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AN ULTRAFILTER COMPLETION OF A NEARNESS SPACE

BY

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ABSTRACT: An ultrafilter completion is constructed for a nearness space. It is shown to preserve the T_1 separation axiom. Characterizing conditions are given for it to be topological or for its topology to be compact. It is shown to have the simple extension topology and for a given Hausdorff space a compatible nearness structure is found for which its ultrafilter completion is homeomorphic to the Katetov H-closed extension.

One of the most unifying concepts to surface in topology in recent years is the concept of a nearness space provided by Herrlich [7]. The categories of symmetric topological space, uniform spaces, proximity spaces, and contiguity spaces are all embedded in the category NEAR, of nearness spaces and nearness maps.

Nearness spaces have had an impact on the study of extensions of a topological space: see for example [2], [3] and [12]. The study of topology from the categorical viewpoint has been enhanced by the advent of nearness spaces as demonstrated in [4], [8], [9] and [12].

Herrlich defines a nearness space to be complete if every maximal near collection has an adherence point. He constructs a completion for a nearness space in [7]. Two of the most powerful theorems to be developed in this arena deal directly with his completion. The first appears in [8], where Herrlich shows how the Smirnov compactification, the Samuel compactification, the Wallman compactification, the Čech-Stone compactification, and the Hewitt realcompactification may be constructed as a nearness space completion of the original space equipped with certain specified compatible nearness structures. The second theorem appears in Bentley and Herrlich [3], where they show that every strict T_1 extension, up to the usual equivalence, of a T_1 topological space may be generated as a completion of the original space, equipped with a compatible concrete nearness structure.

In order to characterize various topological properties of the underlying

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topology of a nearness space it became apparent that it was quite useful to study the behavior of filters and ultrafilters that were also near collections. From this a second notion of completeness was developed; namely that every near ultrafilter had an adherent point. In [5] this was called B-complete. It seems appropriate now to call such a space ultrafilter complete; restating the result in [5] we have that the underlying topology of a nearness space is compact if and only if the nearness space is ultrafilter complete and totally bounded.

It is natural that one should attempt a completion of a nearness space with respect to this second notion of completeness. Attempting to mimic Herrlich's completion by using ultrafilters rather than clusters fails; one actually constructs a semi-nearness space instead of a nearness space, but this is another story.

In this paper an ultrafilter completion for a nearness space is constructed. Various properties of this space are studied; it is shown that it preserves the T_1 separation axiom. Those nearness spaces whose ultrafilter completions are topological are characterized as well as those whose ultrafilter completion have a compact topology. For a Hausdorff topological space a compatible nearness structure is isolated such that its ultrafilter completion is homeomorphic to the Katětov H-closed extension.

2. Preliminaries

Let X be a set; then $\mathcal{P}^n(X)$ will denote the power set of $\mathcal{P}^{n-1}(X)$ for each natural number n and $\mathcal{P}^0(X)=X$. Let ξ be a subset of $\mathcal{P}^2(X)$ and $\mathcal{A}$ and $\mathcal{B}$ subsets of $\mathcal{P}(X)$. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of X. Then the following notation is used:

(1) "$\mathcal{A}$ is near" or $\xi\mathcal{A}$ means $\mathcal{A}\in\xi$; and "$\mathcal{A}$ is far" or $\bar{\xi}\mathcal{A}$ means $\mathcal{A}\notin\xi$.

(2) $A\xi B$ means $\{A, B\}\in\xi$.

(3) $\mathrm{cl}_\xi A=\{x\in X:\{\{x\}, A\}\in\xi\}$.

(4) $\mathcal{A}\vee\mathcal{B}=\{A\cup B: A\in\mathcal{A}, B\in\mathcal{B}\}$.

(5) $\mathcal{A}$ corefines $\mathcal{B}$ means that for each $A\in\mathcal{A}$ there exists a $B\in\mathcal{B}$ such that $B\subset A$.

Definition 2.1. Let X be a set and $\xi\subset\mathcal{P}^2(X)$. Then (X,ξ) is called a nearness space provided:

(N1) $\cap\mathcal{A}\neq\phi$ implies $\mathcal{A}\in\xi$.

(N2) If $\mathcal{A}\in\xi$ and for each $B\in\mathcal{B}$ there exists $A\in\mathcal{A}$ with $A\subset\mathrm{cl}_\xi B$, then $\mathcal{B}\in\xi$.

