gamma'(p, r)$. In particular, each apartment $\Sigma \supseteq \gamma$ and $\Sigma' \supseteq \gamma'$ does. It follows that the choices for $f(r)$ induced by γ and γ' are the same.

We claim that if (r, s) is an edge in D, then

$$d(r, s) = d(f(r), f(s)). \tag{7}$$

By Lemma 5.3, there are three cases: either (r, s) occurs in a complete geodesic from p to some q, or it occurs in a diamond move between two such geodesics γ and γ', or r and s are both on the boundary and (r, s) occurs in a triangle move between two geodesics γ and γ'. In the first case, (7) is true by construction. In the second case, $f(\gamma)$ and $f(\gamma')$ are contained in a single apartment, because every apartment contains all geodesics from $f(p)$ to $f(q)$. In the third case, there is an apartment containing p and (r, s) by the axioms for a building, since these are both simplices. In both cases, the existence of this common apartment implies (7). □

Now let $\vec{\lambda}$ be a minuscule dominant sequence, and let $\vec{\mu}$ be a minuscule path of type $\vec{\lambda}$. Then there is a corresponding non-elliptic web $w(\vec{\lambda}, \vec{\mu})$ with dual diskoid $D(\vec{\lambda}, \vec{\mu})$. There is also a corresponding component $\overline{Q(A(\vec{\lambda}, \vec{\mu}))}$ of $F(\vec{\lambda})$.

We have two bases for $H_{\mathrm{top}}(F(\vec{\lambda}))$, one given by $[\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$ and the other given by $\Psi(w(\vec{\lambda}, \vec{\mu}))$, and both bases are indexed by the minuscule path $\vec{\mu}$. Under the isomorphism

$$H_{\mathrm{top}}(F(\vec{\lambda})) \cong \mathrm{Inv}(V(\vec{\lambda})),$$

these become the Satake and web bases, respectively, the first by definition and the second by Theorem 4.6. Our purpose in this section is to prove that the transition matrix between these two bases is unitriangular. Define a partial order on minuscule paths by the rule that $\vec{\nu} \leqslant \vec{\mu}$ if $\nu_i \leqslant \mu_i$ for all i.

Theorem 5.6. *The transition matrix between the Satake and web bases is unitriangular with respect to the partial order $\leqslant$.*

In the next section, we will use this result to deduce Theorem 1.5, which concerns a weaker partial order and is thus a stronger statement.

We divide the proof of Theorem 5.6 into the following two lemmas.

Lemma 5.7. *Suppose that $\vec{\nu} \nleqslant \vec{\mu}$. Then the coefficient of $[\overline{Q(A(\vec{\lambda}, \vec{\nu}))}]$ in $\Psi(w(\vec{\lambda}, \vec{\mu}))$ is 0.*

Proof. By Corollary 4.7, it suffices to show that if $Q(A(\vec{\lambda}, \vec{\nu}))$ is contained in $\pi(Q(D(\vec{\lambda}, \vec{\mu})))$, then $\vec{\nu} \leqslant \vec{\mu}$.

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Let $f \in Q(D(\vec{\lambda}, \vec{\mu}))$. If q_i is the ith boundary vertex of the diskoid $D(\vec{\lambda}, \vec{\mu})$, then $f(q_i) \in \overline{\mathrm{Gr}(\mu_i)}$. On the other hand, if $\pi(f) \in Q(A(\vec{\lambda}, \vec{\nu}))$, then $f(q_i) \in \mathrm{Gr}(\nu_i)$. Thus $\nu_i \leqslant \mu_i$ for all i as desired. □

LEMMA 5.8. *The coefficient of* $[\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$ *in* $\Psi(w(\vec{\lambda}, \vec{\mu}))$ *is* 1.

Proof. Let $Z = \overline{\pi^{-1}(Q(A(\vec{\lambda}, \vec{\mu}))}$. Then Z is a component of $Q(D(\vec{\lambda}, \vec{\mu}))$, and it has dimension $d(\vec{\lambda})$ by Theorem 5.5. Recall from Theorem 4.6 that we have a homology class $c(w) \in H_{d(\vec{\lambda})}(Q(D))$ such that $\pi_*(c(w)) = \Psi(w)$. Using the notation in the proof of Theorem 4.6,

$$Q(D(\vec{\lambda}, \vec{\mu})) = \pi_{0,1}^{-1}(X_1) \cap \cdots \cap \pi_{n-1,n}^{-1}(X_n)$$

and

$$c(w) = [\pi_{0,1}^{-1}(X_1)] \cap \cdots \cap [\pi_{n-1,n}^{-1}(X_n)].$$

Since Z is a component of the expected dimension, we see that the coefficient of $[Z]$ in $c(w)$ is the length of the local ring of $Q(D(\vec{\lambda}, \vec{\mu}))$ along Z (by [Ful98, Proposition 8.2]). This length equals 1, since the next lemma shows that the scheme $\pi^{-1}(Q(A(\vec{\lambda}, \vec{\mu})))$ is isomorphic to the reduced scheme $Q(A(\vec{\lambda}, \vec{\mu}))$.

The degree of $\pi|_Z$ is 1, so $\pi_*([Z]) = [\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$. Moreover, Z is the only component of $Q(D(\vec{\lambda}, \vec{\mu}))$ which maps onto $\overline{Q(A(\vec{\lambda}, \vec{\mu}))}$, so we conclude that the coefficient of $[\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$ in $\pi_* c(w)$ is also 1, as desired. □

LEMMA 5.9. *The restriction of the map* $\pi : Q(D(\vec{\lambda}, \vec{\mu})) \to F(\vec{\lambda})$ *to* $\pi^{-1}(Q(A(\vec{\lambda}, \vec{\mu}))$ *is an isomorphism of schemes onto the reduced scheme* $Q(A(\vec{\lambda}, \vec{\mu}))$.

