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SINGULAR COTANGENT MODEL

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Abstract. Any singular level of a completely integrable system (c.i.s.) with non-degenerate singularities has a singular affine structure. We shall show how to construct a simple c.i.s. around the level, having the above affine structure. The cotangent bundle of the desingularized level is used to perform the construction, and the c.i.s. obtained looks like the simplest one associated to the affine structure. This method of construction is used to provide several examples of c.i.s. with different kinds of non-degenerate singularities.

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1. Introduction. Let (M^{2n}, ω, F) be a non-degenerate integrable system. This means that (M^{2n}, ω) is a symplectic manifold and $F = (f_1, \ldots, f_n)$ is a proper moment map, which is non-singular almost everywhere, and its singularities are of Morse-Bott type.

The $\mathbb{R}^n$-action generated by the Hamiltonian vector fields $H_{f_1}, \ldots, H_{f_n}$ gives a singular Lagrangian foliation on (M^{2n}, ω). Any leaf is an orbit of this actions. At the same time, any connected component of $F^{-1}(c)$ is a level. We know that the regular levels are n-dimensional tori and the singular levels are finite union of several leaves.

A semilocal classification of such integrable systems is still open. It consists in finding a complete system of invariants describing symplectically a neighborhood of a level. Some approaches to solve this question has been currently made, see [**4**, **13**, **15**–**17**]. As the local description of non-degenerate singularities is given in terms of products of elliptic, hyperbolic and focus-focus components, the number of elliptic, hyperbolic and focus-focus components at each point of the level will play an essential role in the classification problem.

As in the regular case, the Hamiltonian vector fields $H_{f_1}, \ldots, H_{f_n}$ endow any leaf and any level with an affine structure with singularities. In studying the semi-local classification, we have seen that this affine structure gives strong conditions on the set of invariants that we have found. Our proposal in this paper is to prove that from this affine structure we can construct a completely integrable system around a given level L_0 of (M^{2n}, ω, F), such that the induced affine structure on L_0 is the given one. This construction looks like the simplest one with the given affine structure on L_0.

As the '1-jet' of the former completely integrable system (c.i.s. from now on) and the one constructed in this way coincide, it makes sense to denote this c.i.s. as the linearized c.i.s. of the initial one.

This process of construction of the linearized c.i.s. indicates us a way to construct c.i.s. with prescribed non-degenerate singularities along a given singular level. Some examples of construction are given.

I would like to express my gratitude to Pierre Molino for interesting and fruitful conversations on this subject.

2. Definition and basic properties.

2.1. Local expressions. Let $(M^{2n}, \omega, f_1, \ldots, f_n)$ be an integrable system, i.e. (M^{2n}, ω) is a symplectic manifold. The functions $f_1, \ldots, f_n$ are Poisson commuting (first integrals of a given Hamiltonian system), such that $df_1 \wedge \cdots \wedge df_n \neq 0$ on a dense subset of M^{2n}, and the moment map $F : M^{2n} \longrightarrow \mathbb{R}^n$, $F = (f_1, \ldots, f_n)$, is proper.

Such an integrable system is said to be non-degenerate if, in a neighborhood of each point $p_0 \in M^{2n}$, there exist canonical coordinates $(x_1, y_1, \ldots, x_n, y_n)$, and n local functions $h_1, \ldots, h_n$, which have one of the following expressions:

$$h_i = y_i \qquad \text{(regular terms)},$$
$$h_i = (x_i)^2 + (y_i)^2 \qquad \text{(elliptic terms)},$$
$$h_i = x_i y_i \qquad \text{(hyperbolic terms)},$$
$$\begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1} \\ h_{i+1} = x_i y_{i+1} - y_i x_{i+1} \end{cases} \qquad \text{(focus-focus terms)},$$

such that:

(1) $f_1, \ldots, f_n$ Poisson commute with $h_1, \ldots, h_n$.
(2) $\{j^2_{p_0} f_1, \ldots, j^2_{p_0} f_n\}$ and $\{j^2_{p_0} h_1, \ldots, j^2_{p_0} h_n\}$ generate the same space of 2-jets at p_0.

This notion of non-degeneracy implies obvious conditions on the space of 2-jets generated at each point by $f_1, \ldots, f_n$. Conversely, assuming these infinitesimal conditions, the existence of such adapted coordinates is an important result, due to H. Eliasson [**9**]. The demonstration has been completed by E. Miranda [**13**] and E. Miranda & V. N. San [**14**]. Adapted coordinates will be referred to as Eliasson coordinates, or simply E-coordinates.

We denote as $\mathcal{L}$ the Lagrangian singular foliation associated to this c.i.s., i.e. the leaves of $\mathcal{L}$ are the orbits of the $\mathbb{R}^n$-action generated by $H_{f_1}, \ldots, H_{f_n}$.

From a practical point of view, one can locally look at $(M^{2n}, \omega, \mathcal{L})$ as $(\mathbb{R}^{2n}, \omega_0, \mathcal{L}_0)$, where ω_0 is the standard symplectic two-form on $\mathbb{R}^{2n}$ and $\mathcal{L}_0$ is given by $dh_i = 0$, $i = 1, \ldots, n$.

Let L_0 be a singular level, and p_0 a point of L_0. By taking E-coordinates around p_0 we have the following characteristic numbers of the point p_0: the numbers k_e, k_h and k_f correspond to the number of elliptic, hyperbolic and focus-focus terms of the set $(h_1, \ldots, h_n)$. The leaf through p_0 is $\mathbb{T}^c \times \mathbb{R}^o$, and the numbers c and o are called the degrees of closedness and openness of the leaf, respectively.

Following Zung (see [**20**]), the 5-tuple (k_e, k_h, k_f, c, o) is called the leaf-type and (k_e, k_h, k_f) the Williamson type of p_0. In general one has $k_e + k_h + 2k_f + c + o = n$. In [**20**] it is proved that the three numbers k_e, $k_f + c$, $k_h + k_k + o$, are invariants of each level. These numbers are known as the degrees of ellipticity, closedness and openness of the level.

Summarizing, one can say that a singular level L_0 is a compact $(n-k_e)$-manifold, with self-intersections provided with a manifold structure of dimension less than $(n-k_e)$. This level is endowed with a non-degenerate $\mathbb{R}^{n-k_e}$-action. Non-degenerate action means that the isotropy of this action at each point is linearizable.