(N3) If $\mathcal{A}\notin\xi$ and $\mathcal{B}\notin\xi$ then $\mathcal{A}\vee\mathcal{B}\notin\xi$.

(N4) $\phi\in\mathcal{A}$ implies $\mathcal{A}\notin\xi$.

A nearness space is called a N1-space provided:

(N5) $\{x\}\xi\{y\}$ implies $x=y$.

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Given a nearness space (X, ξ), the operator cl_ξ is a closure operator on X. Hence there exists a topology associated with each nearness space in a natural way. This topology is denoted by $t(\xi)$. This topology is symmetric. (Recall that a topology is symmetric provided $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$.) Conversely, given any symmetric topological space (X, t) there exists a compatible nearness structure ξ_t given by $\xi_t = \{\mathcal{A} \subset \mathcal{P}(X) : \cap \bar{\mathcal{A}} \neq \phi\}$. To say that a nearness structure ξ is compatible with a topology t on a set X means that $t = t(\xi)$.

DEFINITION 2.2. Let (X, ξ) be a nearness space.

(1) (X, ξ) is called topological provided $\mathcal{A} \in \xi$ implies $\cap \bar{\mathcal{A}} \neq \phi$.

(2) (X, ξ) is called totally bounded provided $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{A}$ has the finite intersection property implies $\mathcal{A} \in \xi$.

(3) (X, ξ) is called ultrafilter complete if each near ultrafilter converges. The following result, [5], is stated here and referred to later.

THEOREM A. *Let (X, ξ) be a nearness space.*

(1) *The underlying topology is compact if and only if ξ is ultrafilter complete and totally bounded.*

Let (Y, t) be a topological space and $\bar{X} = Y$. $t(X)$ will denote the subspace topology on X. For each $y \in Y$, set $\mathcal{O}y = \{O \cap X : y \in O \in t\}$. Then $\{\mathcal{O}y : y \in Y\}$ is called the filter trace of y on X.

Let t(strict) be the topology on Y generated by the base $\{O^* : O \in t(X)\}$ where $O^* = \{y \in Y : O \in \mathcal{O}y\}$. Let t(simple) be the topology on Y generated by the base $\{O \cup \{y\} : O \in \mathcal{O}y, y \in Y\}$. Then t(strict) and t(simple) are such that Y with either of these topologies is an extension of $(X, t(X))$, called a strict extension, or simple extension of X, respectively. Note that

$$t(\text{strict}) \leq t \leq t(\text{simple}).$$

Moreover; a topology s on Y with the same filter traces as t, forms an extension of $(X, t(X))$ if and only if it satisfies the above inequality. (See Banaschewski [1].)

In a nearness space (X, ξ), a nonempty collection of subsets of X is called an X-cluster if it is maximal in ξ with respect to inclusion. The nearness space is called complete if every X-cluster has a non-empty adherence.

Herrlich's completion of a nearness space was presented in [7]. A brief description of it appears in [3] which we provide here for the convenience of the reader. Let (X, ξ) be a nearness space and let Y be the set of all X-clusters $\mathcal{A}$ with empty adherence. Set $X^* = X \cup Y$. For each $A \subset X$, define $\mathrm{cl}\, A = \{y \in Y : A \in y\} \cup \mathrm{cl}_\xi A$. A nearness structure ξ^* is defined on X^* as follows: $\mathcal{B} \in \xi^*$ provided $\mathcal{A} = \{A \subset X$: there exists $B \in \mathcal{B}$ with $B \subset \mathrm{cl}\, A\} \in \xi$. (X^*, ξ^*) is a complete nearness space with $\mathrm{cl}_{\xi^*} X = X^*$. Also, for $A \subset X$, $\mathrm{cl}_{\xi^*} A = \mathrm{cl}\, A$.

The following important theorem is due to Herrlich and Bentley [3].

THEOREM B. *For any T_1 nearness space (X, ξ) the following conditions are equivalent:*

(1) *ξ is a nearness structure induced on X by a strict extension.*

(2) *The completion (X^*, ξ^*) of (X, ξ) is topological.*

(3) *Every non-empty X-near collection is contained in some X-cluster.*

A nearness space satisfying the above equivalent conditions is called concrete.

