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# ENDOPRIMAL ABELIAN GROUPS

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*Dedicated to Ervin Fried on his 70th birthday*

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**Abstract**

A group $A$ is said to be endoprimal if its term functions are precisely the functions which permute with all endomorphisms of $A$. In this paper we describe endoprimal groups in the following three classes of abelian groups: torsion groups, torsionfree groups of rank at most 2, direct sums of a torsion group and a torsionfree group of rank 1.

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## 1. Introduction

An algebra is called primal if every finitary function defined on it is a term function. The most common primal algebra is the 2-element Boolean algebra. Primal algebras were introduced when studying categorical properties of the variety of Boolean algebras. Subsequently, several generalisations have been investigated: algebras in which the term functions are exactly the functions that preserve some derived structure. Following this line, an algebra is called *endoprimal* if its term functions are precisely those functions which permute with all endomorphisms.

Endoprimal algebras have arisen, however, in a different way: in the course of investigations into duality theory. Without using this name, Davey [2] proved in 1976

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that every finite chain is endoprimal as a Heyting algebra. In 1985 Davey and Werner [5] proved that the Heyting algebra $\mathbf{2}^2 \oplus \mathbf{1}$ is also endoprimal, and this paper marks the appearance of the name 'endoprimal'. The next result appeared in 1993, when Márki and Pöschel [9] proved that a distributive lattice is endoprimal if and only if it is not relatively complemented. In 1996 Davey [3] showed that the occurrence of endoprimal algebras in duality theory is not an accident: every endodualisable finite algebra (that is, every finite algebra which admits a particular kind of natural duality) is endoprimal. And, finally, in a recent paper Davey and Pitkethly [4] developed, based on an idea from [9], a general method to construct endoprimal algebras in several varieties. They described endoprimal members in the varieties of vector spaces, semilattices, Boolean algebras, and Stone algebras, and obtained partial results for Heyting algebras and abelian groups. In particular, they proved the following two results.

THEOREM 1.1 ([4, Theorem 6.2]). *An abelian group $A$ of exponent $m$ is endoprimal if and only if $\mathbb{Z}_m^2$ embeds into $A$.*

THEOREM 1.2 ([4, Theorem 6.4]). *An abelian group $A \in \mathbb{ISP}(\mathbb{Z})$ is endoprimal if and only if $\mathbb{Z}^2$ embeds into $A$.*

In the present paper, work on which started parallelly to the investigations of Davey and Pitkethly, we observe that all endoprimal abelian torsion groups are bounded. This result together with the results of [4] yields a complete description of endoprimal torsion groups. One of our central results describes endoprimal members in the class of abelian groups having a nonzero free homomorphic image. This result is considerably stronger than Theorem 1.2. A similar idea of proof is used in several other situations, in particular, it gives an alternative proof for the sufficiency part of Theorem 1.1. Our basic idea in the case of torsionfree groups is to embed a group $A$ into its injective hull, which is a vector space over the field of rationals, and then apply linear algebra. So far we have been able to handle completely the torsionfree abelian groups of rank at most 2. For mixed groups, only partial results have been obtained. The most important of them gives a complete description of endoprimal members in the class of direct sums of a torsionfree abelian group of rank 1 and a torsion abelian group.

## 2. General observations

In what follows, group will mean abelian group. We shall use the notations $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$ for the (additive) groups (or sometimes, by abuse, for the sets or rings) of integers, integers mod $n$, and rational numbers, respectively. The letter $p$ will denote an arbitrary prime number. For undefined notions and notations we refer to [7].

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For short, we call a function $f$ of finite arity on a group $A$ an *endofunction* of $A$ if it permutes with all endomorphisms of $A$, that is,

$$\phi(f(a_1, \ldots, a_n)) = f(\phi(a_1), \ldots, \phi(a_n))$$

for every endomorphism $\phi$ of $A$. Hence a group $A$ is endoprimal if and only if all its endofunctions are term functions (that is, functions of the form $f(x_1, \ldots, x_n) = k_1x_1 + \cdots + k_nx_n$ with some integers $k_1, \ldots, k_n$).

PROPOSITION 2.1. *If* End $A$ *is isomorphic to a subring of* $\mathbb{Q}$ *then the group* $A$ *is not endoprimal.*

PROOF. As is well known, if End $A$ is isomorphic to a subring of $\mathbb{Q}$ then $A$ must be a torsionfree group. Consider the function

$$f(x, y) = \begin{cases} 0 & \text{if} \quad y = 0, \\ x & \text{if} \quad y \neq 0 \end{cases}$$

in $A$. Clearly, $f$ is not a term function. We prove that it is an endofunction. Indeed, let $\phi$ be an arbitrary endomorphism of $A$. Then $\phi(x) = rx$ with an appropriate $r \in \mathbb{Q}$ for all $x$ in $A$, and by straightforward checking we see that $\phi f(x, y) = f(\phi(x), \phi(y))$ for all $x, y \in A$. □

COROLLARY 2.2. *No torsionfree group of rank* 1 *is endoprimal.*

The following proposition shows that in Corollary 2.2 one cannot manage with unary functions. This follows easily from the fact that every endofunction permutes with all endomorphisms of the form $\phi(x) = kx$ where $k$ is a fixed rational number. In particular, every unary endofunction preserves the zero.

PROPOSITION 2.3. *All unary endofunctions of a torsionfree group of rank* 1 *have the form* $f(x) = rx$ *where* $r$ *is a fixed rational number.*

COROLLARY 2.4. *Let* $C$ *be a torsionfree group of rank* 1. *Then all unary endofunctions of* $C$ *are term functions if and only if* $C$ *is not* $p$*-divisible for any prime* $p$.

PROOF. If $C$ is $p$-divisible then the function $f(x) = (1/p)x$ is an endofunction which is not a term function. Conversely, if $f$ is a unary endofunction but not a term function then it is of the form $rx$ where $r \in \mathbb{Q} \setminus \mathbb{Z}$. Hence $C$ must be $p$-divisible for any prime divisor $p$ of the denominator of $r$. □

Another very useful observation is that endofunctions of a direct sum of two groups are sums of endofunctions of the direct summands.