Proof. First, note that $Q(A(\vec{\lambda}, \vec{\mu}))$ is reduced since it is isomorphic to a iterated fibred product of varieties by the proof of Theorem 4.1.

Let $X = \pi^{-1}(Q(A(\vec{\lambda}, \vec{\mu})))$ and $Y = Q(A(\vec{\lambda}, \vec{\mu}))$. We have already shown in Theorem 5.5 that the map $\pi : X \to Y$ gives a bijection at the $\mathbb{C}$-points. Now, let S be any scheme of finite type over $\mathbb{C}$. The proof of Theorem 5.5 uses some building-theoretic arguments which don't obviously work for S-points. However, the argument in the first paragraph of the proof does work for any S, as follows. Following the notation in that paragraph, let γ be a geodesic in Γ from the base point p of $D(\vec{\lambda}, \vec{\mu})$ to the kth boundary vertex q and let $\vec{\nu}$ be the lengths along this geodesic (by definition $\sum \nu_i = \mu_k$). Let $f \in X(S)$. Then the restriction of the map $m : \mathrm{Gr}(\vec{\nu}) \to \mathrm{Gr}$ to $m^{-1}(\mathrm{Gr}(\mu_k))$ is an isomorphism of schemes and, in particular, an injection on S-points. Hence we see that $f(r)$ is determined by $f(q)$ for all r along the geodesic. Since every internal vertex of the diskoid lies on some geodesic, $f \in X(S)$ is determined by its restriction to the boundary. Thus, the map $X(S) \to Y(S)$ is injective.

So we have a map from a scheme to a smooth variety which is a bijection on $\mathbb{C}$-points and an injection on S-points. By the following lemma, the map is an isomorphism. □

LEMMA 5.10. *Let* X *and* Y *be finite-type schemes over* $\mathbb{C}$. *Assume that* Y *is reduced and normal. Let* $\phi : X \to Y$ *be a morphism which induces a bijection on* $\mathbb{C}$*-points and an injection on* S*-points for all finite-type* $\mathbb{C}$*-schemes* S. *Then* ϕ *is an isomorphism.*

Proof. Consider the maps

$$X_{\mathrm{red}} \to X \to Y.$$

The composition $X_{\mathrm{red}} \to Y$ is a bijection on $\mathbb{C}$-points and hence an isomorphism [Kum02, Theorem A.11]. This allows us to construct a map $\psi : Y \to X$ such that $\phi\psi = \mathrm{id}_Y$.

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The fact that ϕ induces an injection on S-points means that the map

$$\mathrm{Hom}_{\mathrm{Sch}}(X, X) \xrightarrow{\phi\circ} \mathrm{Hom}_{\mathrm{Sch}}(X, Y)$$

is injective. Consider what happens to id_X and $\psi\phi$ under this map: they are sent to ϕ and $\phi\psi\phi$, respectively. But since $\phi\psi = \mathrm{id}_Y$, these two elements of $\mathrm{Hom}_{\mathrm{Sch}}(X, Y)$ are equal. Hence, by the injectivity, $\mathrm{id}_X = \psi\phi$ and thus ϕ is an isomorphism. □

5.3 Consequences of the cyclic action

The goal of this section is to prove Theorem 1.4 and then derive some corollaries. The proof is based on Theorem 5.5. However, we first need to understand the cyclic action on webs and Satake fibres, i.e. the action that results from changing the base point of a polygon or diskoid.

Fix a minuscule sequence $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and consider the corresponding Satake fibre $F(\vec{\lambda})$. Also, regard the indices of the sequence $\vec{\lambda}$ as lying in $\mathbb{Z}/n$. For each $i \in \mathbb{Z}/n$, we define

$$\vec{\lambda}^{(i)} = (\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_{n-1}, \lambda_0, \lambda_1, \ldots, \lambda_i)$$

to be the ith cyclic permutation of $\vec{\lambda}$ (so that $\vec{\lambda}^{(0)} = \vec{\lambda}$).

Let Z be an irreducible component of $F(\vec{\lambda})$. Since $G(\mathscr{O})$ is connected and acts on $F(\vec{\lambda})$, we see that Z is $G(\mathscr{O})$-invariant. Define

$$Z_1 = \{([g_1^{-1} g_2], \ldots, [g_1^{-1} g_n], t^0) \mid ([g_1], \ldots, [g_n]) \in Z \subset F(\vec{\lambda}^{(1)})\},$$

and by iteration define $Z_i \subseteq F(\vec{\lambda}^{(i)})$ for all $i \in \mathbb{Z}/n$. This yields a bijection

$$\mathrm{Irr}(F(\vec{\lambda})) \cong \mathrm{Irr}(F(\vec{\lambda}^{(1)})),$$

which we call 'geometric rotation of components'. Another way to think about Z_i is to consider the unbased configuration space of $P(\vec{\lambda})$ and note that it fibres over Gr in n different ways, by choosing each of the n vertices of $P(\vec{\lambda})$ as the base point. (However, the geometry of these fibrations is subtle, because the fibres do not have to be isomorphic algebraic varieties.)

A straightforward calculation in convolution algebras (where all intersections are transverse) shows that geometric rotation matches pivotal rotation in $\mathbf{hconv}(\mathrm{Gr})$. At the same time, Theorem 1.2 tells us that the diagram

$$\begin{array}{ccc} H_{\mathrm{top}}(F(\vec{\lambda})) & \xrightarrow{\Phi} & \mathrm{Inv}(V(\vec{\lambda})) \\ \downarrow R & & \downarrow R \\ H_{\mathrm{top}}(F(\vec{\lambda}^{(1)})) & \xrightarrow{\Phi} & \mathrm{Inv}(V(\vec{\lambda}^{(1)})) \end{array} \tag{8}$$

commutes, where the invariant spaces on the right are in $\mathbf{rep}(G)^u_{\min}$.