Let L_0 be a singular level of a c.i.s. (M^{2n}, ω, F); it is known that the degree of ellipticity k_e is the same at all the points of L_0. Associated to this degree of ellipticity, we have a Hamiltonian $\mathbb{T}^{k_e}$-action in a neighborhood of L_0, such that the isotropy of this action contains L_0. Due to this, one can see, using for instance coordinates adapted to this $\mathbb{T}^{k_e}$-action, that the isotropy subset of this action is a symplectic submanifold $M^{2(n-k_e)} \subset M^{2n}$. This symplectic submanifold is obviously a neighborhood of L_0 and one has an induced c.i.s. on $M^{2(n-k_e)}$ such that L_0 is a singular level.

From this point, we can regard L_0 as a singular level in the c.i.s. defined on $M^{2(n-k_e)}$, The degree of ellipticty in L_0 is obviously zero. So, from now on we shall restrict our attention to the case of singular levels with $k_e = 0$. Once obtained the linearized model in this case, a simple product with k_e copies of the standard model in the unit disk E_2 will produce, in general, the model that we are searching for.

2.2. Desingularized level. We assume that on the singular level L_0, with singular affine structure ∇_0, the degree of ellipticity vanishes. In order to give a construction of a 'standard' c.i.s., such that L_0 is a singular level, and the induced singular affine structure on it coincides with ∇_0, we start by giving the construction of the so called desingularized level.

As $k_e = 0$ on L_0, dim.$L_0 = n$. Let ψ be a differentiable embedding of M^{2n} in an euclidean space $\mathbb{R}^l$. Let $L_0^r \subset L_0$ be the subset of regular points of L_0, i.e., a point $p \in L_0$ lies in L_0^r if and only if the rank of dF is equal to n at p. Note that L_0^r is not, in general, a connected submanifold.

Let $G_n(\mathbb{R}^l)$ be the n-dimensional Grassmann manifold of $\mathbb{R}^l$. For any n-dimensional vector subspace V of $\mathbb{R}^l$, let $[V]$ be corresponding class in $G_n(\mathbb{R}^l)$.

We consider the following embedding of L_0^r in $M^{2n} \times G_n(\mathbb{R}^l)$ given by:

$$\varphi : L_0^r \hookrightarrow M^{2n} \times G_n(\mathbb{R}^l), \quad p \mapsto (p, [T_{\psi(p)}(\psi(L_0))]).$$

We define the desingularized level, $\hat{L}_0$ as the closure of $\varphi(L_0^r)$ in $M^{2n} \times G_n(\mathbb{R}^l)$.

Claim 2.1. $\hat{L}_0$ is a n-dimensional submanifold of $M^{2n} \times G_n(\mathbb{R}^l)$

Proof. Let q be a point of $\varphi(L_0^r)$. As φ is an embedding, we have a n-dimensional natural chart defined around q.

Let $q = (p, [V])$ be a point in $\mathrm{Cl}(\varphi(L_0^r)) \setminus \varphi(L_0^r)$. Obviously $p \in L_0 \setminus L_0^r$. We have to provide a system of coordinates in a neighborhood of $(p, [V])$. Let (k_h, k_f) be the Williamsom type of the point p. We know that there is a chart U in M^{2n}, with coordinates $(x_1, y_1, \ldots, x_n, y_n)$, centered at p, and $L_0 \cap U$ is given by $h_i = 0$, $i = 1, \ldots, n$, where the functions h_i are in the form of Section 2.1. Let us have a look at the tangent space to L_0 at a point near to p: from the expressions of the functions h_i, we see

$$L_0 \cap U = \prod_{j=1}^{s} (L_0^j \cap U^j),$$

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where each $L_0^j \cap U^j$ is of the form

(a) $U^j = D_2$, with coordinates (x_j, y_j), $h_j = y_j$, and $L_0^j \cap U^j$ is given by $h_j = 0$. These are the regular terms, and there are $n - k_h - 2k_f$ regular terms.
(b) $U^j = D_2$, with coordinates (x_j, y_j), $h_j = x_j y_j$, and $L_0^j \cap U^j$ is given by $h_j = 0$ (hyperbolic term).
(c) $U_j = D_2 \times D_2$, with coordinates $(x_j, y_j, x_{j+1}, y_{j+1})$, $h_j = x_j y_j + x_{j+1} y_{j+1}$, $h_{j+1} = x_j y_{j+1} - x_{j+1} y_j$, and $L_0^j \cap U^j$ is given by $h_j = h_{j+1} = 0$ (focus-focus term).

According to this, we give a system of coordinates around $(p, [V])$, by considering the above three possibilities:

In the case (a)

$$T_{(x_j,0)}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial x_j} \right\rangle,$$

where $\langle \frac{\partial}{\partial x_j} \rangle$ means the vector subspace generated by $\frac{\partial}{\partial x_j}$.

In this case $L_0^j \cap U^j$ is a regular submanifold,

$$T_{(0,0)}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial x_j} \right\rangle$$

so, $\langle \frac{\partial}{\partial x_j} \rangle \subset V$ and we take the function x_j as the coordinate, coming from the regular term, in a neighborhood of $(p, [V])$.

In the case (b)

$$T_{(0,y_j)}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial y_j} \right\rangle$$

or

$$T_{(x_j,0)}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial x_j} \right\rangle.$$

At the point $(0, 0) \in (L_0^j \cap U^j)$ we have $\left\langle \frac{\partial}{\partial y_j} \right\rangle \subset V$ or $\left\langle \frac{\partial}{\partial x_j} \right\rangle \subset V$, so y_j or x_j will be used as the coordinate, coming from the hyperbolic term, in a neighborhood of $(p, [V])$. In the case (c)

$$T_{(0,y_j,0,y_{j+1})}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_{j+1}} \right\rangle,$$

$$\text{or} \quad T_{(x_j,0,x_{j+1},0)}(L_0^j \cap U^j) = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_{j+1}} \right\rangle,$$

at the point $(0, 0, 0, 0) \in (L_0^j \cap U^j)$ we have $\left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_{j+1}} \right\rangle \subset V$ or $\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_{j+1}} \right\rangle \subset V$, so (y_j, y_{j+1}) or (x_j, x_{j+1}) will be used as the coordinates, coming from the focus-focus term, in a neighborhood of $(p, [V])$.

The system of coordinates around $(p, [V])$ is the result of summing up the above coordinates, and it is quite obvious the compatibility of the charts defined in this form.