If a nearness space is concrete then its completion is topological and consequently ultrafilter complete. This is not the case in general; for a complete nearness space is its own completion, and there exists nearness spaces that are complete but not ultrafilter complete (space X of Example 3 in [3]). Also, there exists nearness spaces that are ultrafilter complete but not complete (Example 2.5 in [5]). The ultrafilter completion constructed in this paper is quite distinct in general from the completion constructed by Herrlich; for example the underlying topology of the completion of a nearness space is a strict extension of the original space while the ultrafilter completion constructed here has the simple extension topology.

3. An ultrafilter completion

Let (X, ξ) be a nearness space. For any $A \subset X$, $\bar{A}$ will denote the closure of A in X, even when X is embedded in a larger space. For any ultrafilter $\mathcal{F}$ on X let:

$$\mathcal{O}(\mathcal{F}) = \{O : O \in \mathcal{F} \text{ and } O \text{ is open in } X\},$$

$$\mathcal{G}(\mathcal{F}) = \{A : \bar{A} \in \mathcal{F}\}.$$

LEMMA 3.1. *Let (X, ξ) be a nearness space. Let $\mathcal{F}$ and $\mathcal{H}$ be free ultrafilters of X. Then $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\mathcal{H})$ if and only if $\mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{H})$.*

The following lemma provides a general method for extending certain types of nearness structures to a larger space; we use this technique to construct an ultrafilter completion.

LEMMA 3.2. *Let X be a subspace of a symmetric space Z and ξ a compatible nearness structure on X such that if $\mathcal{A} \subset \mathcal{P}(X)$ and $\bigcap \mathrm{cl}_Z \mathcal{A} \neq \phi$ then $\mathcal{A} \in \xi$. Set*

$$\eta = \{\mathcal{A} \subset \mathcal{P}(Z) : \bigcap \mathrm{cl}_Z \mathcal{A} \neq \phi \quad \text{or} \quad \{A \cap X : A \in \mathcal{A}\} \in \xi\}.$$

Then:

(1) *η is a compatible nearness structure on Z that extends ξ, and*

(2) *If (Y, δ) is a nearness space and $f : Z \to Y$ is a continuous mapping such that $(f \mid X) : (X, \xi) \to (Y, \delta)$ is a nearness map then $f : (Z, \eta) \to (Y, \delta)$ is a nearness map.*

Proof. (1). (N1) and (N4) are obvious and (N3) follows from easy calculations. Let $A \subset Z$. Claim: $\mathrm{cl}_\eta A = \mathrm{cl}_Z A$. Easily $\mathrm{cl}_Z A \subset \mathrm{cl}_\eta A$. Let $t \in \mathrm{cl}_\eta A$. Then

$\{\{t\}, A\} \in \eta$. Either $\mathrm{cl}_Z\{t\} \cap \mathrm{cl}_Z A \neq \phi$; in which case $t \in \mathrm{cl}_Z A$ since A is symmetric, or $\{\{t\} \cap X, A \cap X\} \in \xi$ in which case $t \in \mathrm{cl}_X (A \cap X) \subset \mathrm{cl}_Z A$. Thus $\mathrm{cl}_Z A = \mathrm{cl}_\eta A$.

To see that (N2) holds let $\mathscr{A} \in \eta$ and for each $B \in \mathscr{B}$ there exists $A \in \mathscr{A}$ such that $A \subset \mathrm{cl}_\eta B$. If $\cap \mathrm{cl}_Z \mathscr{A} \neq \phi$ then $\cap \mathrm{cl}_Z \mathscr{B} = \cap \mathrm{cl}_\eta \mathscr{B} \neq \phi$ and thus $\mathscr{B} \in \eta$. On the other hand, suppose $\{A \cap X : A \in \mathscr{A}\} \in \xi$. Since ξ is a nearness structure on X it follows that $\{\mathrm{cl}_Z B \cap X : B \in \mathscr{B}\} \in \xi$. Thus $\{\mathrm{cl}_X (B \cap X) : B \in \mathscr{B}\} \in \xi$ and thus $\{B \cap X : B \in \mathscr{B}\} \in \xi$. Hence $\mathscr{B} \in \eta$.

Easily $\xi = \mathscr{P}(X) \cap \eta$ and hence η extends ξ.