By Theorem 4.1, $Z = \overline{Q(A(\vec{\lambda}, \vec{\mu}))}$ for some minuscule path $\vec{\mu}$ of type $\vec{\lambda}$. From $(\vec{\lambda}, \vec{\mu})$ we obtain a diskoid $D = D(\vec{\lambda}, \vec{\mu})$. In D, the distances from the base point to the other boundary vertices are given by $\vec{\mu}$. Now, for each $i \in \mathbb{Z}/n$, let $\vec{\mu}^{(i)}$ denote the sequence of distances from the ith boundary vertex to the rest of the boundary. Since a rotated CAT(0) diskoid is still a CAT(0) diskoid, we see that $D = D(\vec{\lambda}^{(i)}, \vec{\mu}^{(i)})$ as well.

Lemma 5.11. *For each i, $Z_i = \overline{Q(A(\vec{\lambda}^{(i)}, \vec{\mu}^{(i)}))}$.*

Although this lemma may look purely formal, it is (as far as we know) a non-trivial identification of two different cyclic actions. The cyclic action used to define Z_i is defined directly from the geometric Satake correspondence; it comes from the fact that the unbased configuration space of $P(\vec{\lambda})$ fibres over Gr in more than one way. The cyclic action on the right, in particular the definition of $\vec{\mu}^{(i)}$, comes instead from rotating webs. The two cyclic actions 'should be' the same because the diagram analogous to (8) for webs commutes (since $\mathbf{spd}(\mathrm{SL}(3))$ is equivalent to $\mathbf{rep}^u(\mathrm{SL}(3))$). However, the lemma is non-trivial because it is not true that the invariant vector $\Psi(w(\vec{\lambda},\vec{\mu}))$ coming from the web equals the fundamental class of the corresponding component.

Proof. Our proof uses Theorem 5.6, the unitriangularity theorem. Let M be the unitriangular change-of-basis matrix; the rows of M are labelled by the web basis, while the columns are indexed by the geometric Satake basis. Since both bases are cyclically invariant as in the diagram (8), there is a combinatorial cyclic action on the rows and columns of M that takes M to itself.

Suppose for the moment that M is an abstractly unitriangular matrix whose rows and columns are labelled by two sets A and B. In other words, there exist an unspecified bijection $A \cong B$ and a linear or partial order of A that makes M unitriangular. Then the partial order may not be unique, but the bijection is. If we choose any compatible linear order, then it is easy to see that the expansion of $\det M$ has only one non-zero term. This term selects the unique compatible bijection. Since it is unique, it intertwines the two cyclic actions in our case. □

We say that $\vec{\nu} \leqslant_S \vec{\mu}$ when $\vec{\nu}^{(i)} \leqslant \vec{\mu}^{(i)}$ for all $i \in \mathbb{Z}/n$. If D and E are the diskoids of $w(\vec{\nu})$ and $w(\vec{\mu})$, then this condition says that $d_D(p,q) \leqslant d_E(p,q)$ for every two vertices on their common boundary. Theorem 1.5 follows by combining Theorem 5.6 with Lemma 5.11.

We define a subset $U \subseteq Z$ as follows:

$$U = \{(L_i)_{i\in\mathbb{Z}/n} \in F(\vec{\lambda}) \mid d(L_i, L_j) = \mu_j^{(i)}\}.$$

Lemma 5.11 shows that U is dense in Z. The following proposition then completes the proof of Theorem 1.4.

Proposition 5.12. *Restricting the configuration to the boundary gives an isomorphism*

$$\pi : Q_g(D) \xrightarrow{\cong} U.$$

Proof. By definition, U consists of those configurations of D that preserve all distances between boundary vertices. By Lemma 5.2, these are exactly the configurations that preserve all distances in D. □

If $f \in Q_g(D)$ is a global isometry, then in particular it is an embedding of D into the affine building Δ. This has an interesting area consequence.

Lemma 5.13. *Let K be a two-dimensional simplicial complex with trivial homology, $H_*(K,\mathbb{Z}) = H_*(\mathrm{pt})$. Then every simplicial 1-cycle α in K is the homology boundary of a unique 2-chain β.*

Proof. If β_1 and β_2 are two such 2-chains, then $\beta_1 - \beta_2$ is closed and therefore null-homologous. Since K has no 3-simplices, the only way for β_1 and β_2 to be homologous is if they are equal. □

Theorem 5.14. *If a $\mathrm{CAT}(0)$, type-A_2 diskoid D is embedded in an affine building Δ, then it is the unique least-area diskoid that extends the embedding of its boundary P.*

Proof. Let f be the embedding. Then $f_*([D])$ is a 2-chain whose 1-norm is the area of D. If $f' : D' \to \Delta$ is another extension of P, then $f'_*([D']) = f_*([D])$ and the area of D' cannot be smaller than the area of D. Moreover, if they have equal area, then $f^{-1} \circ f$ is a bijection between the faces of D' and the faces of D. The faces of D' must be connected in the same way as those of D and also attached to P in the same way, because each edge in Δ has at most two faces of $f(D)$. □

By contrast, the A_2 spider relations (4) reduce the area of a diskoid. The following proposition is easy to check, as well as inevitable given Proposition 5.13 and Theorem 1.4.

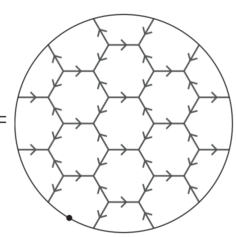
Proposition 5.15. *If w is a web with a face having two or four sides, so that the dual diskoid D has a vertex with two or four triangles, then in any configuration $f : D \to \mathrm{Gr}$ these triangles land on top of each other in pairs.*

Proposition 5.15 thus motivates the relations (4) as moves that locally remove area from a configuration f.