PROPOSITION 2.5. *Let $A = B \oplus C$ and $f$ be an ($n$-ary) endofunction of $A$. Let $f_B$ and $f_C$ be the restrictions of $f$ to $B$ and $C$, respectively. Then $f_B$ and $f_C$ are endofunctions of $B$ and $C$, respectively, and $f = (f_B, f_C)$, that is,*

$$f(b_1 + c_1, \dots, b_n + c_n) = f_B(b_1, \dots, b_n) + f_C(c_1, \dots, c_n)$$

*for all $b_i \in B$, $c_i \in C$, $i = 1, \dots, n$.*

PROOF. Denote by $\pi_B$ and $\pi_C$ the projection maps from $A$ to $B$ and $C$, respectively, followed by embedding into the same components of $A$. Clearly, $\pi_B$ and $\pi_C$ are endomorphisms of $A$ and $\pi_B + \pi_C$ is the identical mapping of $A$. Hence

$$\begin{aligned} & f(b_1 + c_1, \dots, b_n + c_n) \\ &\quad = \pi_B(f(b_1 + c_1, \dots, b_n + c_n)) + \pi_C(f(b_1 + c_1, \dots, b_n + c_n)) \\ &\quad = f(\pi_B(b_1 + c_1), \dots, \pi_B(b_n + c_n)) + f(\pi_C(b_1 + c_1), \dots, \pi_C(b_n + c_n)) \\ &\quad = f(b_1, \dots, b_n) + f(c_1, \dots, c_n) = f_B(b_1, \dots, b_n) + f_C(c_1, \dots, c_n) \end{aligned}$$

for all $b_i \in B$, $c_i \in C$, $i = 1, \dots, n$.

Every pair of endomorphisms $\phi \in \operatorname{End} B$, $\psi \in \operatorname{End} C$ defines via componentwise action an endomorphism of $A$. Since $f$ must permute with this endomorphism, we immediately have that $f_B$ and $f_C$ are endofunctions of $B$ and $C$, respectively. □

COROLLARY 2.6. *Let $A$ be the direct sum of nonzero groups $B$ and $C$ with* $\operatorname{Hom}(B, C) = \operatorname{Hom}(C, B) = 0$. *Then*

(i) *the set of endofunctions of $A$ consists of all pairs $(f_B, f_C)$ where $f_B$ and $f_C$ are endofunctions of $B$ and $C$, respectively;*

(ii) *the group $A$ is endoprimal if and only if $B$ and $C$ are bounded and endoprimal.*

PROOF. (i) By our assumption, all endomorphisms of $A$ act componentwise and therefore all pairs $(f_B, f_C)$ of endofunctions of the components are endofunctions of $A$.

(ii) If $f_B$ is any endofunction of $B$ then by (i) it extends to the endofunction $(f_B, 0)$ of $A$. Since $A$ is endoprimal, this function must be a term function and so $f_B$ is a term function as well. Thus $B$, and similarly $C$, is endoprimal.

If, say, $B$ is unbounded then the function $f = (f_B, f_C)$ with $f_B = 1_B$, the identity function on $B$, and $f_C = 0$ is an endofunction of $A$ which is not a term function. On the other hand, if both $B$ and $C$ are bounded then the Chinese Remainder Theorem applies to show that every pair of $n$-ary term functions of the components is induced by a common $n$-ary term $t$. □

We conclude the section with a positive result that proves the existence of a large class of endoprimal groups. Recall that a group $A$ admits $\mathbb{Z}$ as a direct summand if and only if it has $\mathbb{Z}$ as a homomorphic image.

THEOREM 2.7. *Let a group $A$ have $\mathbb{Z}$ as a direct summand: $A = B \oplus \mathbb{Z}$, and assume that $B$ is unbounded. Then $A$ is endoprimal.*

PROOF. Let $f$ be an $n$-ary endofunction on $A$. By Proposition 2.5 we have $f = (f_B, f_{\mathbb{Z}})$ where $f_B$ and $f_{\mathbb{Z}}$ are $n$-ary endofunctions of $B$ and $\mathbb{Z}$, respectively. We shall prove by induction on $n$ that $f$ is a term function.

Firstly, let $n = 1$. Then by Proposition 2.3 $f_{\mathbb{Z}}$ is a term function. We may assume, without loss of generality, that $f_{\mathbb{Z}} = 0$. For every $b \in B$, there exists a $\phi \in \operatorname{End} A$ such that $\phi(1) = b$. Then

$$f_B(b) = f(b) = f(\phi(1)) = \phi(f(1)) = \phi(f_{\mathbb{Z}}(1)) = 0,$$

hence $f_B = 0$ and $f = 0$.

It remains to make the induction step. Let $n \geq 2$ and, given an arbitrary element $d \in B$, let $\psi$ be the endomorphism of $A$ such that $\psi(1) = d$ and $\psi|_B = 1_B$. This endomorphism $\psi$ is given by the formula

$$\psi(b + k) = b + kd,$$

where $b \in B$ and $k \in \mathbb{Z}$ are arbitrary. Then

$$\begin{aligned} f(\psi(b_1 + k_1), \ldots, \psi(b_n + k_n)) &= f(b_1 + k_1 d, \ldots, b_n + k_n d) \\ &= f_B(b_1 + k_1 d, \ldots, b_n + k_n d) \end{aligned}$$

and

$$\begin{aligned} \psi(f(b_1 + k_1, \ldots, b_n + k_n)) &= \psi(f_B(b_1, \ldots, b_n) + f_{\mathbb{Z}}(k_1, \ldots, k_n)) \\ &= f_B(b_1, \ldots, b_n) + f_{\mathbb{Z}}(k_1, \ldots, k_n)d, \end{aligned}$$

hence

$$f_B(b_1 + k_1 d, \ldots, b_n + k_n d) = f_B(b_1, \ldots, b_n) + f_{\mathbb{Z}}(k_1, \ldots, k_n)d \tag{1}$$

for all $b_1, \ldots, b_n \in B$ and $k_1, \ldots, k_n \in \mathbb{Z}$.