(2). Let $\mathscr{A} \in \eta$. If $\cap \mathrm{cl}_Z \mathscr{A} \neq \phi$ then $\cap \mathrm{cl}_T f(\mathscr{A}) \neq \phi$ since f is continuous and thus $f(\mathscr{A}) \in \delta$. Otherwise; $\{A \cap X : A \in \mathscr{A}\} \in \xi$ and $\{(f \mid X)(A \cap X) : A \in \mathscr{A}\} \in \delta$ since $f \mid X$ is a nearness map. For $A \in \mathscr{A}$; $(f \mid X)(A \cap X) = f(A \cap X) \subset f(A)$. Therefore $f(\mathscr{A}) \in \delta$ and f is a nearness map.

Definition 3.1. Let (X, ξ) be a nearness space and $Y = \{\mathcal{O}(\mathscr{F}) : \mathscr{F}$ a free near ultrafilter on $X\}$. Set $X' = X \cup Y$. Let t' be the simple extension topology on X' generated by the trace filters $\mathcal{O}(\mathscr{F})$ for $\mathcal{O}(\mathscr{F})$ in Y and the open neighborhood filters for points in X.

Lemma 3.3. *Let (X, ξ) be a nearness space. Then:*

(1) *(X', t') is a symmetric topological space.*

(2) *Let $A \subset X'$, then $\mathrm{cl}_{X'} A = \mathrm{cl}_X (A \cap X) \cup \{\mathcal{O}(\mathscr{F}) : A \cap X \in \mathscr{G}(\mathscr{F})\}$.*

(3) *If $\mathscr{A} \subset \mathscr{P}(X)$ and $\cap \mathrm{cl}_{X'} \mathscr{A} \neq \phi$ then $\mathscr{A} \in \xi$.*

Proof. (1). X is symmetric and open in X'. If $\mathcal{O}(\mathscr{F})$ and $\mathcal{O}(\mathscr{G})$ belong to Y and $\mathcal{O}(\mathscr{F}) \neq \mathcal{O}(\mathscr{G})$ then $\mathcal{O}(\mathscr{F}) \notin \mathrm{cl}_{X'}(\mathcal{O}(\mathscr{G}))$. Let $\mathcal{O}(\mathscr{F}) \in Y$ and $x \in X$. Since X is open in X', $x \notin \mathrm{cl}_{X'}(\mathcal{O}(\mathscr{F}))$. Since $\mathscr{F}$ is a free ultrafilter there exists an open set O containing x such that $O \notin \mathscr{F}$. Since X is symmetric, $\mathrm{cl}_X\{x\} \subset 0$. Now $X - \mathrm{cl}_X\{x\} \in \mathscr{F}$ and hence $\mathcal{O}(\mathscr{F}) \notin \mathrm{cl}_{X'}\{x\}$. Thus X' is symmetric.

The proofs of (2) and (3) are straightforward and thus omitted.

Definition 3.2. Let (X, ξ) be a nearness space and (X', t') be as defined in Definition 3.1. Let ξ' be the nearness structure that extends ξ as constructed in Lemma 3.2.

Theorem 3.4. *Let (X, ξ) be a nearness space. Then (X', ξ') is an ultrafilter completion of (X, ξ).*

Proof. By Lemmas 3.2 and 3.3, we have that (X', ξ') is a nearness space and $\xi' \cap \mathscr{P}(X) = \xi$. Easily X is dense in X'; and X is open in X' since X' has the simple extension topology. We now show that (X', ξ') is ultrafilter complete.

Let $\mathscr{F}'$ be a near ultrafilter on X'. Then $\mathscr{F}' \in \xi'$. Suppose $\cap \mathrm{cl}_{X'} \mathscr{F}' = \phi$, then $\mathscr{F} = \{F \cap X : F \in \mathscr{F}'\} \in \xi$, and $\cap \bar{\bar{\mathscr{F}}} = \phi$. Since $\mathscr{F} \in \xi$, $\phi \notin \mathscr{F}$. Let $F \cap X \in \mathscr{F}$ and $G \cap X \in \mathscr{F}$ where F and G belong to $\mathscr{F}'$; then $F \cap G \in \mathscr{F}'$ and thus $(F \cap X) \cap (G \cap X) = (F \cap G) \cap X \in \mathscr{F}$. Suppose $X \supset A \supset (F \cap X)$ for some $F \in \mathscr{F}'$. Then $A \cup (X' - X) \supset F \in \mathscr{F}'$ and hence $A \in \mathscr{F}$. Suppose $A \cup B \in \mathscr{F}$ then