5.4 Web bases are not Satake

In § 5.2, we showed that the transformation between the web basis and the Satake basis is unitriangular with respect to the given order. Thus it is reasonable to ask if this transformation is the identity. As with Lustzig's dual canonical basis, there is an early agreement between the two. For any web with no internal faces, i.e. whose dual diskoid has no internal vertices, the image of the map π is $\overline{Q(A(\vec{\lambda}, \vec{\mu}))}$ by Theorem 4.3, and π is injective. It follows from Corollary 4.7 and Lemma 5.8 that $[\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$ is the web vector.

Now consider the following web $w(\vec{\mu})$, with the indicated base point.

$w(\vec{\mu}) =$ 

In [KK99], it was shown that this is the first web whose invariant vector is not dual canonical. This is the web associated with the minuscule path

$$\vec{\mu} = (0,\ \omega_1,\ \omega_1 + \omega_2,\ \omega_1 + 2\omega_2,\ 3\omega_2,\ \omega_1 + 3\omega_2,\\ 2\omega_1 + 2\omega_2,\ 3\omega_1 + \omega_2,\ 3\omega_1,\ 2\omega_1 + \omega_2,\ \omega_1 + \omega_2,\ \omega_2,\ 0)$$

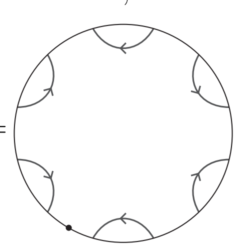
of type

$$\vec{\lambda} = (\omega_1, \omega_2, \omega_2, \omega_1, \omega_1, \omega_2, \omega_2, \omega_1, \omega_1, \omega_2, \omega_2, \omega_1).$$

Let

$$\vec{\nu} = (0, \omega_1, 0, \omega_2, 0, \omega_1, 0, \omega_2, 0, \omega_1, 0, \omega_2, 0).$$

This is another minuscule path also of type $\vec{\lambda}$; the corresponding web $w(\vec{\nu})$ is much simpler and is both a Satake vector and a dual canonical vector, as shown below.

$w(\vec{\nu}) =$ 

In [KK99], it was shown that

$$\Psi(w(\vec{\mu})) = b(\vec{\mu}) + b(\vec{\nu}),$$

where $b(\vec{\mu})$ denotes the dual canonical basis vector indexed by $\vec{\mu}$.

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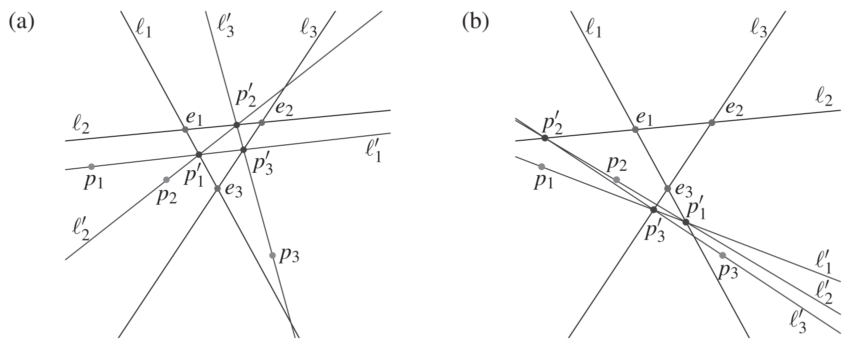
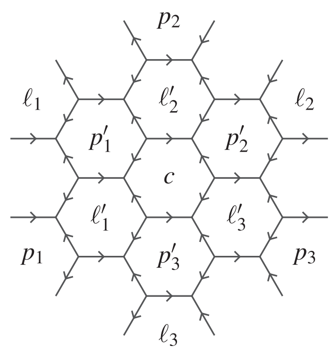


FIGURE 12. The two solutions to the problem for the given ℓ_i and p_i.

THEOREM 5.16. *Let $w(\vec{\mu})$, $\vec{\lambda}$, $\vec{\mu}$ and $\vec{\nu}$ be as above. Then the invariant vector $\overline{\Psi(w(\vec{\mu}))}$ is not in the Satake basis. More precisely, it has a coefficient of 2 for the basis vector $[\overline{Q(A(\vec{\lambda},\vec{\nu}))}]$.*

Proof. We will show that the general fibre of π over $Q(A(\vec{\lambda},\vec{\nu}))$ is of size 2. We label the faces of the web as in the following diagram.



If $f \in Q(D(\vec{\lambda},\vec{\mu}))$, then $\pi(f) \in Q(A(\vec{\lambda},\vec{\nu}))$ if and only if f assigns $p_i \in \mathrm{Gr}(\omega_1)$ and $\ell_i \in \mathrm{Gr}(\omega_2)$ on those faces and assigns $t^0 \in \mathrm{Gr}(0)$ to all empty faces. In order to determine the fibre of π over a point in $Q(A(\vec{\lambda},\vec{\nu}))$, we must calculate the possible choices for p'_i, ℓ'_i and c satisfying the appropriate conditions. Since $p_i \in \mathrm{Gr}(\omega_1)$ and $\ell_i \in \mathrm{Gr}(\omega_2)$, this forces $p'_i \in \mathrm{Gr}(\omega_1)$, $\ell'_i \in \mathrm{Gr}(\omega_2)$ and $c \in \overline{\mathrm{Gr}(\omega_1+\omega_2)}$. We can think of the points of $\mathrm{Gr}(\omega_1)$ and $\mathrm{Gr}(\omega_2)$ as being, respectively, the points and lines in $\mathbb{CP}^2$. Then the conditions given by the edges of the web are as follows: p'_i is a point on the line ℓ_i, and ℓ'_i is a line containing the points p_i, p'_{i-1} and p'_i; see Figure 12.