Taking in (1) $b_1 = 0$, $k_1 = 1$ and $k_2 = \cdots = k_n = 0$, we get

$$f_B(d, b_2, \ldots, b_n) = f_B(0, b_2, \ldots, b_n) + f_{\mathbb{Z}}(1, 0, \ldots, 0)d. \tag{2}$$

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The function $f(0, x_2, \ldots, x_n)$ is an $(n-1)$-ary endofunction of $A$, which must be a term function by the induction hypothesis. Then $f_B(0, x_2, \ldots, x_n)$ is a term function as well and by (2), since $d \in B$ was arbitrary, $f_B$ itself is a term function, say $m_1x_1 + \cdots + m_nx_n$ where $m_1 = f_{\mathbb{Z}}(1, 0, \ldots, 0)$. Now we can rewrite (1) as follows:

$$m_1(b_1 + k_1d) + \cdots + m_n(b_n + k_nd) = m_1b_1 + \cdots + m_nb_n + f_{\mathbb{Z}}(k_1, \ldots, k_n)d\,,$$

which implies

$$m_1k_1d + \cdots + m_nk_nd = f_{\mathbb{Z}}(k_1, \ldots, k_n)d\,.$$

Since the latter holds for arbitrary $d \in B$ and $B$ is unbounded, we have that $f_{\mathbb{Z}}$ is also the term function $m_1x_1 + \cdots + m_nx_n$. Since $f = (f_B, f_{\mathbb{Z}})$, the same applies to $f$. □

REMARK. 1. In Corollary 5.2 we shall see that the converse statement is also true: if B is bounded then $B \oplus \mathbb{Z}$ is not endoprimal.

2. Actually the proof of Theorem 2.7 goes through in the case of modules over an arbitrary commutative ring with identity. The result reads as follows: *Let $R$ be a commutative ring with identity and $B$ be a faithful unital $R$-module. Then the $R$-module $B \oplus R$ is endoprimal.*

## 3. Torsion groups

Theorem 1.1 describes bounded endoprimal groups. The rest of the torsion groups is covered by the following theorem.

THEOREM 3.1. *Every endoprimal torsion group is bounded.*

PROOF. Let $A$ be an endoprimal torsion group. Then $A$ is the direct sum of its components $A_p$, and since there is no nonzero homomorphism between different components, by Corollary 2.6 all $A_p$ must be endoprimal and there can be only finitely many nonzero $A_p$'s. If $A$ is not a $p$-group for some prime $p$, then Corollary 2.6 implies that all primary components of $A$ are bounded and then so is $A$. Finally, every endoprimal $p$-group is bounded, for on an unbounded $p$-group the action of a non-rational $p$-adic integer is clearly an endofunction but not a term function (for the latter see [8, Proposition 4.1]). □

We conclude the section with a short alternative proof of Theorem 1.1. Our proof does not use the ideas of duality theory.

PROOF (of Theorem 1.1). Suppose that a group $A$ is of exponent $m$ and first assume that $A$ has a subgroup isomorphic to $\mathbb{Z}_m^2$. Then $A$ is isomorphic to the direct sum of $\mathbb{Z}_m$ and a group of exponent $m$. That $A$ is endoprimal can be proved exactly in the same manner as in Theorem 2.7 (see Remark 2 after its proof).

Now assume that $A$ has no subgroup isomorphic to $\mathbb{Z}_m^2$. Then A must have a primary component, say of exponent $p^k$, which does not contain any subgroup isomorphic to $(\mathbb{Z}_{p^k})^2$. In view of Corollary 2.6, in order to prove that $A$ is not endoprimal it suffices to show that this component is not endoprimal. In other words, we have reduced the proof to the case when $m$ is a prime power $p^k$.

By assumption we have a decomposition $A = B \oplus C$ where the exponent of $B$ is less than $p^k$ and $C$ is cyclic of order $p^k$. Define a function $f : A^2 \to A$ as follows:

$$f(b_1 + c_1, b_2 + c_2) = \begin{cases} 0 & \text{if} \quad p^{k-1}c_2 = 0, \\ p^{k-1}c_1 & \text{if} \quad p^{k-1}c_2 \neq 0. \end{cases}$$

We prove that $f$ is an endofunction but not a term function. First we show that $f$ is an endofunction, that is,

$$\phi(f(x, y)) = f(\phi(x), \phi(y)) \tag{3}$$

for every endomorphism $\phi$ of $A$.

If $b \in B$ and $c \in C$ then $p^{k-1}c = 0$ is equivalent to $p^{k-1}(b + c) = 0$ and the latter implies $p^{k-1}(\phi(b + c)) = 0$ for every $\phi \in \operatorname{End} A$. Hence the equality (3) holds if $p^{k-1}c_2 = 0$. Similarly, the equality (3) holds if $p^{k-1}\phi(b_2 + c_2) \neq 0$ or, equivalently, $p^{k-1}\phi(c_2) \neq 0$. So the only case that needs to be handled is $p^{k-1}c_2 \neq 0$, $p^{k-1}\phi(c_2) = 0$. It follows from these two conditions that $c_2$ is a generator of $C$ and $p^{k-1}\phi(A) = p^{k-1}\phi(C) = 0$. Now

$$f(\phi(b_1 + c_1), \phi(b_2 + c_2)) = 0$$

and

$$\phi(f(b_1 + c_1, b_2 + c_2)) = \phi(p^{k-1}c_1) = p^{k-1}\phi(c_1) = 0.$$

It remains to show that the function $f$ is not a term function. Now, $f$ preserves $C$, and it suffices to prove that the restriction of $f$ to $C$ is not a term function. Suppose that, for certain fixed $l, n \in \mathbb{Z}$, the equality $f(x, y) = lx + ny$ holds for all $x, y \in C$. Since $f(x, 0) = 0$ for every $x \in C$, we have $l \equiv 0 \mod p^k$ and hence $f(x, y) = ny$. The latter, however, contradicts $f(x, y) = p^{k-1}x$ what we have if $y$ is a generator of $C$. □

## 4. Torsionfree groups

Firstly, observe that a nonzero torsionfree endoprimal group cannot be $p$-divisible for any $p$ because otherwise the function $(1/p)x$ would be an endofunction and, obviously, this function is not a term function in a torsionfree group.