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$A \cup B \supset X \cap F$ for some $F \in \mathcal{F}'$. Let $A' = (X' - X) \cup A$ and $B' = (X' - X) \cup B$. Then $A' \cup B' \supset F \in \mathcal{F}'$ and since $\mathcal{F}'$ is an ultrafilter either A' or B' belongs to $\mathcal{F}'$, thus either A or B belongs to $\mathcal{F}$. Hence $\mathcal{F}$ is a nonconvergent near ultrafilter on X and thus $\mathcal{O}(\mathcal{F}) \in X'$. Let O' be an open set in X' containing $\mathcal{O}(\mathcal{F})$. Then there exists $O \in \mathcal{O}(\mathcal{F})$ with $O \cup \{\mathcal{O}(\mathcal{F})\} \subset O'$. Now $O \in \mathcal{F}$ and there exists $F' \in \mathcal{F}'$ such that $O = F' \cap X$. Let $Q = F' \cap O'$ and $P = F' - O'$. Then $F' = P \cup Q \in \mathcal{F}'$. Since $\mathcal{F}'$ is an ultrafilter either P or Q belongs to $\mathcal{F}'$. Suppose $P \in \mathcal{F}'$. Then $P \cap X \in \mathcal{F}$ and thus $X - O \in F$ which is impossible. Hence $Q = F' \cap O' \in \mathcal{F}'$. Thus $O' \in \mathcal{F}'$ and $\mathcal{F}'$ converges to $\mathcal{O}(\mathcal{F})$ and we have a contradiction. Therefore, (X', ξ') is ultrafilter complete.

4. Properties of X'

(X', ξ') will always denote the ultrafilter completion of the nearness space (X, ξ) constructed in the previous section. A nearness space that is a subspace of a topological nearness space is called subtopological. In Bentley and Herrlich [3] it is shown that a nearness space is subtopological if and only if each near collection is contained in a near grill. They also show that the completion (X^*, ξ^*) of a nearness space is topological if and only if each near collection is contained in a maximal near collection. (See Section 2, Theorem B.) The following theorem characterizes the nearness spaces whose ultrafilter completion (X', ξ') is topological.

THEOREM 4.1. *Let (X, ξ) be a nearness space. The following are equivalent.*
(1). *(X', ξ') is topological.*
(2). *For each $\mathcal{A} \in \xi$ there exists a near ultrafilter $\mathcal{F}$ such that $\mathcal{A} \subset \mathcal{G}(\mathcal{F})$.*

Proof. (Note: (2) does not require the ultrafilter $\mathcal{F}$ to be free; also each $\mathcal{G}(\mathcal{F})$ is a special type of grill.) (1) implies (2). Let $\mathcal{A} \in \xi$. Then $\mathcal{A} \in \xi'$ and $\cap \operatorname{cl}_{X'} \mathcal{A} \neq \phi$ since (X', ξ') is topological. Then either there exists an $x \in X$ with $x \in \bar{A}$ for each $A \in \mathcal{A}$, in which case $\mathcal{F}_x = \{F \subset X : x \in F\}$ is a near ultrafilter and $\mathcal{A} \subset \mathcal{G}(\mathcal{F}_x)$; or there exists $\mathcal{O}(\mathcal{F}) \in X'$ with $A \in \mathcal{G}(\mathcal{F})$ for each $A \in \mathcal{A}$. Thus $\mathcal{A} \subset \mathcal{G}(\mathcal{F})$. (2) implies (1). Let $\mathcal{A} \in \xi'$, then either $\cap \operatorname{cl}_X{}' \mathcal{A} \neq \phi$, in which case we are through, or $\tilde{\mathcal{A}} = \{A \cap X : A \in \mathcal{A}\{ \in \xi$. Then there exists a near ultrafilter $\mathcal{F}$ on X with $\tilde{\mathcal{A}} \subset \mathcal{G}(\mathcal{F})$. If $\mathcal{F}$ has a nonempty adherence then there exists an $x \in X$ with $x \in \bar{F}$ for each $F \in \mathcal{F}$. Then $x \in \operatorname{cl}_{X'}(A)$ for each $A \in \mathcal{A}$. If on the other hand $\operatorname{adh} \mathcal{F} = \phi$, then $\mathcal{O}(\mathcal{F}) \in X'$ and since $A \cap X \in \mathcal{G}(\mathcal{F})$ for each $A \in \mathcal{A}$ we have $\mathcal{O}(\mathcal{F}) \in \operatorname{cl}_{X'}(A)$ for each $A \in \mathcal{A}$. Hence (X', ξ') is topological.