Suppose that either the p_i are not collinear or the ℓ_i are not concurrent. Then, by the duality of points and lines, we may assume that the ℓ_i are not concurrent. Let e_i be the intersection of ℓ_i and ℓ_{i+1}. Then we can express the points p'_i in barycentric coordinates given by e_i:

$$p'_1 = (t_1, 0, 1-t_1),$$
$$p'_2 = (1-t_2, t_2, 0),$$
$$p'_3 = (0, 1-t_3, t_3).$$

Note that by doing this, we restrict ourselves to an affine subspace of $\mathbb{P}^2$, so we may lose but we don't gain solutions. The collinearity condition results in the equations

$$p_i = (1-s_i)p'_i + s_i p_{i-1}.$$

Solving this problem amounts to solving

$$(1-s_1)t_1 = p_{11}, \quad s_1(1-t_3) = p_{12},$$
$$(1-s_2)t_2 = p_{22}, \quad s_2(1-t_1) = p_{23},$$
$$(1-s_3)t_3 = p_{33}, \quad s_3(1-t_2) = p_{31},$$

where p_{ij} are the barycentric coordinates of the p_i. If none of these coordinates are 0, then we can eliminate all but one variable to get the relation

$$t_1 = \cfrac{p_{11}}{1 - \cfrac{p_{12}}{1 - \cfrac{p_{33}}{1 - \cfrac{p_{31}}{1 - \cfrac{p_{22}}{1 - \cfrac{p_{23}}{1-t_1}}}}}}.$$

The right-hand side of this equation is a composition of fractional linear transformations that condenses to a single fractional linear transformation

$$t_1 = \frac{\alpha_{11}t_1 + \alpha_{12}}{\alpha_{21}t_1 + \alpha_{22}}$$

with generic coefficients. Thus, generically, we obtain a quadratic equation for t_1 with two solutions.

It remains to determine the face c, which lies in $\mathrm{Gr}(\omega_1 + \omega_2)$. If $c \notin \mathrm{Gr}(0)$, then the conditions given by the edges of the web would be $p'_i = p'_j$ and $\ell'_i = \ell'_j$ for all i and j, which cannot happen since either the p_i are not collinear or the ℓ_i are not concurrent. Thus, for any solution of the above equations, we get exactly one element in $Q(D(\vec{\lambda}, \vec{\mu}))$. And for any generic point $p \in Q(A(\vec{\lambda}, \vec{\nu}))$, the fibre $\pi^{-1}(p)$ has two points.

Let X denote the closure in $Q(D(\vec{\lambda}, \vec{\mu}))$ of the union of all fibres $\pi^{-1}(p)$ with two points. Then X is either a component of $Q(D(\vec{\lambda}, \vec{\mu}))$ or a union of two components. Moreover, X contains all components of $Q(D(\vec{\lambda}, \vec{\mu}))$ which map onto $Q(A(\vec{\lambda}, \vec{\nu}))$. Since the above argument shows that the scheme-theoretic fibre of π over a general point of $Q(A(\vec{\lambda}, \vec{\nu}))$ is two reduced points, we also know that X is generically reduced. Hence the coefficient of $[X]$ in the homology class $c(w)$ from Theorem 4.6 is 1. Since the map $\pi : X \to Q(A(\vec{\lambda}, \vec{\nu}))$ is of degree 2 and X contains all components mapping to $\underline{Q(A(\vec{\lambda}, \vec{\nu}))}$, the coefficient of $[Q(A(\vec{\lambda}, \vec{\nu}))]$ in $\pi_*(c(w))$ is 2. In particular, $\pi_*(c(w))$ differs from $[\overline{Q(A(\vec{\lambda}, \vec{\mu}))}]$, as desired. □

In fact, we suspect that $Q(D(\vec{\lambda}, \vec{\mu}))$ only has two components, which would imply that

$$\Psi(w(\vec{\mu})) = [\overline{Q(A(\vec{\lambda}, \vec{\mu}))}] + 2[\overline{Q(A(\vec{\lambda}, \vec{\nu}))}];$$

otherwise, $\Psi(w(\vec{\mu}))$ has these two terms and perhaps others. Either way, the coefficient of 2 is different from what arises in the dual canonical basis [KK99],

$$\Psi(w(\vec{\mu})) = b(\vec{\mu}) + b(\vec{\nu}).$$

Thus we have the following result.

Theorem 5.17. *The geometric Satake bases for invariants of $G = \mathrm{SL}(3)$ are eventually not dual canonical.*

This is not such a surprising statement in light of the well-known fact that the canonical and semicanonical bases do not coincide (as a consequence of the work of Kashiwara and

Saito [KS97]). In both Theorem 5.17 and the canonical/semicanonical situation, a homology basis does not coincide with a basis defined using a bar-involution. The analogy between these two results could perhaps be made precise using skew Howe duality (SL(3), SL(n)-duality).

It is known that $\Psi(w(\vec{\mu}))$ is the first basis web that is not dual canonical, i.e. the only basis web up to rotation with 12 or fewer minuscule tensor factors. We conjecture that it is also the first basis web for SL(3) that is not geometric Satake. Equivalently, we conjecture that all three bases first diverge at the same position.

Question 5.18. For arbitrary G, is the dual canonical basis of an invariant space $\mathrm{Inv}_G(V(\lambda))$ positive unitriangular in the geometric Satake basis?

6. Euler convolution of constructible functions

In this section, we switch from convolution in homology to convolution in constructible functions. The idea of defining convolution algebras using constructible functions is common in geometric representation theory (see, for example, [Lus03]).

More specifically, we will define a new category $\mathbf{econv}(\mathrm{Gr})_0$, which conjecturally is equivalent to $\mathbf{rep}^u_{-1}(G)_{\min}$; we will prove this conjecture for $G = \mathrm{SL}(2)$ and $G = \mathrm{SL}(3)$. When computing invariant vectors from webs, the construction is a state model as in § 4.2, where the counting is done using Euler characteristic.