The above considerations show us the C^∞-structure of $\hat{L}_0$, and provide us good systems of coordinates to use in the sequel. □

The map $j : \hat{L}_0 \longrightarrow M^{2n}$ given by $j(p, [V]) = p$ is differentiable, and we see that $j(\hat{L}_0) = L_0$. The preimage of any singular point on L_0 consists of $2^{k_h+k_f}$ points on $\hat{L}_0$. This map is a Lagrangian immersion in (M^{2n}, ω)

3. The singular cotangent model.

3.1. Affine structure on the desingularized level. The Poisson action of the local F-basic functions defines a singular affine structure ∇_0 on L_0. On each leaf of the level, the affine structure is defined by considering the infinitesimal generators of the $\mathbb{R}^n$-action as parallel vector fields. This $\mathbb{R}^n$-action can be lifted in a natural way to $\hat{L}_0$. Note that this lift is possible because the map $j : \hat{L}_0 \longrightarrow M^{2n}$ is an immersion. As the affine structure on L_0 is provided by the Hamiltonian vector fields $H_{f_1}|_{L_0} = X_1, \ldots, H_{f_n}|_{L_0} = X_n$ it seems natural to write as $\hat{X}_i$, $i = 1, \ldots, n$, the vector fields on $\hat{L}_0$ induced through the immersion j, and $\hat{\nabla}_0$ this affine structure.

By using the affine structure on $\hat{L}_0$ we are going to construct a completely integrable system on $(T^*\hat{L}_0, \hat{\omega}_0)$, where $\hat{\omega}_0$ is the standard symplectic two-form on $T^*\hat{L}_0$, and such that the affine structure on $\hat{L}_0$, induced by this c.i.s. is $\hat{\nabla}_0$.

In an ultimate step, by a process of gluing, using natural local identifications, we will obtain a c.i.s., such that L_0 is one level, and the affine structure on L_0 will be ∇_0.

3.2. A completely integrable system on $T^*\hat{L}_0$. We define n differentiable functions $g_1,\ldots,g_n$ on $T^*\hat{L}_0$ by

$$g_i((p, [V]), w) := \langle \hat{X}_i(p, [V]), w\rangle$$

and we will prove that $(T^*\hat{L}_0, \hat{\omega}_0, (g_1, \ldots, g_n))$ is completely integrable. To do it, we need only prove that $\{g_i, g_j\}|_0 = 0$, where $\{.,.\}|_0$ means the standard Poisson bracket in $T^*\hat{L}_0$. By a continuity argument, it will be sufficient to prove it at the points $((p, [V]), -) \in T^*\hat{L}_0$, where p is a regular point. We can take Eliasson coordinates $(x_1, y_1, \ldots, x_n, y_n)$ around the point p, and as any basic function only depends on $(y_1, \ldots, y_n)$ in a neighborhood of p, the expression of H_{f_i} in this neighborhood will be of the form $\sum_{j=1}^n \frac{\partial f_i}{\partial y_j}(0, \ldots, 0)\frac{\partial}{\partial x_j}$, i.e. is a vector fields with constant coefficients (which is obvious because the vector field is affine parallel).

Let α be the Liouville form on $T^*\hat{L}_0$. In this neighborhood, $\alpha = \sum_{i=1}^n x_i^* dx_i$, $g_i = \sum_{k=1}^n x_k^* \frac{\partial f_i}{\partial y_k}(0, \ldots, 0)$, so $dg_i = \sum_{k=1}^n \frac{\partial f_i}{\partial y_k}(0, \ldots, 0)dx_k^*$, and obviously

$$\Lambda^0(dg_i, dg_j) = 0.$$

Once we know that $(T^*\hat{L}_0, \hat{\omega}_0, (g_1, \ldots, g_n))$ is completely integrable, we remark that, in general, is not proper: let us assume, for instance that $\dim . L_0 = 1$, and let p be a hyperbolic point, we can write $\omega_0 = dx \wedge dy$, and $h = xy$, then $\{x = 0\}$ is a leaf.

For further considerations it seems suitable to look at the local expressions of the above functions g_i $(i = 1, \ldots, n)$: let $(p, [V])$ be a point in $\hat{L}_0$, where p is a point of L_0 of Williamson type (k_h, k_f), and let r be the number of regular terms at p. This means

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that in a system of E-coordinates $(x_1, \ldots, y_n)$ around p, we have

$$h_i = y_i, \quad i = 1, \ldots, r \quad \text{(regular terms)},$$

$$h_i = x_i y_i, \quad i = r+1, .., r+k_h \quad \text{(hyperbolic terms)},$$

$$\begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1} \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} \quad i = r+k_h+1, \ldots, r+k_h+2k_f-1 \quad \text{(focus-focus terms)}.$$

According to the proof of Claim 2.1, we can assume, without loss of generality, that

$$V = \bigoplus_{i=1}^{r} \left\langle \frac{\partial}{\partial x_i} \right\rangle \bigoplus_{i=r+1}^{r+k_h} \left\langle \frac{\partial}{\partial x_i} \right\rangle \bigoplus_{i=r+k_h+1}^{r+k_h+2k_f} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{i+1}} \right\rangle$$

So, the system of coordinates that we consider around $(p, [V])$ is

$$(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+k_h}; x_{r+k_h+1}, \ldots, x_{r+k_h+2k_f}).$$

In the above neighborhood of $p \in M$, any function f_i can be expressed as

$$f_i = f_i(h_1, \ldots, h_n),$$

so the expression of df_i is

$$df_i = \sum_{j=1}^{n} D_j f_i \cdot dh_j$$

and as the coefficients of this last expression are constant on each level, we have that in L_0

$$H_{f_i}|_{L_0} = \sum_{j=1}^{n} a_j^{(i)} \cdot H_{h_j}|_{L_0}, \quad a_j^{(i)} \in \mathbb{R}.$$

The determinant of the matrix $(a_j^{(i)})$ being non zero.