THEOREM 4.2. *Let (X, ξ) be a nearness space. The following are equivalent.*
(1) *The underlying topology of (X', ξ') is compact.*
(2) *(X', ξ') is totally bounded.*
(3) *(X, ξ) is totally bounded and $X' - X$ is finite.*

Proof. By theorem A, and the fact that (X', ξ') is ultrafilter complete it

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follows that (1) and (2) are equivalent. (2) implies (3). Each set $S \subset X' - X$ is closed in X'. If $X' - X$ is infinite, then there exists a infinite sequence $\{\mathcal{O}(\mathcal{F}_i)\}$ of distinct elements in $X' - X$. Let $A_k = \{\mathcal{O}(\mathcal{F}_i) : i \geq k\}$ and $\mathcal{A} = \{A_k : k \in N\}$. Then $\cap \operatorname{cl}_{X'} \mathcal{A} = \phi$ and $\mathcal{A} \notin \xi'$. But $\mathcal{A}$ has the finite intersection property; which is impossible since (X', ξ') is totally bounded. Hence $X' - X$ must be finite. (X, ξ) is totally bounded since this property is preserved by nearness subspaces.

(3) implies (2). Let $\mathcal{S} \subset \mathcal{P}(X')$ and $\mathcal{S} \notin \xi'$. Then $\cap \operatorname{cl}_{X'} \mathcal{S} = \phi$ and $\{S \cap X : S \in \mathcal{S}\} \notin \xi$. Let $X' - X = \{\mathcal{O}(\mathcal{F}_1), \ldots, \mathcal{O}(\mathcal{F}_n)\}$. Then there exists $A_k \in \mathcal{S}$ with $\mathcal{O}(\mathcal{F}_k) \notin \operatorname{cl}_{X'}(A_k)$ for $1 \leq k \leq n$. Set $\mathcal{A} = \{A_k : 1 \leq k \leq n\}$. Since ξ is totally bounded there exists a finite subcollection, say $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$ of $\mathcal{S}$ such that $\cap \{B_i \cap X : 1 \leq i \leq m\} = \phi$. Then $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is a finite subcollection of $\mathcal{S}$ with empty intersection. Hence (X', ξ') is totally bounded.

THEOREM 4.3. *Let (X, ξ) be a nearness space. If X is T_1 then X' is T_1.*

Proof. Let $x \in X$, then $\operatorname{cl}_{X'}\{x\} = \{x\} \cup \{\mathcal{O}(\mathcal{F}) : \{x\} \in \mathcal{F}\} = \{x\}$. Let $\mathcal{O}(\mathcal{F}) \in X'$. Then $\operatorname{cl}_{X'}\{\mathcal{O}(\mathcal{F})\} = \{\mathcal{O}(\mathcal{F})\}$. Hence X' is T_1.

The following example shows that there exists Hausdorff nearness spaces for which no ultrafilter completion, however constructed, is Hausforff. (A Hausdorff nearness space is a nearness space for which the underlying topology is Hausdorff.)

Let (X, t) be a symmetric topological space with $\mathcal{F}$ and $\mathcal{H}$ free ultrafilters on X. Set

$$\xi(\mathcal{F}, \mathcal{H}) = \{\mathcal{A} \subset \mathcal{P}(X) : \cap \bar{\mathcal{A}} \neq \phi \text{ or } \bar{\mathcal{A}} \subset \mathcal{F} \text{ or } \bar{\mathcal{A}} \subset \mathcal{H}\}.$$

Then $\xi(\mathcal{F}, \mathcal{H})$ is a compatible nearness structure; a special case of a nearness structure generated by a class of ultrafilters.