Our first result in this section gives a sufficient condition for endoprimality for groups in which a rank 1 subgroup splits off. By $\chi(a)$ we denote the characteristic sequence of the element $a$ (that is, the sequence of the $p$-heights of $a$ under a fixed ordering of the primes). The members of such a sequence are non-negative integers and $\infty$. Two characteristic sequences are said to be equivalent if they differ only at finitely many places, and not at the places where $\infty$ occurs. The equivalence classes of characteristic sequences are called *types*. The type $\mathbf{t}(a)$ of an element $a$ is the type containing $\chi(a)$. It is well known that all nonzero elements of a torsionfree group $A$ of rank 1 have the same type. This type is denoted by $\mathbf{t}(A)$.

THEOREM 4.1. *Let a torsionfree group $A$ decompose into $A = B \oplus C$ where $B$ is nontrivial, $C$ has rank* 1 *and its type does not contain infinity, and suppose that* $\mathbf{t}(C) \leq \mathbf{t}(b)$ *for every $b \in B$. Then $A$ is endoprimal.*

PROOF. The proof follows the lines of Theorem 2.7, and we shall refer to the latter for some computations. Without loss of generality, assume that $C$ is a subgroup of $\mathbb{Q}$ and $1 \in C$.

As in Theorem 2.7, an arbitrary endofunction $f$ of $A$ has the form $f = (f_B, f_C)$ where $f_B$ and $f_C$ are endofunctions on $B$ and $C$, respectively. We shall prove by induction on the arity of $f$ that $f$ is a term function.

Firstly, let $f$ be a unary function. Then by Corollary 2.4 $f_C$ is a term function and we may assume, without loss of generality, that $f_C = 0$. Now, if $\phi \in \operatorname{End} A$ is such that $\phi(1) \in B$ then $f_B(\phi(1)) = 0$ follows exactly as in the proof of Theorem 2.7. The difference is that not every $b \in B$ can occur in the role of $\phi(1)$. However, in view of the condition on types, every cyclic subgroup of $B$ contains nonzero elements which can be in role of $\phi(1)$, that is, for every $b \in B$ there exist a nonzero integer $k$ and an endomorphism $\phi \in \operatorname{End} A$ such that $\phi(1) = kb$. Then $kf_B(b) = f_B(kb) = 0$ and $f_B(b) = 0$ because $B$ is torsionfree.

Now we turn to the proof of the induction step. We assume that all $(n-1)$-ary endofunctions of $A$ are term functions and consider an $n$-ary endofunction $f$. The formulas (1) and (2), with $f_{\mathbb{Z}}$ replaced by $f_C$, follow exactly as in the proof of Theorem 2.7. The basic difference with the latter is that $d$ cannot be assumed to be an arbitrary element of $B$. However, similarly to the unary case, we may claim that for any $d \in B$ there exists $0 \neq k \in \mathbb{Z}$ such that the formulas (1) and (2) hold when $d$ replaced by $kd$. The reason is that because of the condition on types some nonzero integral multiple of

$d$ must be a homomorphic image of 1 under a suitable $\phi \in \operatorname{Hom}(B, C)$. Hence there also exists $\psi \in \operatorname{End} A$ such that $\psi|_B = 1_B$ and $\psi(1) = kd$.

Consequently, taking in account that $f_B$ permutes with multiplication by integers, we get

$$\begin{aligned} kf_B(d, b_2, \dots, b_n) &= f_B(kd, kb_2, \dots, kb_n) \\ &= f_B(0, kb_2, \dots, kb_n) + f_C(1, 0, \dots, 0)kd \\ &= kf_B(0, b_2, \dots, b_n) + kf_C(1, 0, \dots, 0)d. \end{aligned}$$

Since $B$ is torsionfree, this implies

$$f_B(d, b_2, \dots, b_n) = f_B(0, b_2, \dots, b_n) + f_C(1, 0, \dots, 0)d$$

for all $d, b_2, \dots, b_n \in B$. The rest of the proof repeats that of Theorem 2.7 almost literally. We only have to notice that $f_C(1, 0, \dots, 0) \in \mathbb{Z}$ because the unary endofunctions of $C$ are term functions. □

Before turning to groups of rank 2, let us advance a general observation. If $A$ is a torsionfree group then its injective hull $D$ is a vector space over $\mathbb{Q}$ and every endomorphism of $A$ extends in a unique way to an endomorphism of $D$. Let $R$ be the $\mathbb{Q}$-algebra of linear transformations of $D$ generated by the extensions of all endomorphisms of $A$. Clearly, $R$ has an identity element, and it is easy to see that any element of $R$ is of the form $r\phi$ where $r \in \mathbb{Q}$ and $\phi \in \operatorname{End} A$.

Let now $f$ be an $n$-ary endofunction of $A$. If $d_1, \dots, d_n$ are arbitrary elements of $D$ then there are $a_1, \dots, a_n \in A$ and $r \in \mathbb{Q}$ such that $d_i = ra_i$, $i = 1, \dots, n$. Now the formula $f(d_1, \dots, d_n) = rf(a_1, \dots, a_n)$ extends $f$ to $D$ in a unique way, and it is easy to see that this function permutes with all members of $R$. Hence, if we are able to prove that the only functions on $D$ which permute with all members of $R$ are multiplications by rational numbers, and $A$ is not $p$-divisible for any $p$, then $A$ is endoprimal. On the other hand, if the centre of $R$ is nontrivial, that is, it contains a nonscalar linear transformation of $D$ then the centre of $\operatorname{End} A$ is also nontrivial, whence $A$ has an endofunction which is not a term function. If $\dim R = 1$, that is, $R$ consists of scalar maps, then $A$, too, has only scalar endomorphisms and thus it is not endoprimal by Proposition 2.1.

We now focus on groups of rank 2. If $\dim R \leq 2$ then $R$ is commutative, hence by the above arguments $A$ is not endoprimal. We are going to show that if $\dim R \geq 3$ and $A$ is endoprimal then necessarily $A$ is a direct sum of two groups of rank 1.