Finally, in the cotangent space of the above neighborhood of $(p, [V])$, endowed with canonical coordinates

$$(x_1, x_1^*, \ldots, x_r, x_r^*; x_{r+1}, x_{r+1}^*, \ldots; x_{r+k_h+1}, x_{r+k_h+1}^*, x_{r+k_h+2}, x_{r+k_h+2}^*, \ldots)$$

the expression of g_i is

$$\begin{aligned} g_i = a_1^{(i)} x_1^* + \cdots + a_{r+1}^{(i)} x_{r+1} x_{r+1}^* + \cdots + a_{r+k_h+1}^{(i)} (x_{r+k_h+1} x_{r+k_h+1}^* + x_{r+k_h+2} x_{r+k_h+2}^*) \\ + a_{r+k_h+2}^{(i)} (x_{r+k_h+1} x_{r+k_h+2}^* - x_{r+k_h+2} x_{r+k_h+1}^*). \end{aligned} \quad (1)$$

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3.3. The singular cotangent model. In order to have a proper completely integrable system around the level L_0 we have to define appropriate identifications between the germs of $T^*\hat{L}_0$ at the different preimages $j^{-1}(p)$ for any singular point $p \in L_0$, and we will obtain a germ of c.i.s. around L_0. To do it, we start by describing a neighborhood in L_0 of any singular point. We shall see that such a neighborhood admits a canonical decomposition as product of three terms, one corresponding to the regular part, one corresponding to the hyperbolic terms and the third one corresponding to the focus-focus terms.

Let p be a singular point of rank $r = n - (k_h + 2k_f)$. We know that there is a system of E-coordinates around p, and centered at p,

$$(x_1, \dots, x_r, y_1, \dots, y_r; x_{r+1}, y_{r+1}, \dots; x_{r+k_h+1}, x_{r+k_h+2}, y_{r+k_h+1}, y_{r+k_h+2}, \dots)$$

such that $(x_1, \dots, x_r, y_1, \dots, y_r)$ corresponds to the regular part, (x_{r+1}, y_{r+1}), $\dots, (x_{r+k_h}, y_{r+k_h})$ are associated to the hyperbolic branches, and $(x_{r+k_h+1}, x_{r+k_h+2}, y_{r+k_h+1}, y_{r+k_h+2}), \dots, (x_{r+k_h+2k_f-1}, x_{r+k_h+2k_f}, y_{r+k_h+2k_f-1}, y_{r+k_h+2k_f})$ correspond to the focus-focus branches.

We point out the following important remark: for any two systems of E-coordinates around the same point p, one can see that **the axes through p, given by the following expressions, do not depend on the chosen system of E-coordinates around p.** These axes are given by

$$\begin{array}{ccc} \{(0, \dots, 0, x_{r+1}, 0, \dots, 0)\} & \cup & \{(0, \dots, 0, 0, y_{r+1}, 0, \dots, 0)\} \\ & \dots\dots & \\ \{(0, \dots, 0, x_{r+k_h}, 0, \dots, 0)\} & \cup & \{(0, \dots, 0, y_{r+k_h}, 0, \dots, 0)\} \end{array}$$

these pairs of axes correspond to the hyperbolic components.

To see it one can observe that if $\{x'_1, y'_1, \dots\}$ is another system of E-coordinates around p. We can assume that a local affine parallel vector field like

$$X' = x'_{r+1}\frac{\partial}{\partial x'_{r+1}} - y'_{r+1}\frac{\partial}{\partial y'_{r+1}}$$

restricted to the level through p is expressed by a linear combination with more than one hyperbolic or focus-focus affine parallel vector field, let us assume, for instance, that its expression on the level is

$$X' = a\left(x_i\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial y_i}\right) + b\left(x_j\frac{\partial}{\partial x_j} - y_j\frac{\partial}{\partial y_j}\right), \quad i, j = r+1, \dots, r+k_h$$

for some non-zero constants a and b. By considering the locus of points where the vector field vanishes we see that necessarily a or b must be zero. A similar argument will exclude expressions of the type focus-focus in a decomposition like the above one. So the axes above described, and characterized by the fact that the point p is in the closure of the orbits of such vector field, so do not depend on the chosen E-coordinates.

We have just seen that any local affine vector field of the form

$$X' = x'_j\frac{\partial}{\partial x'_j} - y'_j\frac{\partial}{\partial y'_j}, \quad j = r+1, \dots, r+k_h$$

restricted to L_0 verifies

$$X' = a_l \left(x_l \frac{\partial}{\partial x_l} - y_l \frac{\partial}{\partial y_l} \right) \quad \text{(for a constant} \quad a_l), \quad l = r+1, \dots k_h.$$

Summarizing this information for all the hyperbolic components, we have seen that on the level L_0, and after a reordering of the hyperbolic components, the relation between the corresponding hyperbolic E-coordinates is

$$x'_i = b_i x_i^{\frac{1}{a_i}}, y'_i = c_i y_i^{\frac{1}{a_i}}, \quad i = r+1, \dots, r+k_h,$$

where a_i, b_i, c_i are constants. Looking at the expression of $dx_i \wedge dy_i$ which must coincide with $dx'_i \wedge dy'_i$ when $x_i = y_i = x'_i = y'_i = 0$, we see that $a_i = 1$ and $b_i = \frac{1}{c_i}$.

In order to attain the main result, we will use that on L_0 the coordinates x'_i, y'_i only depend on x_i, y_i, $(i = r+1, \dots, r+k_h)$, respectively.

Now we focus our study on the focus-focus part, and we shall give a similar result. We shall prove that the pairs of planes

$$\{(0, \dots, 0, x_{r+k_h+1}, x_{r+k_h+2}, 0, \dots, 0)\} \cup \{(0, \dots, 0, y_{r+k_h+1}, y_{r+k_h+2}, 0 \dots, 0)\}$$
$$\dots$$
$$\{0, \dots, 0, x_{r+k_h+2k_f-1}, 0, x_{r+k_h+2k_f}, 0)\} \cup \{(0, \dots, 0, y_{r+k_h+2k_f-1}, 0, y_{r+k_h+2k_f})\}$$

do not depend on the chosen E-coordinates. The proof is quite similar to the above one. We just point out the main steps of the proof: recall that for each component of the focus-focus part given by

$$\{(0, \dots, 0, x_l, x_{l+1}, 0, \dots, 0)\} \cup \{(0, \dots, 0, y_l, y_{l+1}, 0 \dots, 0)\},$$
$$l = r + k_h + 1, \dots, r + k_h + 2k_f - 1$$

the corresponding local basic functions are

$$h_l = x_l y_l + x_{l+1} y_{l+1}, \quad h_{l+1} = x_l y_{l+1} - y_l x_{l+1.}$$

By using the complex coordinates (z_l, z_{l+1}), with $z_l := x_l + i x_{l+1}$, $z_{l+1} := y_l + i y_{l+1}$. We have

$$h_l + i h_{l+1} = \bar{z}_l \cdot z_{l+1.}$$

The term of the canonical symplectic two-form is written as

$$dx_l \wedge dy_l + dx_{l+1} \wedge dy_{l+1} = \text{Re} \quad (d\bar{z}_l \wedge dz_{l+1}).$$

The Lagrangian foliation is obtained by considering the level surfaces of the function $\bar{z}_l \cdot z_{l+1}$.