Let (X, t) be a Hausdorff topological space with $\mathcal{F}$ and $\mathcal{H}$ free ultrafilters on X. Suppose $(\tilde{X}, \tilde{\xi}(\mathcal{F}, \mathcal{H}))$ is a Hausdorff ultrafilter completion for $(X, \xi(\mathcal{F}, \mathcal{H}))$. Now there are two cases; either $\mathcal{F}$ and $\mathcal{H}$ converge to distinct points in X in which case they contain disjoint open sets, or they converge to the same point in which case $\mathcal{F} \cup \mathcal{H} \in \xi(\mathcal{F}, \mathcal{H})$. But this can happen only if $\mathcal{F} \subset \mathcal{G}(\mathcal{H})$ or $\mathcal{H} \subset \mathcal{G}(\mathcal{F})$. Hence in order to show that there exists a Hausdorff nearness space that does not have a Hausdorff ultrafilter completion it suffices to find a Hausdorff topological space (X, t) containing two free ultrafilters $\mathcal{F}$ and $\mathcal{H}$ such that:

(1) Every open set in $\mathcal{F}$ meets every open set in $\mathcal{H}$; and
(2) There exists $F \in \mathcal{F}$ such that $\bar{F} \notin \mathcal{H}$; and
(3) There exists $H \in \mathcal{H}$ such that $\bar{H} \notin \mathcal{F}$.

This is accomplished in the following example.

EXAMPLE 4.1. Let R denote the set of real numbers. The collection of all the usual open sets on the reals together with all sets whose complements are countable forms a subbase for a topology t. Then $O \in t$ if and only if $O = Q - C$

where Q is open in the usual topology and C is countable. Let a and b be elements of R with $b>0$; set $S_b(a)=(a-b, a+b)$. Set

$$S=\bigcup\{S_{\pi/n}(n): n>5\} \quad \text{and} \quad T=\left\{n\pm\frac{\pi}{n}: n>5\right\}.$$

Let Ir denote the set of irrational numbers. Let $\mathscr{F}$ by any ultrafilter containing the collection:

$$\{(a, \infty): a\in R\}\cup\{T, \text{Ir}\}.$$

Let O be any open set in $\mathscr{F}$. Then $O\cap S\neq \phi$. Let $\mathcal{O}^*(\mathscr{F})$ denote the collection of all the usual open sets in $\mathscr{F}$. Then each of these sets meets Q, the set of rational numbers. Let $\mathscr{H}$ be any ultrafilter on R containing the collection:

$$\mathcal{O}^*(\mathscr{F})\cup\{S, Q\}.$$

We now show that the three statements immediately preceding this example hold.

(1) Let $O\in\mathscr{F}$ and $P\in\mathscr{H}$ with O and P open sets. Now $O=O_1-C_1$ where O_1 is open in the usual topology and C_1 is countable. Then $O_1\in\mathscr{H}$. Now $P\cap O_1\neq\phi$ and $P=O_2-C_2$ where O_2 is open in the usual topology and C_2 is countable. Now $O_1\cap O_2$ is a nonempty open set in the usual topology on R. Hence $O_1\cap O_2$ is not countable and therefore $O\cap P$ is nonempty.

(2) $T\in\mathscr{F}$ and $\bar{T}=T$ and $T\cap S=\phi$. Hence $\bar{T}\notin\mathscr{H}$.

(3) $Q\in\mathscr{H}$ and Q is closed. Now $Q\cap\text{Ir}=\phi$ and hence $\bar{Q}\notin\mathscr{F}$.

On the surface it seems slightly unsatisfactory that there exists Hausdorff nearness spaces with no Hausdorff ultrafilter completion. Yet, if two ultrafilters are sufficiently tangled together in the sense that their open sets meet and yet essentially different in the sense that each contains a closed set not contained in the other, then an ultrafilter completion of such a space reflects this situation by allowing the two filters to converge to distinct points but these points are tangled together in the sense that they can not be separated with open sets.

A slight modification of the ultrafilter completion constructed in this paper yields a Hausdorff ultrafilter completion for a special class of nearness spaces. In a nearness space (X, ξ), $\mathcal{O}(\mathscr{H})$ will be called minimal if it is minimal in the collection of all $\mathcal{O}(\mathscr{F})$, for $\mathscr{F}$ a free near ultrafilter. $\mathscr{F}$ and $\mathscr{H}$ are said to have the open intersection property if each open set in $\mathscr{F}$ meets every open set in $\mathscr{H}$.