Recall that a torsionfree group is called *homogeneous* of type $\tau$ if all its elements are of the same type $\tau$. We shall call a torsionfree group $A$ *almost homogeneous* if either it is homogeneous or it has a rank 1 subgroup $B \leq A$ such that all elements in $A \setminus B$ are of the same type $\tau$. In the latter case we call $\tau$ the *type* of $A$ and write $\tau = \mathbf{t}(A)$.

LEMMA 4.2. *Let $A$ be an almost homogeneous torsionfree group of rank 2 which is not $p$-divisible for any $p$. If $A$ has a non-nilpotent endomorphism of rank 1 then $A$ has a nontrivial direct summand.*

PROOF. First observe that the type $\tau$ of $A$ does not contain the infinity. Indeed, if it did then, for some $p$, we would have two linearly independent $p$-divisible elements of $A$, which would imply that $A$ is $p$-divisible. Let $\phi \in \operatorname{End} A$ be non-nilpotent and of rank 1 and put $B = \phi(A)$. Suppose first that $B$ is contained in a rank 1 subgroup of $A$ of type $\tau$. Then obviously $\mathbf{t}(B) \leq \tau$. On the other hand, there are elements of type $\tau$ that do not belong to $\operatorname{Ker}\phi$. Hence, $\phi$ embeds some group of type $\tau$ into $B$, which implies $\tau \leq \mathbf{t}(B)$. Thus, $\mathbf{t}(B) = \tau$.

Now suppose that $B$ is not contained in any rank 1 subgroup of $A$ of type $\tau$. Then, however, there is another non-nilpotent endomorphism $\psi$ of rank 1 whose image is contained in such a subgroup of $A$. Indeed, since $\phi$ is not nilpotent, we have $\phi^2 \neq 0$. Since $B$ is of rank 1, there is a nonzero $r \in \mathbb{Q}$ such that $\phi(x) = rx$ for all $x \in B$. Replacing $\phi$ by a suitable integral multiple of it we may assume that $r \in \mathbb{Z}$. Now put $\psi = r \cdot 1_A - \phi$. We see that $\operatorname{Ker}\psi$ consists of the elements $a \in A$ such that $ra = \phi(a)$, and this is the maximal rank 1 subgroup of $A$ containing $B$. Hence $\psi$ is of rank 1. An easy calculation shows that $\psi^2 = r\psi$, hence $\psi$ is not nilpotent. Therefore the image of $\psi$ cannot be contained in $\operatorname{Ker}\psi$. Since $A$ is almost homogeneous and $\operatorname{Ker}\psi$ is not of type $\tau$, the image of $\psi$ must be of type $\tau$. This proves that under our assumptions there exists $\phi \in \operatorname{End} A$ such that $B = \phi(A)$ is a group of rank 1 and type $\tau$.

Let $C$ be any maximal subgroup of rank 1 in $A$. Then either $C = \operatorname{Ker}\phi$ or $C \cap \operatorname{Ker}\phi = 0$, and in the latter case $\phi$ embeds $C$ into $B$. Assume first that $\mathbf{t}(C) \neq \tau$. This means that the group $A$ is not homogeneous and $C$ is the only maximal rank 1 subgroup of $A$ whose type is not $\tau$. Then, since $A$ is of rank 2 and almost homogeneous of type $\tau$, some element $0 \neq c \in C$ can be written as a sum of two elements of type $\tau$. So $\mathbf{t}(c) \geq \tau$ in $A$. But $C$ is a maximal rank 1 subgroup, hence if a multiple of an element belongs to $C$ then so does the element itself. Therefore $\mathbf{t}(c) \geq \tau$ in $C$, whence $\mathbf{t}(C) > \tau$ must hold by the assumption. Then $C$ cannot be embedded into $B$, so $C$ must be the kernel of $\phi$.

If $\mathbf{t}(C) = \tau$ and $C \neq \operatorname{Ker}\phi$ then $\phi$ embeds $C$ into $B$. Now $\phi(C)$ is a subgroup of $B$ isomorphic to $B$, so it must be of the form $rB$ for some nonzero rational number $r$. Since there is no infinity in $\tau$, $r$ must be an integer. Thus there is an $m \in \mathbb{Z}$ such that $\phi(C) = mB$.

Since $A$ is the union of its maximal rank 1 subgroups, the above discussion implies that $B$ is the union of its subgroups of the form $\phi(C_i)$ where all $C_i$ are maximal rank 1 subgroups of type $\tau$. Moreover, each of them has the form $m_i B$ where $m_i \in \mathbb{Z}$. Let $m > 0$ be the greatest common divisor of all $m_i$'s and suppose that $m \neq 1$. Then, $\phi(A)$ is contained in $mB$ which is a proper subset of $B$ because there is no infinity

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in $\tau$, a contradiction.

Consequently, $m = 1$ and there exist finitely many of the $m_i$'s, say $m_1, \dots, m_s$, and integers $k_1, \dots, k_s \in \mathbb{Z}$ such that

$$1 = k_1 m_1 + \cdots + k_s m_s .$$

Fix a nonzero element $b \in B$ and in every $C_i$ take the element $c_i$ such that $\phi(c_i) = m_i b$. Let $c = k_1 c_1 + \cdots + k_s c_s$, then $\phi(c) = b$ and hence $c \neq 0$. Denote by $C$ the maximal rank 1 subgroup of $A$ containing $c$.

Now, if $b'$ is another element of $B$, it can be written as $rb$ for some $r \in \mathbb{Q}$. Again, in every $C_i$ there exists a unique element $c_i'$ such that $\phi(c_i') = m_i b'$ and it is easy to see that $c_i' = rc_i$ for every $i$. Putting $c' = k_1 c_1' + \cdots + k_s c_s'$, we have

$$\begin{aligned} \phi(c') &= \phi(k_1 c_1' + \cdots + k_s c_s') = \phi(k_1 r c_1 + \cdots + k_s r c_s) \\ &= r(k_1 \phi(c_1) + \cdots + k_s \phi(c_s)) = r(k_1 m_1 b + \cdots + k_s m_s b) = rb = b'. \end{aligned}$$

We proved that the restriction of $\phi$ to $C$ is a bijection from $C$ to $B$. Denoting by $\psi$ the inverse of this bijection, we have an idempotent endomorphism $\psi\phi$ of rank 1 proving that $A$ has the required decomposition. □

THEOREM 4.3. *A torsionfree group $A$ of rank 2 is endoprimal if and only if $A = B \oplus C$ where $B$ and $C$ are groups of rank 1, $\mathbf{t}(B) \geq \mathbf{t}(C)$, and $C$ (or, equivalently, $A$) is not $p$-divisible for any prime $p$.*

PROOF. The sufficiency of the conditions follows from Theorem 4.1. As we have noticed at the beginning of this section, if $A$ is $p$-divisible for some $p$ then it is not endoprimal.