Using this complex coordinates (z_l, z_{l+1}), the level through the point $(0, 0)$ is the union of the planes $\{z_l = 0\}$ and $\{z_{l+1} = 0\}$.

The Hamiltonian vector field H_{h_l} is

$$H_{h_l} = z_l \frac{\partial}{\partial z_l} + \bar{z}_l \frac{\partial}{\partial \bar{z}_l} - z_{l+1} \frac{\partial}{\partial z_{l+1}} - \bar{z}_{l+1} \frac{\partial}{\partial \bar{z}_{l+1}},$$

its flow is

$$\varphi_t(z_l, z_{l+1}) = (e^t z_l, e^{-t} z_{l+1}).$$

With respect to h_{l+1}, we see

$$H_{h_{l+1}} = iz_l \frac{\partial}{\partial z_l} - i\bar{z}_l \frac{\partial}{\partial \bar{z}_l} + iz_{l+1} \frac{\partial}{\partial z_{l+1}} - i\bar{z}_{l+1} \frac{\partial}{\partial \bar{z}_{l+1}}$$

Thus, its flow is

$$\psi_s(z_l, z_{l+1}) = (e^{is} z_l, e^{is} z_{l+1}).$$

Now, by considering a new system of E-coordinates, we can refer any focus-focus component for these new E-coordinates, in complex notation, as (z'_l, z'_{l+1}), and then look at the expressions of the local Hamiltonian vector fields corresponding to h'_l and h'_{l+1}. A similar argument referred to the locus of points where such vector fields vanish shows the coincidence up to order of the pair of planes (transverse planes) corresponding to both systems of E-coordinates.

With this argument we have seen that on the level L_0, and for any two pairs of 'focus-focus' functions corresponding to two systems of E-coordinates around the point p, and after reordering it is verified

$$H_{h_l} = aH_{h'_l} + bH_{h'_{l+1}}, \quad H_{h_{l+1}} = cH_{h'_l} + dH_{h'_{l+1}}, \quad l = r + 2k_h + 1, \ldots$$

for some constants a, b, c, d.

As the vector fields H'_{l+1} and H_{l+1} are associated to a Hamiltonian S^1-action, we have $c = 0$, $d = \pm 1$, we may assume $d = 1$.

This means, by using inner contraction with the symplectic two-form, that $\mathrm{Im}(d(\bar{z}_l z_{l+1}))= \mathrm{Im}(d(\bar{z}'_l z'_{l+1}))$.

By using once more the inner contraction of $H_{h_l} = aH_{h'_l} + bH_{h'_{l+1}}$ with the symplectic two-form we have

$$\mathrm{Re}(d(\bar{z}_l z_{l+1})) = a \cdot \mathrm{Im}(d(\bar{z}'_l z'_{l+1})) + b \cdot \mathrm{Re}(d(\bar{z}'_l z'_{l+1}))$$

In the plane $z_l = 0$ we have

$$\mathrm{Im}(z_{l+1} d\bar{z}_l) = \mathrm{Im}(z'_{l+1} d\bar{z}'_l)$$
$$\mathrm{Re}(z_{l+1} d\bar{z}_l) = a \cdot \mathrm{Im}(z'_{l+1} d\bar{z}'_l) + b \cdot \mathrm{Re}(z'_{l+1} d\bar{z}'_l)$$

In the plane $z_{l+1} = 0$ we have

$$\mathrm{Im}(z_l d\bar{z}_{l+1}) = \mathrm{Im}(z'_l d\bar{z}'_{l+1})$$
$$\mathrm{Re}(z_l d\bar{z}_{l+1}) = a \cdot \mathrm{Im}(z'_l d\bar{z}'_{l+1}) + b \cdot \mathrm{Re}(z'_l d\bar{z}'_{l+1})$$

By considering, for instance, the above equations corresponding to the plane $z_l = 0$, we obtain

$$\mathrm{Re}(z_l d\bar{z}_{l+1}) = a \cdot \mathrm{Im}(z_l d\bar{z}_{l+1}) + b \cdot \mathrm{Re}(z'_l d\bar{z}'_{l+1}) = a \cdot \mathrm{Im}(z_l d\bar{z}_{l+1}) + b \cdot \mathrm{Re}(z_l d\bar{z}_{l+1})$$

because $\mathrm{Re}(z_l d\bar{z}_{l+1}) = \mathrm{Re}(z'_l d\bar{z}'_{l+1})$, thus, we obtain

$$(1 - b) \cdot \mathrm{Re}(z_l d\bar{z}_{l+1}) = a \cdot \mathrm{Im}(z_l d\bar{z}_{l+1})$$

so, necessarily $a = 0$ and $b = 1$.

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We have proved that on the union of planes $\{z_l = 0\} \cup \{z_{l+1} = 0\}$ it is verified:

$$H_{h_l} = H_{h'_l} \quad \text{and} \quad H_{h_{l+1}} = H_{h'_{l+1}}.$$

On $\{z_l = 0 = z'_l\}$, this gives

$$iz_{l+1}\frac{\partial}{\partial z_{l+1}} - i\bar{z}_{l+1}\frac{\partial}{\partial \bar{z}_{l+1}} = iz'_{l+1}\frac{\partial}{\partial z'_{l+1}} - i\bar{z}'_{l+1}\frac{\partial}{\partial \bar{z}'_{l+1}}$$

$$z_{l+1}\frac{\partial}{\partial z_{l+1}} + \bar{z}_{l+1}\frac{\partial}{\partial \bar{z}_{l+1}} = z'_{l+1}\frac{\partial}{\partial z'_{l+1}} + \bar{z}'_{l+1}\frac{\partial}{\partial \bar{z}'_{l+1}},$$

by taking the inner contraction of the above two terms with the symplectic two form we see

$$z_{l+1}d\bar{z}_l = z'_{l+1}d\bar{z}'_l, \quad \text{on} \quad \{z_l = 0\}.$$

In a similar way, we see that on $\{z_{l+1} = z'_{l+1} = 0\}$ we have

$$iz_l\frac{\partial}{\partial z_l} - i\bar{z}_l\frac{\partial}{\partial \bar{z}_l} = iz'_l\frac{\partial}{\partial z'_l} - i\bar{z}'_l\frac{\partial}{\partial \bar{z}'_l}$$

which gives

$$\bar{z}_l dz_{l+1} = \bar{z}'_l dz'_{l+1}, \quad \text{on} \quad \{z_{l+1} = 0\}.$$