THEOREM 4.6. *Let (X, ξ) be a Hausdorff nearness space satisfying the following:*

(1) *For each near ultrafilter $\mathscr{F}$ there exists a near ultrafilter $\mathscr{H}$ for which $\mathcal{O}(\mathscr{H})$ is minimal and $\mathcal{O}(\mathscr{F})\supset\mathcal{O}(\mathscr{H})$.*

(2) *If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two near ultrafilters for which $\mathcal{O}(\mathscr{H}_1)$ and $\mathcal{O}(\mathscr{H}_2)$ are distinct and minimal then $\mathscr{H}_1$ and $\mathscr{H}_2$ do not have the open intersection property. Then there exists a Hausdorff ultrafilter completion $(\tilde{X}, \tilde{\xi})$ for (X, ξ).*

Proof. The construction is identical to that for (X', ξ') except we set

$$Y = \{\mathcal{O}(\mathcal{F}) : \mathcal{F} \text{ is a free near ultrafilter and } \mathcal{O}(\mathcal{F}) \text{ is minimal}\}.$$

The proof is essentially identical to the proof of Theorem 3.4 with only a slight modification which is handled by hypothesis (1). Hypothesis (2) guarantees that $(\tilde{X}, \tilde{\xi})$ is Hausdorff.

Easily more can be said about constructing Hausdorff ultrafilter completions but we terminate our discussion of this topic with the following theorem.

Let (X, t) be a Hausdorff topological space. Let M be the collection of all free open ultrafilters on X. Set $Y = X \cup M$. Let κX be the set Y with the topology generated by the base $\{U : U \in t\} \cup \{\{\mathfrak{M}\} \cup U : \mathcal{M} \in M \text{ and } U \in \mathcal{M}\}$. Then κX is called the Katětov H-closed extension of (X, t), [11].

THEOREM 4.5. *Let (X, t) be a Hausdorff topological space. Set $\xi_h = \{\mathcal{A} \subset \mathcal{P}(X) : \cap \bar{\mathcal{A}} \neq \phi$ or there exists a free open ultrafilter $\mathcal{M}$ with $A \cap O \neq \phi$ for each $A \in \mathcal{A}$ and $O \in \mathcal{M}\}$. Let X' denote the underlying topological space of the ultrafilter completion (X', ξ_h'). Then X' is homeomorphic to the Katetov H-closed extension κX.*

Proof. The proof follows easily from the fact that the free open ultrafilters are precisely the $\mathcal{O}(\mathcal{F})$ for the free near ultrafilters in ξ_h.

5. Ultrafilter complete is almost reflective in NEAR

THEOREM 5.1. *Let (X, ξ) be a nearness space and (Y, n) an ultrafilter complete nearness space. If $f : (X, \xi) \to (Y, \eta)$ is a nearness map then there exists a nearness map $\tilde{f} : (X', \xi') \to (Y, \eta)$ such that $\tilde{f} \mid X = f$.*

Proof. Let $\mathcal{F}$ be a free near ultrafilter in (X, ξ). Then $f(\mathcal{F})$ is a near ultrafilter in (Y, η) and hence converges to some $y \in Y$. Choosing one such limit for each $\mathcal{O}(\mathcal{F}) \in X' - X$ provides an extension $\tilde{f}$ of f that is continuous. $\tilde{f}$ is a nearness map by Lemma 3.2.

If the mapping $\tilde{f}$ in the above theorem were unique then the ultrafilter complete nearness spaces and nearness maps would form a reflective subcategory in NEAR. Apparently this weaker concept of reflective subcategory has not been developed. The following corollary shows that a nearness map between nearness spaces can be lifted to their ultrafilter completions.

COROLLARY 5.2. *Let (X, ξ) and (Y, η) be nearness spaces and $f : (X, \xi) \to (Y, \eta)$ a nearness map. Then there exists a nearness map $\tilde{f}$ such that the following diagram commutes.*

$$\begin{array}{ccc} (X', \xi') & \xrightarrow{\tilde{f}} & (Y', \eta') \\ \uparrow i & & \uparrow i \\ (X, \xi) & \xrightarrow{f} & (Y, \eta) \end{array}$$

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