Next we prove that any torsionfree endoprimal group $A$ of rank 2 satisfies the assumptions of Lemma 4.2. So we have to prove that $A$ is almost homogeneous and has an endomorphism $\phi$ of rank 1 such that $\phi^2 \neq 0$. By the above remarks, the $\mathbb{Q}$-algebra $R$ must have dimension at least 3.

Suppose first that the injective hull $D$ of $A$ is an irreducible $R$-module with the $R$ defined above. By the density theorem, $R$ must be isomorphic to a full matrix ring over the division ring $K = \mathrm{End}_R D$ which contains $\mathbb{Q}$, and $D$ is a vector space over $K$. Since $A$ is of rank 2, we have $\dim_{\mathbb{Q}} D = 2$, hence $R$ is not isomorphic to $K$ and therefore $\dim_K R \geq 4$. In view of $\dim_{\mathbb{Q}} R \leq 4$, we must have $K = \mathbb{Q}$ and $\dim_{\mathbb{Q}} R = 4$. Obviously, then $A$ is homogeneous and has an endomorphism $\phi$ with the required properties.

If the $R$-module $D$ is not irreducible, then it has a proper nonzero submodule $S$ and we may take a basis $e_1, e_2$ of the vector space ${}_{\mathbb{Q}}D$ such that $e_1 \in S$. This gives a representation of all elements of $R$ in the form of upper triangular matrices of order

2 over $\mathbb{Q}$. Since the dimension of $R$ over $\mathbb{Q}$ cannot be less than 3, we see that $R$ is isomorphic to the ring of all upper triangular matrices of order 2. In particular, for every two elements $a, b \in A \setminus (S \cap A)$, there exists $\phi \in \operatorname{End} A$ such that $\phi(a) \neq 0$ and $\phi(a)$ linearly depends on $b$. This easily implies that $A$ is almost homogeneous. Also, there exists a nonzero endomorphism $\phi$ of $A$ such that $\phi(e_1) = 0$ and $\phi(e_2)$ linearly depends on $e_2$. Obviously, such $\phi$ is of rank 1 and $\phi^2 \neq 0$.

Hence Lemma 4.2 applies and $A$ decomposes into a direct sum of two (rank 1) subgroups. If the types of these subgroups are incomparable then there is no homomorphism between them in either direction, whence $A$ is not endoprimal in view of Corollary 2.6. □

## 5. Mixed groups

Most of our results about mixed groups concern the case when the torsion part of the group splits off. We start, however, with a few results where a different sort of direct decomposition is assumed.

PROPOSITION 5.1. *Let $A$ be the direct sum of a nontorsion group $B$ and a bounded group $T$ and suppose that $T$ has no $p$-component for primes $p$ which occur as orders of some elements of $B$. If $A$ is endoprimal then $B$ is also endoprimal.*

PROOF. Let $f$ be an $n$-ary endofunction of $B$ and $e = \exp(T)$. Define the function $g$ on $A$ by putting

$$g(b_1 + t_1, \dots, b_n + t_n) = f(eb_1, \dots, eb_n),$$

and denote by $\pi$ the projection map $A \to B$. Then, for an arbitrary endomorphism $\phi$ of $A$, both $\phi(g(a_1, \dots, a_n))$ and $g(\phi(a_1), \dots, \phi(a_n))$ are equal to $e(\phi(f(\pi(a_1), \dots, \pi(a_n))))$ for any $a_1, \dots, a_n \in A$, hence $g$ is an endofunction of $A$.

Since $A$ is endoprimal, the function $g$ must be a term function. Moreover, since $A$ is not a torsion group, the term $k_1x_1 + \dots + k_nx_n$ which determines $g$ is unique. Obviously the function $g$ vanishes on $T$, so $e$ must divide all $k_i$. Let $k_i = el_i$, $i = 1, \dots, n$. Since $g$ induces on $B$ the function $ef$, we get

$$e(l_1b_1 + \dots + l_nb_n - f(b_1, \dots, b_n)) = 0$$

for all $b_1, \dots, b_n \in B$. By our assumption this implies $f(b_1, \dots, b_n) = l_1b_1 + \dots + l_nb_n$. □

REMARK. Proposition 5.1 always applies if the torsion part of $A$ is bounded. Namely, by a theorem of Baer [1] and Fomin [6] (see also [7, Theorem 100.1]) if an abelian group has bounded torsion part then the latter is a direct summand of the group. In particular, in view of Corollary 2.2 we have the following result.

COROLLARY 5.2. *Groups of torsionfree rank* 1 *with bounded torsion part are not endoprimal.*

Together with Theorem 2.7, this yields

THEOREM 5.3. *Let $A$ be the direct sum of a group $B$ and the infinite cyclic group. Then $A$ is endoprimal if and only if $B$ is not bounded.*

Now we prove a theorem which shows that endoprimality of a direct summand sometimes yields endoprimality of the whole group.

THEOREM 5.4. *Let $A = B \oplus T$ where $B$ is an endoprimal non-torsion group and $T$ is a torsion group such that $T$ has no $p$-component for primes $p$ for which the torsionfree part of $B$ is $p$-divisible. Then $A$ is endoprimal.*

PROOF. Let $n$ be a natural number which has no prime factor $p$ for which the torsionfree part of $B$ is $p$-divisible. Then $B/nB$ is a group of exponent $n$, and we obtain that the cyclic group of order $n$ is a homomorphic image of $B$.