By a standard process of integration, one can see that the relations of the above two systems of complex coordinates on $\{z_l = z'_l = 0\} \cup \{z_{l+1} = z'_{l+1} = 0\}$ is

$$z\cdot_l = e^{\alpha+i\beta}\cdot z_l, \quad \text{for some constants} \quad \alpha \quad \text{and} \quad \beta, \quad \text{on} \quad z_{l+1} = 0$$

$$z'_{l+1} = e^{\gamma+i\delta}\cdot z'_{l+1}, \quad \text{for some constants} \quad \gamma \quad \text{and} \quad \delta, \quad \text{on} \quad z_l = 0.$$

Finally, we remark that as the 1-forms $\bar{z}_l dz_{l+1}$ and $\bar{z}'_l dz'_{l+1}$ coincide on $\{z_{l+1} = 0\}$, we have that $d\bar{z}_l \wedge dz_{l+1} = d\bar{z}'_l \wedge dz'_{l+1}$ at the point $z_l = z_{l+1} = 0$, so necessarily $\alpha + \gamma = 0$ and $\beta = \delta$.

So, for each focus-focus component, we have seen that the relation, on the level, of two systems of E-coordinates is, up to order, given by

$$z'_l = e^{\alpha+i\beta}z_l, \quad z'_{l+1} = e^{-\alpha+i\beta}z_{l+1}.$$

So, for any two systems of E-coordinates, we know the coincidence of the pairs of axes corresponding to the hyperbolic components, and the coincidence of the pairs of planes corresponding to the focus-focus components; **we are going to see that the regular parts also coincide**. To prove it we have to recall that on the level L_0, the relations between the corresponding hyperbolic and focus-focus components are given, as we have just seen, by

$$x'_i = b_i x_i, \quad y'_i = \frac{1}{b_i}y_i, \quad b_i \quad \text{constant}, \quad i = r+1, \ldots, r+k_h$$

for each hyperbolic component, and

$$z'_l = e^{\alpha_l + i\beta_l} z_l, \quad z'_{l+1} = e^{-\alpha_l + i\beta_l} z_{l+1}, \quad \alpha_l \text{ and } \beta_l \text{ constants}, \quad l = r + 2k_h + 1, \ldots$$

for each focus-focus component.

As a direct consequence of this fact, we have that the regular parts of the level, for both systems of E-coordinates are just the same. Of course, we can use the same functions $y_1, \ldots, y_r$ for both systems of E-coordinates.

As the singular composition of each point is, in fact, a product of regular, hyperbolic and focus-focus components, and we know the uniqueness up to order of this decomposition, we have to show now how this identification is done in hyperbolic and focus-focus cases. In the regular case the identification used will be trivial. To explain it in a detailed form, we start by considering two simple type of singularities, the hyperbolic and focus-focus cases, and finally we consider the general case.

3.3.1. Hyperbolic identification. Let $(p, [V_1]), (p, [V_2]) \in \hat{L}_0$, where $p \in L_0$ is a purely hyperbolic point (degree of hyperbolicity equal to one). Then in Eliasson coordinates (x, y) around the point p in M, we may assume $[V_1] = [\langle \frac{\partial}{\partial x} \rangle]$ and $[V_2] = [\langle \frac{\partial}{\partial y} \rangle]$. Recalling Claim 2.1, we shall use x and y as a system of coordinates in the chosen neighborhoods of $(p, [V_1])$ and $(p, [V_2])$ in $\hat{L}_0$.

We take the corresponding canonical coordinates in $T^*\hat{L}_0$ in two neighborhoods U_1 of $((p, [V_1]), 0)$, U_2 of $((p, [V_2]), 0)$, and we denote them by (x, x^*) and (y, y^*), respectively. Looking at the local expressions, that we have found previously for each function g_i, the basic functions will depend on xx^* and yy^*, respectively, and the canonical symplectic two forms are $dx \wedge dx^*$ and $dy \wedge dy^*$, respectively.

So, the symplectomorphism from U_1 onto U_2, we are searching for, is expressed in these coordinates by

$$\begin{cases} y = -x^* \\ y^* = x \end{cases} .$$

Note that this identification is determined by the fact that $\omega_0(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) > 0$.

We have to check that this identification does not depend on the system of Eliasson coordinates that we have chosen. At this point, we observe that two systems of E-coordinates (x, y) and (x', y') around p, and restricted to the axes $\{x = 0\} \cup \{y = 0\}$ are related unless a rotation of angle $n\frac{\pi}{2}$, by a change of the form

$$\begin{cases} x' = b \cdot x \\ y' = \frac{1}{b} \cdot y \end{cases}$$

and one can check easily that the above gluing is well defined, i.e. it is independent on the E-coordinates that we have chosen.

3.3.2. Focus-focus identification. We can proceed in a similar way for two points $(p, [V_1]), (p, [V_2])$, in the preimage of a focus-focus point $p \in L_0$. Regarding to the proof

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of Claim 2.1 one can consider

$$[V_1] = \left[\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle\right], \quad [V_2] = \left[\left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right\rangle\right].$$

We take canonical coordinates in $T^*\hat{L}_0$ in two neighborhoods U_1 of $(p, [V_1])$ and U_2 of $(p, [V_2])$, and we denote them by (x_1, x_2, x_1^*, x_2^*) and (y_1, y_2, y_1^*, y_2^*), respectively. As in the hyperbolic case, and attending to the local expressions of the functions g_i, the basic functions depend on $x_1x_1^* + x_2x_2^*$, $-x_1^*x_2 + x_1x_2^*$ and $y_1y_1^* + y_2y_2^*$, $-y_1^*y_2 + y_1y_2^*$, respectively. So, the symplectomorphism from U_1 to U_2 we need is given by

$$\begin{cases} y_1 = -x_1^* \\ y_2 = -x_2^* \\ y_1^* = x_1 \\ y_2^* = x_2 \end{cases}.$$

By using complex notation, as we made in the above considerations, we can take $z_1 = x_1 + ix_2$, $z_2 = y_1 + iy_2$. We know that

$$h_1 + ih_2 = \bar{z}_1 \cdot z_2$$
$$\omega_0 = \mathrm{Re} \quad d\bar{z}_1 \wedge dz_2$$

The Lagrangian foliation is given by the level surfaces of

$$\bar{z}_1 \cdot z_2.$$

The level through the point $(0, 0)$ is the union of the planes $\{z_1 = 0\}$ and $\{z_2 = 0\}$. The neighborhoods U_1 and U_2 are neighborhoods of the origin at the planes $\{z_2 = 0\}$ and $\{z_1 = 0\}$, respectively. The identification map we consider is given by

$$\begin{cases} z_2 = -z_1^* \\ z_2^* = z_1 \end{cases}.$$

Note that, as in the hyperbolic case, this identification is determined by the fact that $\omega_0(\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_2}) > 0$.