6.1 Generalities on constructible functions

If X is a proper complex algebraic variety over $\mathbb{C}$ and $f : X \to \mathbb{C}$ is a constructible function, then we define the *Euler characteristic integral*

$$\int_X f \, d\chi \in \mathbb{C}$$

(see [Mac74] or [Joy06]) by linear extension starting with the characteristic functions of closed subvarieties. Specifically, if $f = f_Y$ is the characteristic function of a closed subvariety $Y \subseteq X$, then we define

$$\int_X f_Y \, d\chi \stackrel{\mathrm{def}}{=} \chi(Y).$$

If $\pi : X \to Y$ is a proper morphism between algebraic varieties and $f : X \to \mathbb{C}$ is a constructible function on X, then we define the pushforward of f under π by integration along fibres:

$$(\pi_* f)(p) \stackrel{\mathrm{def}}{=} \int_{\pi^{-1}(p)} f \, d\chi.$$

If $\mathbb{C}_c(X)$ denotes the vector space of constructible functions on X, this pushforward is then a linear map

$$\pi_* : \mathbb{C}_c(X) \to \mathbb{C}_c(Y).$$

The following result is well known; see, for example, [Mac74, Proposition 1] and [Joy06, Theorem 3.8].

Theorem 6.1. *The Euler characteristic integral pushforward of constructible functions is a well-defined covariant functor from the category of proper morphisms between algebraic varieties over $\mathbb{C}$ to the category of complex vector spaces.*

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6.2 Construction of the categories

Given G simple and simply connected as before, we can define a pivotal category $\mathbf{econv}(\mathrm{Gr})$ in a similar fashion to $\mathbf{hconv}(\mathrm{Gr})$, except that we will replace homology with constructible functions throughout.

The objects of $\mathbf{econv}(\mathrm{Gr})$ are the $\mathrm{Gr}(\vec{\lambda})$ where $\vec{\lambda}$ is a sequence of minuscule weights. As in $\mathbf{hconv}(\mathrm{Gr})$, the tensor product is defined by convolution.

We define the invariant space of $\mathrm{Gr}(\vec{\lambda})$ to be the vector space of constructible functions on the Satake fibre:

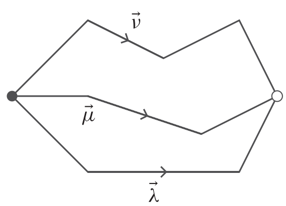
$$\mathrm{Inv}_{\mathbf{econv}(\mathrm{Gr})}(\mathrm{Gr}(\vec{\lambda})) \overset{\mathrm{def}}{=} \mathbb{C}_c(F(\vec{\lambda})).$$

The hom spaces are defined in an equivalent way:

$$\mathrm{Hom}_{\mathbf{econv}(\mathrm{Gr})}(\mathrm{Gr}(\vec{\lambda}), \mathrm{Gr}(\vec{\mu})) \overset{\mathrm{def}}{=} \mathbb{C}_c(Z(\vec{\lambda}, \vec{\mu})).$$

We define the convolution of two hom spaces by convolution, as in $\mathbf{hconv}(\mathrm{Gr})$. We could proceed exactly as in $\mathbf{hconv}(\mathrm{Gr})$, but the 'local' nature of constructible functions permits a simpler definition.

Fix three minuscule sequences $\vec{\lambda}, \vec{\mu}$ and $\vec{\nu}$. Let Γ be a graph homeomorphic to a theta (θ) with three arcs that are polylines of type $\vec{\lambda}$, $\vec{\mu}$ and $\vec{\nu}$ with a common base point.

$\Gamma =$ 

Then there are projections

$$\pi_{\vec{\lambda},\vec{\mu}} : Q(\Gamma) \to Z(\vec{\lambda}, \vec{\mu}),$$
$$\pi_{\vec{\lambda},\vec{\nu}} : Q(\Gamma) \to Z(\vec{\lambda}, \vec{\nu}),$$
$$\pi_{\vec{\mu},\vec{\nu}} : Q(\Gamma) \to Z(\vec{\mu}, \vec{\nu}).$$

Given

$$f \in \mathrm{Hom}_{\mathbf{econv}(\mathrm{Gr})}(\mathrm{Gr}(\vec{\lambda}), \mathrm{Gr}(\vec{\mu})),$$
$$g \in \mathrm{Hom}_{\mathbf{econv}(\mathrm{Gr})}(\mathrm{Gr}(\vec{\mu}), \mathrm{Gr}(\vec{\nu})),$$

we can define their composition by Euler characteristic integration over configurations of the middle polyline $L(\vec{\mu})$, using the fact that constructible functions pull back and multiply as well as push forward:

$$g \circ f \overset{\mathrm{def}}{=} (\pi_{\vec{\lambda},\vec{\nu}})_*(\pi^*_{\vec{\lambda},\vec{\nu}}(f)\pi^*_{\vec{\mu},\vec{\nu}}(g)).$$

It is routine to check that these structures define a pivotal category.

The hom spaces in the category $\mathbf{econv}(\mathrm{Gr})$ are too large for our purposes. We will restrict them by just looking at those constructible functions generated by the constant functions on the Satake fibres corresponding to trivalent vertices. More precisely, define a pivotal functor

$$E : \mathbf{fsp}(G) \to \mathbf{econv}(\mathrm{Gr})$$

which takes the generating vertex in

$$\mathrm{Inv}_{\mathbf{fsp}(G)}(\lambda, \mu, \nu)$$

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to the identity function on $F(\lambda,\mu,\nu)$. Again, λ, μ and ν are all minuscule, and we are assuming that there is a vertex, so

$$\mathrm{Inv}_G(V(\lambda,\mu,\nu)) \neq 0.$$

Let $\mathbf{econv}(\mathrm{Gr})_0$ denote the image of the functor E; it has the same objects as $\mathbf{econv}(\mathrm{Gr})$ but smaller hom spaces.

6.3 Equivalence with the representation category

Before stating the main conjecture and result, we can describe more explicitly how the functor E expresses an Euler characteristic state model. The following result can be seen by chasing through the definitions.