Take an arbitrary endofunction $f$ of $A$. Then $f = (f_B, f_T)$ where $f_B$ and $f_T$ are endofunctions of $B$ and $T$, respectively. Since $B$ is endoprimal, $f_B$ is a term function. By subtracting this term function from $f$ we see that one can assume $f_B = 0$ without loss of generality, and we have to prove that $f_T = 0$ as well.

Suppose first that $f$ is unary, and pick any $t \in T$. By our first observation there is a $\phi \in \operatorname{End} A$ such that $\phi(B)$ is the cyclic group generated by $t$. Let $b \in B$ be such that $\phi(b) = t$. Since $f$ is an endofunction which vanishes on $B$, we have $f(t) = f(\phi(b)) = \phi(f(b)) = 0$ as required.

Suppose now that the statement is true for $(n-1)$-ary functions, let $f$ be an $n$-ary endofunction which vanishes on $B$, and take an arbitrary $n$-tuple $(t_1, \dots, t_n)$ from $T$. Again, in view of our first observation, we can pick an element $b_1 \in B$ and an endomorphism $\phi$ of $A$ such that $\phi(b_1) = -t_1$ and $\phi|_T = 1_T$. Take any $b_2, \dots, b_n \in B$. Now

$$\begin{aligned} f_T(t_1, \dots, t_n) &= \phi(f_T(t_1, \dots, t_n)) = \phi(f_B(b_1, \dots, b_n) + f_T(t_1, \dots, t_n)) \\ &= \phi(f(b_1 + t_1, \dots, b_n + t_n)) = f(\phi(b_1 + t_1), \dots, \phi(b_n + t_n)) \\ &= f(0, \phi(b_2) + t_2, \dots, \phi(b_n) + t_n). \end{aligned}$$

However, $f(0, x_2, \dots, x_n)$ is an $(n-1)$-ary endofunction which vanishes on $B$, so it is the zero map by the induction hypothesis. Thus $f_T(t_1, \dots, t_n) = 0$ as required. □

In view of Proposition 5.1 and the subsequent remark we have now:

COROLLARY 5.5. *A group with bounded torsion part is endoprimal if and only if its largest torsionfree factor is endoprimal.*

Our final result characterizes endoprimal groups of the form $B \oplus T$ where $B$ is torsionfree of rank 1 and $T$ is torsion. Before stating and proving this theorem we present several auxiliary results. The proof of the first of them again follows the lines of the proof of Theorem 2.7.

PROPOSITION 5.6. *Let $A = B \oplus T$ where $B$ is a torsionfree group of rank 1 whose type does not contain infinity and $T$ is an unbounded torsion group. Then $A$ is endoprimal.*

PROOF. Let $T = T_p \oplus T'$ where $T_p$ is the $p$-component of $T$. We shall identify $B$ with a subgroup of $\mathbb{Q}$. Since the type of $B$ does not contain the infinity, we may assume that $1 \in B$ but $(1/p) \notin B$. This agreement implies that all multiples $bt$ where $b \in B$ and $t \in T_p$ are well defined and, for every $t \in T_p$, there exists $\phi \in \operatorname{Hom}(B, T_p)$ (in fact unique) such that $\phi(1) = t$.

An arbitrary endofunction $f$ of $A$ is represented as $(f_B, f_{T_p}, f_{T'})$ where $f_B, f_{T_p}$ and $f_{T'}$ are endofunctions of $B$, $T_p$ and $T'$, respectively. Moreover, the pair $(f_B, f_{T_p})$ is an endofunction of $B \oplus T_p$. We shall prove by induction on the arity of $f$ that $f$ is a term function.

Assume first that $f$ is unary. Then by Corollary 2.4 $f_B$ is a term function $kx$, and by $f_B(x) = f_B(1)x$ (for every $x \in B$) we get $k = f_B(1)$ (that is, $f_B(1)$ does not depend on the choice of the embedding of $B$ into $\mathbb{Q}$). Let $t \in T_p$ and $\phi \in \operatorname{End} A$ be such that $\phi(1) = t$. Then $f(t) = f_{T_p}(\phi(1)) = \phi(f_B(1)) = f_B(1)t$. We see that on every primary component of $A$ the function $f$ induces the same function $kx$ where $k = f_B(1)$. Hence $f(x) = kx$ on the whole $A$.

Assume now that $f$ is $n$-ary and all $(n-1)$-ary endofunctions of $A$ are term functions. Following the lines of proof of Theorem 2.7, we have:

$$f_{T_p}(t, t_2, \ldots, t_n) = k_1 t + k_2 t_2 + \cdots + k_n t_n, \tag{4}$$

$$(f_B(b_1, \ldots, b_n) - k_1 b_1 - \cdots - k_n b_n)t = 0 \tag{5}$$

where $t, t_2, \ldots, t_n \in T_p$, $b_1, \ldots, b_n \in B$, $k_2, \ldots, k_n \in \mathbb{Z}$ and $k_1 = f_B(1, 0, \ldots, 0)$. Now observe that $f_B(x, 0 \ldots, 0)$ is a unary endofuction on $B$, hence it is a term function $kx$ for some $k \in \mathbb{Z}$ and then obviously $k = f_B(1, 0 \ldots, 0)$ holds, no matter how the embedding of $B$ to $\mathbb{Q}$ is chosen. This implies that the coefficient $k_1$ in (4) and (5) is an integer which does not depend on the prime $p$. In particular, since $T$ is unbounded and $t$ in (5) is an arbitrary element of $A$ whose order is a prime power,

(5) implies that $f$ induces the same term function on all primary components of $T$. Taking in account also the formula (4), we have that $f$ is a term function. □

PROPOSITION 5.7. *Let $A = B \oplus T$ where $B$ is a $p$-divisible torsionfree group of rank 1 and suppose that the $p$-component $T_p$ of $T$ is reduced. Then $A$ is not endoprimal.*

PROOF. If $T_p = 0$ then $(1/p)x$ is an endofunction of $A$ which is not a term function. If $T_p \neq 0$ then by the assumption we can write $A = B \oplus T_p \oplus T'$ with a torsion group $T'$ which has no $p$-component. Clearly, $T_p$ has no nontrivial homomorphisms to $B \oplus T'$, nor has $T'$ to $T_p$. But there is no nontrivial homomorphism from $B$ to $T_p$ either for $B$ is $p$-divisible and $T_p$ is a reduced $p$-group. Hence Corollary 2.6 applies. □