As in the hyperbolic case, we have to check that this identification does not depend on the chosen system of E-coordinates.

We recall that the relations of the E-coordinates on the level in this case is given by

$$z_1' = e^{\alpha+i\beta}z_l, \quad z_2' = e^{-\alpha+i\beta}z_2, \quad \alpha \quad \text{and} \quad \beta \quad \text{constants},$$

and one can check easily that the above gluing is well defined, i.e. it is independent on the E-coordinates that we have chosen.

3.3.3. General identification. Let p be a point of L_0 of Williamson type (k_h, k_f). Any system of E-coordinates around this point consists of r regular terms; following the notation introduced in Section 2.1, there are r functions $h_i = y_i$, defined in a regular block described by the coordinates $(x_1, y_1, \ldots, x_r, y_r)$. There are k_h hyperbolic function $h_i = x_i \cdot y_i$, this gives k_h hyperbolic blocks, described by the coordinates (x_i, y_i). Finally,

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there are $2k_f$ functions of the form $h_i = x_i y_i + x_{i+1} y_{i+1}$, $h_{i+1} = x_i y_{i+1} - y_i x_{i+1}$, this gives k_f focus-focus blocks described by the coordinates $(x_i, x_{i+1}, y_i, y_{i+1})$.

According to this, we have a local description (in a neighborhood of p) of the affine structure on a neighborhood of p in L_0, by considering the restriction to L_0 of the local Hamiltonian vector fields corresponding to the functions h_i. These Hamiltonian vector fields are

$$\frac{\partial}{\partial x_i} \quad \text{in the regular case,}$$

$$x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} \quad \text{in the hyperbolic case,}$$

$$\text{and the pair of vector fields} \quad x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}} - y_i \frac{\partial}{\partial y_i} - y_{i+1} \frac{\partial}{\partial y_{i+1}}$$

$$-x_{i+1} \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_{i+1}} - y_{i+1} \frac{\partial}{\partial y_i} + y_i \frac{\partial}{\partial y_{i+1}} \quad \text{in the focus-focus case.}$$

Now let $\hat{p}$ be a point of $j^{-1}(p)$. Recall that $j^{-1}(p)$ contains $2^{k_h+k_f}$ points, and we have a branch of $\hat{L}_0$ through any one of these points. Following Section 2.2 we can express any point $\hat{p} \in j^{-1}(p)$ as $\hat{p} = (p, [V])$,

$$V = \bigoplus_{i=1}^{r} \left\langle \frac{\partial}{\partial x_i} \right\rangle \bigoplus_{i=r+1}^{r+k_h} \left\langle \frac{\partial}{\partial \alpha_i} \right\rangle \bigoplus_{i=r+k_h+1}^{r+k_h+2k_f-1} \left\langle \frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \beta_{i+1}} \right\rangle.$$

Any term $\frac{\partial}{\partial \alpha_i}$ in a hyperbolic element means either $\frac{\partial}{\partial x_i}$ or $\frac{\partial}{\partial y_i}$. By a similar convention, any pair $\frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \beta_{i+1}}$ is either $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{i+1}}$ or $\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_{i+1}}$.

The system of coordinates to take in the branch of $\hat{L}_0$ through $\hat{p}$ is

$$(x_1, \ldots, x_r; \alpha_{r+1}, \ldots, \alpha_{r+k_h}; \beta_{r+k_h+1}, \ldots, \beta_{r+k_h+2k_k}).$$

REMARK 3.1. The branch through $\hat{p}$ is the product of an open neighborhood of 0 in $\mathbb{R}^r$ (the regular part of the branch) with k_h copies of an open interval of $\mathbb{R}$, which are the hyperbolic elements, and finally k_f copies of an open neighborhood of 0 in $\mathbb{R}^2$, which are the focus-focus elements.

Note that the affine structure of the branch is described by the vector fields

$$\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, r \quad \text{the regular part}$$

$$x_i \frac{\partial}{\partial x_i} \quad \text{or} \quad -y_i \frac{\partial}{\partial y_i}, \quad i = r+1, \ldots, r+k_h \quad \text{the hyperbolic part.}$$

And corresponding to the focus-focus part, we have one of the two pairs of vector fields

$$x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}}, \quad -x_{i+1} \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_{i+1}} \quad \text{or}$$

$$-y_i \frac{\partial}{\partial y_i} - y_{i+1} \frac{\partial}{\partial y_{i+1}}, \quad -y_{i+1} \frac{\partial}{\partial y_i} + y_i \frac{\partial}{\partial y_{i+1}}, \quad i = r+k_h+1, \ldots, r+k_h+2k_f-1.$$

Let us define now the local identification between two branches of $\hat{L}_0$ corresponding to two points $\hat{p}_1, \hat{p}_2 \in j^{-1}(p)$. Each one of these branches is a product of regular, hyperbolic and focus-focus components. The identification that we are searching for will act trivially on the regular component. For the common hyperbolic elements, the mapping to consider is obviously trivial, and if not the map is given by $y_i = -x_i^*, y_i^* = x_i$. For the focus-focus elements we proceed in a similar form: we take the identity if they coincide and if not we take the map given by

$$y_i = -x_i^*, \quad y_{i+1} = -x_{i+1}^*, \quad y_i^* = x_i, \quad y_{i+1}^* = x_{i+1}.$$

REMARK 3.2. Note that the regular part and each one of the hyperbolic and focus-focus elements are completely determined by the affine structure on L_0.

Note that this last remark ensures the compatibility of the local identifications defined.