Proposition 6.2. *Given a web $w \in \mathbf{fsp}(G)$ with boundary $\vec{\lambda}$ and dual diskoid D, $E(w)$ is the function on the Satake fibre $F(\vec{\lambda})$ whose value at $p \in F(\vec{\lambda})$ is $\chi(\pi^{-1}(p))$. (Here $\pi : Q(D) \to F(\vec{\lambda})$ is the map which restricts a diskoid configuration to its boundary.)*

So we are indeed producing a function which counts (using Euler characteristic) ways to extend the boundary configuration to a diskoid configuration.

We are now ready to formulate our alternate version of the geometric Satake correspondence.

Conjecture 6.3. There is an equivalence of pivotal categories

$$\mathbf{econv}(\mathrm{Gr}(G^\vee))_0 \cong \mathbf{rep}^u_{-1}(G)_{\min}.$$

Recall from §§ 2.2 and 2.3 that $\mathbf{rep}^u_{-1}(G)_{\min}$ and $\mathbf{rep}^u(G)_{\min}$ have the same information except for a sign correction. We offer the following corollary of Conjecture 6.3 as a stand-alone conjecture.

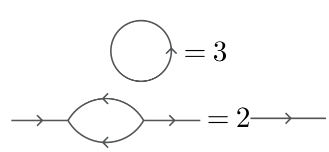
Conjecture 6.4. Let w be any closed web with dual diskoid D. Then

$$\Psi(w) = \pm\chi(Q(D)).$$

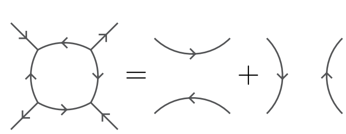
Here $\Psi(w)$ denotes the value of w in the pivotal category $\mathbf{rep}^u(G)_{\min}$, and the sign comes as a result of the sign correction between $\mathbf{rep}^u_{-1}(G)_{\min}$ and $\mathbf{rep}^u(G)_{\min}$.

Theorem 6.5. *Conjecture 6.3 holds when $G = \mathrm{SL}(2)$ and $G = \mathrm{SL}(3)$.*

Proof. We will first argue the more difficult case of $G = \mathrm{SL}(3)$. We argue by checking the skein relations of $\mathbf{spd}_{-1}(\mathrm{SL}(3))$. The first two skein relations



are straightforward, because the relevant fibres are always $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively. The third skein relation



is a bit more work. The diskoid dual on the left consists of four triangles. The configuration space of the quadrilateral $P(\omega_1,\omega_2,\omega_1,\omega_2)$ has two components, corresponding to the two ways

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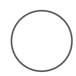
of collapsing the quadrilateral to two edges. In each case, there is a unique extension to the diskoid in which the diskoid collapses to two triangles. The remaining case that should be checked is the intersection of the two components in which the quadrilateral collapses to a single edge. In this case the fibre is $\mathbb{P}^1$, because there is a $\mathbb{P}^1$ of ways to extend the edge to a triangle, and the diskoid can collapse onto this triangle. Thus the local Euler characteristic at the intersection is 2, which matches the sum on the right-hand side of the skein relation.

Thus, either the image of E is equivalent to $\mathbf{rep}^u_{-1}(\mathrm{SL}(3))_{\min}$, or it is a quotient. However, $\mathbf{rep}^u_{-1}(\mathrm{SL}(3))_{\min}$ is simple as a linear-additive, pivotal category, because the pairing of dual invariant spaces is non-degenerate. (Alternatively, Theorem 1.5 also implies that basis webs are linearly independent after applying E, because of unitriangularity.) Therefore the image of E is equivalent to $\mathbf{rep}^u_{-1}(\mathrm{SL}(3))_{\min}$ itself.

In the case where $G = \mathrm{SL}(2)$, we only need to check the following skein relation (3) with $q = -1$:

 $= 2$

In this case the diskoid on the left is a based edge, the diskoid on the right is a point, the fibre is $\mathbb{P}^1$, and its Euler characteristic is 2 as desired. □

It should also be possible to prove Conjecture 6.3 when $G = \mathrm{SL}(m)$. The idea is to use the geometric skew Howe duality of Mirković and Vybornov [MV07b] and the ideas in [Kam11, § 6] to express this conjecture in terms of constructible functions on quiver varieties for the Howe dual $\mathrm{SL}(n)$. Then we are in a position to apply Nakajima's work from [Nak94, § 10]. Note that this approach does not make use of the geometric Satake correspondence.

6.4 Relationship with homological convolutions

A constructible function is constant on a dense open subset of any irreducible variety. If X is an irreducible variety, we write $f(X)$ for the value of f on this dense open subset.

We can define a non-functor Ξ from $\mathbf{econv}(\mathrm{Gr})$ to $\mathbf{hconv}(\mathrm{Gr})$ as follows. On objects, Ξ is the identity, while on morphisms we define

$$\Xi : \mathbb{C}_c(Z(\vec{\lambda}, \vec{\mu})) \longrightarrow H_{\mathrm{top}}(Z(\vec{\lambda}, \vec{\mu}))$$

by the formula

$$\Xi : f \mapsto \sum_{X \in \mathrm{Irr}(Z(\vec{\lambda}, \vec{\mu}))} f(X)[X].$$

The map Ξ is not a functor because it does not respect convolution (as some simple examples will show). However, we offer the following tentative conjecture.

Conjecture 6.6. The map Ξ between hom spaces restricts to an equivalence of pivotal categories from $\mathbf{econv}(\mathrm{Gr})_0$ to $\mathbf{hconv}(\mathrm{Gr})$ up to a sign correction of the tensor and pivotal structures.

This conjecture implies Conjecture 6.3, because the conjectured equivalence is compatible with the functors from $\mathbf{fsp}(G)$. Conjecture 6.6 would also imply the following simple formula for expansion of the invariant vectors coming from webs in the Satake basis, which generalizes Conjecture 6.4.