PROPOSITION 5.8. *Let $A = B \oplus T$ where $B$ is a torsionfree $p$-divisible group of rank 1 and $T$ is a $p$-group which is not reduced. Then every endofunction of $A$ is of the form $r_1x_1 + \cdots + r_nx_n$ where $r_1, \ldots, r_n$ are rational numbers with denominators prime to $p$.*

PROOF. We first assume that the torsion part $T$ is divisible and identify $B$ with a subgroup of $\mathbb{Q}$ containing 1. Observe that every $t \in T$ is the image of 1 under a suitable $\phi \in \mathrm{Hom}(B, T)$. Indeed, since $T$ is $p$-divisible, it suffices to observe that for every nonzero proper subgroup $B' \leq B$ the quotient group $B/B'$ is a torsion group whose $p$-component is isomorphic to $\mathbb{Z}_{p^\infty}$. Moreover, the group $B'$ may be chosen so that the order of 1 modulo $B'$ equals the order of $t$. Consider an arbitrary endofunction $f$ of $A$. Then $f = (f_B, f_T)$ where $f_B$ and $f_T$ are endofunctions of $B$ and $T$, respectively. If the function $f$ is unary then by Proposition 2.3, there exists an $r \in \mathbb{Q}$ such that $f_B(x) = rx$ for every $x \in B$.

We have to show that the denominator of $r$ is prime to $p$. Suppose on the contrary that $r$ is a reduced fraction $k/l$ where $p$ divides $l$. Now pick $0 \neq t \in T$ and assume that the order of $f_T(t)$ is $p^m$. Obviously, for a suitable integral multiple $g$ of a power of $f$ the function $g_B$ is multiplication by $u/p^s$ where $u, s \in \mathbb{Z}$, $u$ is prime to $p$, and $s \geq m$. Then $p^s g_B(1) = u$ and if $\phi$ is an endomorphism of $A$ such that $\phi(1) = t$, we have

$$ut = \phi(u) = \phi(p^s g_B(1)) = p^s \phi(g_B(1)) = p^s g_T(\phi(1)) = p^s g_T(t) = 0,$$

which contradicts the fact that $u$ is prime to the order of $t$.

Thus $p$ does not divide $l$ and the equality $lf_B(1) = k$ implies

$$kt = \phi(k) = \phi(lf_B(1)) = l\phi(f_B(1)) = lf_B(\phi(1)) = lt,$$

which is equivalent to $f_T(t) = rt$.

Assume now that all $(n-1)$-ary endofunctions of $A$ are of the required form and let $f$ be $n$-ary. As in Theorem 2.7, every element of $T$ is a homomorphic image of 1 under $\phi \in \operatorname{End} A$ such that $\phi|_T = 1_T$. This allows again to establish analogues of formulas (1) and (2) and to carry out the induction step. The coefficient $f_B(1, 0, \dots, 0)$ is a rational number of the required form because $f(x, 0, \dots, 0)$ is a unary endofunction of $B$.

Now consider the general case. Let $T = C \oplus D$ where $C$ is reduced and $D$ is divisible. Let $f$ be an $n$-ary endofunction on $A$. Then $f = (f_B, f_C, f_D)$ where $f_B, f_C$, and $f_D$ are endofunctions of $B$, $C$, and $D$, respectively. By the first part of the proof, we may assume that $f_B$ and $f_D$ are zero functions and we have to prove $f_C = 0$. Suppose, on the contrary, that $f_C(c_1, \dots, c_n) \neq 0$ for some $c_1, \dots, c_n \in C$. Applying Zorn's lemma, we take a subgroup $H$ of $C$ maximal with respect to $f_C(c_1, \dots, c_n) \notin H$. The quotient group $C/H$ is subdirectly irreducible, hence it is isomorphic to $\mathbb{Z}_{p^k}$ where $k \in \mathbb{N} \cup \{\infty\}$. Let $\phi$ be the composition of the natural homomorphism $C \to C/H$ and an embedding $C/H \to D$. Clearly $H = \operatorname{Ker}\phi$.

Now take any endomorphism $\psi$ of $A$ whose restriction to $C$ is $\phi$. Then

$$\psi(f_C(c_1, \dots, c_n)) = f_D(\psi(c_1), \dots, \psi(c_n)) = 0$$

implying $f_C(c_1, \dots, c_n) \in H$, a contradiction. □

THEOREM 5.9. *Let $A = B \oplus T$ where $B$ is a torsionfree group of rank 1 and $T$ is a torsion group. Denote by $P$ the set of those prime numbers $p$ for which $\chi_p(B) = \infty$ and by $T_p$ the $p$-component of $T$. Then $A$ is endoprimal if and only if one of the following cases occurs*:

(i) *$P \neq \emptyset$ and $T_p$ is not reduced for any $p \in P$;*
(ii) *$P = \emptyset$ and $T$ is not bounded.*

PROOF. If $A$ is endoprimal then one of the two conditions must occur in view of Proposition 5.7 and Corollary 5.2. Conversely, suppose that $A$ satisfies one of these conditions. If $P = \emptyset$ then $A$ is endoprimal by Proposition 5.6. Suppose $P \neq \emptyset$ and take an arbitrary endofunction $f(x_1, \dots, x_n)$ in $A$. Since $f$ is an endofunction in $B \oplus T_p$ for each $p \in P$, Proposition 5.8 says that $f$ is of the form $r_1x_1 + \dots + r_nx_n$ in $B \oplus T_p$, where $r_1, \dots, r_n$ are rational numbers with denominators prime to $p$. Since $B$ is torsionfree, the coefficients $r_1, \dots, r_n$ must be the same for the different $p$'s and none of the primes $p \in P$ may occur in denominators of $r_i$. But the $r_i$'s cannot have primes $q$ with $q \notin P$ in their denominators either since for such $q$ we have $\chi_q(B) < \infty$ and therefore $(1/q)x$ is not defined over the whole of $B$. Thus $f$ is a term function in $A$. □

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