At this point, we remark that the local expressions of the functions g_i (look at (1) in Section 3.2), are preserved by these gluings.

Summarizing, the gluing is well defined, and we remark that the local expressions of the functions g_i (look at (1) in Section 3.2), are preserved by these gluings.

Once we have shown how to identify the neighborhoods of the points in the same preimage we get a germ of $2n$-dimensional symplectic manifold (N, ω_0) containing L_0 as a singular Lagrangian submanifold. We have just seen that that the functions g_i, $i = 1, \dots, n$, can be projected to N. Let $G_1, \dots, G_n$ be the projected functions.

So, we have proved

THEOREM 3.1. $(N, \omega_0, (G_1, \dots, G_n))$ *is a completely integrable system, having* L_0 *as a singular level, and such that the singular affine structure on* L_0 *is the initial one. This is the c.i.s. that we say the linearized c.i.s. of the given* $(M^{2n}, \omega, (f_1, \dots f_n))$.

3.4. Construction of c.i.s. with prescribed singularities. As a sort of application of the above considerations, let us point out how to give some c.i.s. with prescribed singularities, around a singular level.

The intrinsic geometry of the level, i.e. the number and kind of singular points, is obviously related with the kind of singularities along its singular points.

We show how to obtain a c.i.s. around a 2-dimensional singular level with a focus-focus point and one circle of hyperbolic points:

Let us consider the 2-sphere S^2. This manifold S^2 will play the role of the desingularized level, $\hat{L}_0$, of the above sections. We shall consider T^*S^2 and two vector fields, with singularities, which will be used to define two functions on T^*S^2. These functions have singularities at several points, and by furnishing the local identifications at the corresponding singular points, we will get the c.i.s. around the singular level.

Let θ (longitude) and φ (latitude) be polar coordinates on S^2.The vector fields to consider are: $X = \frac{\partial}{\partial\theta}$ and $Y = h(\varphi)\frac{\partial}{\partial\varphi}$. We have to give a "good"expression for $h(\varphi)$, obviously we take $h(-\frac{\pi}{2}) = 0, h(\frac{\pi}{2}) = 0$, and $h(-\frac{\pi}{4}) = h(\frac{\pi}{4}) = 0$. We consider four open

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subsets of S^2

$$U_1 = \left\{(\theta,\varphi) \mid -\frac{\pi}{2} \leq \varphi < -\frac{\pi}{2} + \varepsilon\right\},$$
$$V_1 = \left\{(\theta,\varphi) \mid -\frac{\pi}{4} - \varepsilon < \varphi < \frac{\pi}{4} + \varepsilon\right\},$$
$$V_2 = \left\{(\theta,\varphi) \mid \frac{\pi}{4} - \varepsilon < \varphi < \frac{\pi}{4} + \varepsilon\right\},$$
$$U_2 = \left\{(\theta,\varphi) \mid \frac{\pi}{2} - \varepsilon < \varphi \leq \frac{\pi}{2}\right\},$$

where $\varepsilon < \frac{\pi}{8}$.

On U_1 and U_2we can take as coordinates the first two cartesian coordinates (x_1, x_2), and consider the vector fields:

on U_1, $x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} = -\frac{\cos\varphi}{\sin\varphi}\cdot\frac{\partial}{\partial\varphi}$ and $-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} = \frac{\partial}{\partial\theta}$,
on U_2, $-x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} = \frac{\cos\varphi}{\sin\varphi}\cdot\frac{\partial}{\partial\varphi}$ and $-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} = \frac{\partial}{\partial\theta}$.
On V_1 and V_2 we can take (θ,φ) as coordinates and the following vector fields:

on V_1, $-(\varphi + \frac{\pi}{4})\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\theta}$,
on V_2, $(\varphi - \frac{\pi}{4})\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\theta}$.
Now we see that the function $h(\varphi)$ we are searching for can be a differentiable function of φ, such that its values in U_1, V_1, U_2, V_2 are the above prescribed and not vanishing on $S^2 \setminus (U_1 \cup V_1 \cup U_2 \cup V_2)$.

Finally, we consider on T^*S^2 the pair of functions f, g associated to the vector fields X, Y, defined as follows: for any point $(z, w) \in T^*S^2$, $f(z, w) := \langle X(z), w\rangle$, $g(z, w) := \langle Y(z), w\rangle$. The c.i.s. we are searching for is obtained by a process of gluing from $(T^*S^2, \omega_0, (f, g))$. This gluing can be easily established by defining two symplectomorphisms: one of them is a local symplectomorphism between $(T^*U_1, (x_1 = 0, x_2 = 0, 0, 0))$ and $(T^*U_2, (X_1 = 0, X_2 = 0, 0, 0))$. We recall (see hyperbolic gluing in Section 3.2) that the mapping is given by

$$(x_1, x_2, y_1, y_2) \longmapsto (X_1 = -y_1, X_2 = -y_2, Y_1 = x_1, Y_2 = x_2).$$

The other symplectomorphism we need to realize the gluing is a semi-local symplectomorphism from $(T^*V_1, (\theta, \varphi = -\frac{\pi}{4}, 0, 0))$ in $T^*V_2, (\theta, \varphi = \frac{\pi}{4}, 0, 0))$, described as follows: on T^*V_1 we take $(\theta, \varphi + \frac{\pi}{4}, \Theta, \Phi)$ as canonical coordinates. In the same form, we take $(\bar\theta = \theta, \bar\varphi - \frac{\pi}{4} = \varphi - \frac{\pi}{4}, \bar\Theta = \Theta, \bar\Phi)$ as canonical coordinates in T^*V_2. The symplectomorphism we need to define the gluing is

$$\left(\theta, \varphi + \frac{\pi}{4}, \Theta, \Phi\right) \longmapsto \left(\bar\theta = \theta, \bar\varphi - \frac{\pi}{4} = \Phi, \bar\Theta = \Theta, \bar\Phi = -\left(\varphi + \frac{\pi}{4}\right)\right).$$

Thus, the quotient of a germ of neighborhood of S^2 in T^*S^2, by using these identifications provides us a germ of completely integrable system around a level having one singular point of focus-focus type and one circle of hyperbolic points.

One sees from this construction that the same arguments can serve to give c.i.s. with several points of focus-focus type; it should be necessary to use different copies of S^2 and define a gluing by using the poles alternatively. The above construction suggests different ways of having circles of hyperbolic points in the level.

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