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Conjecture 6.7 (Corollary of Conjecture 6.6). Let w be a minuscule web with boundary $\vec{\lambda}$ and dual diskoid D. Then we can expand $\Psi(w)$ in the Satake basis as

$$\Psi(w) = \pm \sum_{X \in \mathrm{Irr}(F(\vec{\lambda}))} \chi(\pi^{-1}(x))[X],$$

where x is a generic point of each X and $\pi : Q(D) \to F(\vec{\lambda})$ is the restriction map from a diskoid configuration to its boundary.

As partial evidence for Conjecture 6.6, we note that a similar statement has been conjectured in the quiver variety setting.

7. Future work

This article is hopefully only the beginning of an investigation into configuration spaces of diskoids and their relations to presented pivotal categories, or spiders.

7.1 Basis webs for $\mathrm{SL}(n)$

In future work, the first author will establish the following generalization of Theorem 2.3 and Theorem 1.5 to $\mathrm{SL}(n)$.

Theorem 7.1. *Given a sequence of minuscule weights $\vec{\lambda}$ of $\mathrm{SL}(n)$, there is a map $w(\vec{\mu})$ from the minuscule path $\vec{\mu}$ of type $\vec{\lambda}$ to webs. The image of these webs in $\mathrm{Inv}(V(\vec{\lambda}))$ forms a basis, and the change of basis to the Satake basis is upper unitriangular with respect to the partial order on minuscule paths.*

The geometric results of the current article are used to establish that the webs $w(\vec{\mu})$ form a basis, and, as far as we are aware, no elementary proof is available. This is in sharp contrast to the $\mathrm{SL}(3)$ case, where the basis webs were originally established by elementary means. The webs $w(\vec{\mu})$ themselves are constructed combinatorially using the idea of Westbury triangles [Wes08, Wes12]. Recently, Westbury has combinatorially obtained Theorem 7.1 for the case of a tensor product of standard representations and their duals.

Kim conjectured for $n = 4$ (see [Kim03]), and Morrison for general n (see [Mor07]), a set of generating relations for the kernel of $\mathbf{fsp}(\mathrm{SL}(n)) \to \mathbf{rep}(\mathrm{SL}(n))_{\min}$. Using Theorem 7.1, we hope to establish Kim's and Morrison's conjectures.

7.2 Other rank-2 groups

Since there are established definitions of spiders for B_2 and G_2, it seems quite possible that the results in this paper could be generalized to these two cases, but there are two important problems to resolve. First, the vertex set of the corresponding affine buildings are no longer simply the points of the affine Grassmannian. Second, since we want to study $\mathbf{rep}(G)$ rather than $\mathbf{rep}(G)_{\min}$, it is necessary to look at webs labelled not just by minuscule weights but by fundamental weights. When G is not $\mathrm{SL}(n)$, it is no longer the case that all fundamental weights are minuscule; thus the results of this paper would need to be extended to cover this case.

7.3 Other discrete valuation rings

Our results in this article apply only to the affine Grassmannians of the discrete valuation ring $\mathscr{O} = \mathbb{C}[[t]]$. In fact, the affine Grassmannian Gr exists (as a set), and the Bruhat–Tits building Δ exists and is CAT(0) for any complete discrete valuation ring $\mathscr{O}$. It is a well-known open problem to state and prove a geometric Satake correspondence in this setting; it is only known in the equal-characteristic case $\mathscr{O} = k[[t]]$ for a field k. Since the building geometry is so similar for all choices of $\mathscr{O}$, our results could be interpreted as (further) evidence that a geometric Satake correspondence exists for all $\mathscr{O}$.

7.4 Webs in surfaces

Another possible generalization is from webs in disks to webs in surfaces. If Σ is a closed surface and G has rank 1 or 2, there is an analogous basis of non-elliptic webs on Σ (see [SW07]), which are equivalent to CAT(0) triangulations. (Alternatively, Σ can have boundary circles with marked points, but the closed case is especially interesting.) This web basis is a basis of the skein module of $\Sigma \times [0,1]$, which is also the coordinate ring of the variety of representations $\pi_1(\Sigma) \to G$. Our results suggest an interpretation of this coordinate ring in terms of certain simplicial maps from the universal cover of Σ to the affine building Δ. This should be related to the conjectures of Fock and Goncharov in [FG06].

7.5 Categorification

We would also like to apply our results to categorification and knot homology. According to the philosophy of [CK08], to each web w with dual diskoid D and boundary $\vec{\lambda}$ we could associate an object $A(w)$ in the derived category of coherent sheaves on $\mathrm{Gr}(\vec{\lambda})$. When the configuration space $Q(D)$ has the expected dimension, $A(w)$ should be the pushforward $\pi_*(\mathscr{O}_{Q(D)})$ of the structure sheaf of $Q(D)$. It would also be nice to understand foams (as introduced by Khovanov [Kho04b]) in this language. In particular, it would be interesting to consider the configuration spaces of duals of foams. Some ideas in this direction have been pursued by Frohman.

7.6 Quantum groups

Finally, developing a q-analogue of our theory is also an open problem. As mentioned earlier, there is a functor from the free spider $\mathbf{fsp}(G)$ to $\mathbf{rep}_q(G)$, the representation category of the quantum group for any q that is not a root of unity. However, our geometric Satake machinery only applies in the case where $q = 1$. Hopefully, we can extend to general q using the quantum geometric Satake developed by Gaitsgory (see [Gai08]).

Acknowledgements

The authors would like to thank Dave Anderson, Charles Frohman, Dennis Gaitsgory, André Henriques, Misha Kapovich, Anthony Licata, John Millson, Scott Morrison, Hiraku Nakajima, Alistair Savage, Petra Schwer, and Juliana Tymoczko for useful discussions.

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