

## STURM–LIOUVILLE PROBLEMS WITH REDUCIBLE BOUNDARY CONDITIONS

PAUL A. BINDING<sup>1</sup>, PATRICK J. BROWNE<sup>2</sup> AND BRUCE A. WATSON<sup>3</sup>

<sup>1</sup>*Department of Mathematics and Statistics, University of Calgary,  
Calgary, Alberta T2N 1N4, Canada (binding@ucalgary.ca)*

<sup>2</sup>*Mathematical Sciences Group, Department of Computer Science,  
University of Saskatchewan, Saskatoon, Saskatchewan S7N 5E6,  
Canada (browne@admin.usask.ca)*

<sup>3</sup>*School of Mathematics, University of the Witwatersrand,  
Private Bag 3, PO WITS 2050, South Africa  
(bwatson@maths.wits.ac.za)*

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*Abstract* The regular Sturm–Liouville problem

$$\tau y := -y'' + qy = \lambda y \quad \text{on } [0, 1], \lambda \in \mathbb{C},$$

is studied subject to boundary conditions

$$P_j(\lambda)y'(j) = Q_j(\lambda)y(j), \quad j = 0, 1,$$

where  $q \in L^1(0, 1)$  and  $P_j$  and  $Q_j$  are polynomials with real coefficients. A comparison is made between this problem and the corresponding ‘reduced’ one where all common factors are removed from the boundary conditions. Topics treated include Jordan chain structure, eigenvalue asymptotics and eigenfunction oscillation.

*Keywords:* Sturm–Liouville problem; eigenparameter-dependent boundary conditions; Jordan chain

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### 1. Introduction

This is the first of two papers in which we study certain factorization questions for the regular Sturm–Liouville problem

$$\tau y := -y'' + qy = \lambda y \quad \text{on } [0, 1], \lambda \in \mathbb{C}, \quad (1.1)$$

subject to boundary conditions

$$P_j(\lambda)y'(j) = Q_j(\lambda)y(j), \quad j = 0, 1, \quad (1.2)$$

where  $q \in L^1(0, 1)$  and  $P_j$  and  $Q_j$  are polynomials with real coefficients. Here we discuss the problem from the differential equation viewpoint, and factorization enters, at least

initially, via an appropriately defined characteristic function (see (2.3)). It turns out that some of the ensuing analysis will also be important for our sequel, where we shall consider an operator theoretic setting, and, in particular, operator factorization.

Sturm–Liouville problems with  $\lambda$ -dependent boundary conditions have been studied by many authors. The reference lists of [5, 9, 10, 17] give some idea of the scope, both applied and theoretical, of this activity. While most investigations have concerned polynomials  $P_j$ ,  $Q_j$  of degree 1, various authors (see, for example, [2, 7, 8, 16]) have studied polynomials of higher degree. In general, these conditions have been assumed in reduced form, i.e. such that, for each fixed  $j$ ,  $P_j$  and  $Q_j$  have no common factors. In [3, 13], however, mention was made of the case in which  $P_1$  and  $Q_1$  have a common root. Moreover, as will be seen in the sequel, even when (1.2) is in reduced form, the operator factorization mentioned above naturally leads to associated non-reduced problems.

More specifically, operator factorization and commutation (see, for example, [6]) have provided a modern perspective on the Darboux–Crum transformation, which has been widely used (see, for example, [14]). It turns out that the transformation is ‘almost’ isospectral [6, Theorem 12]. In the sequel, we shall study the analogous factorization and commutation for an operator formulation of (1.1), (1.2) (see [16]). It will be seen that this necessarily leads to problems with reducible boundary conditions even in the classical case when (1.2) is independent of  $\lambda$ . Moreover, the original problem has the same spectrum as the transformed reducible one, whereas the reduced version of the transformed problem is again ‘almost’ isospectral to the original. In the following we shall discuss the relations between these three problems in detail. Here we study reducible and reduced problems in their own right and this will also serve as a foundation for our second paper.

An eigenvalue for our problem is a value of the parameter  $\lambda$  for which (1.1), (1.2) can be solved non-trivially for  $y$ . Even though the boundary conditions are separated, it turns out that the eigenvalues may be (geometrically) non-simple and/or (algebraically) non-semisimple. For example, if  $P_0$  and  $Q_0$  have a common root  $\xi$ , then  $\lambda = \xi$  will be an eigenvalue since (1.2) is satisfied for  $j = 0$  and  $\lambda = \xi$ . It is therefore sufficient for  $y$  to satisfy (1.1), and the terminal condition (1.2) with  $j = 1$ , at  $\lambda = \xi$ . If  $\xi$  is also a root of  $P_1$  and  $Q_1$ , then  $\xi$  will be an eigenvalue of geometric multiplicity 2.

Suppose, for some  $\lambda$ , that it is possible to construct a Jordan chain of length  $m$  of associated eigenfunctions, i.e. that there exist functions  $y^{[0]}, \dots, y^{[m-1]}$  satisfying

$$(\tau - \lambda)y^{[0]} = 0, \quad (\tau - \lambda)y^{[r]} = y^{[r-1]}, \quad (1.3)$$

and the boundary conditions

$$0 = \sum_{i=0}^r \frac{1}{i!} [P_j^{(i)}(\lambda)y^{[r-i]'}(j) - Q_j^{(i)}(\lambda)y^{[r-i]}(j)], \quad j = 0, 1, \quad (1.4)$$

for  $r = 1, \dots, m - 1$ . If  $m$  is the greatest integer with these properties, then  $\lambda$  has algebraic multiplicity  $m$  (see [15, § 2.3] for details). If  $m = 1$ , then  $\lambda$  is called semisimple. As we shall see, regardless of their geometric multiplicity, eigenvalues may or may not be semisimple.

By definition, the ‘reduced’ problem arising from (1.1), (1.2) is that in which (1.2) is replaced by

$$\hat{P}_j(\lambda)y'(j) = \hat{Q}_j(\lambda)y(j), \quad j = 0, 1, \tag{1.5}$$

where  $\hat{P}_j$  and  $\hat{Q}_j$  are obtained from  $P_j$  and  $Q_j$  by the removal of all common factors of the form  $(\lambda - \xi)^k$ . Thus,  $\hat{P}_j$  and  $\hat{Q}_j$  have no common roots.

In § 2 we study the relationship between the Jordan chain structures of (1.1), (1.2) and the reduced problem. It turns out that there are interesting ‘interference’ effects when there are common roots of (1.2) that are also eigenvalues of the reduced problem (see Theorem 2.3). We give asymptotic developments for the eigenvalues of (1.1), (1.2) in § 3. The results resemble those for the standard case of  $\lambda$ -independent boundary conditions but there is now an index shift. The associated oscillation theory for the eigenfunctions is presented in § 4. Again there is an index shift, but it may be different from the previous one. To the best of the authors’ knowledge, these results are new, at least in this generality. We conclude in § 5 with an example which illustrates some of the above ideas.

## 2. Comparison of Jordan structures

Let  $\varphi$  and  $\psi$  be the solutions of (1.1) satisfying the initial conditions

$$\varphi(0, \lambda) = 1, \quad \psi(0, \lambda) = 0, \tag{2.1}$$

$$\varphi'(0, \lambda) = 0, \quad \psi'(0, \lambda) = 1, \tag{2.2}$$

and define the characteristic function  $D$  by

$$D(\lambda) = P_1(\lambda)[Q_0(\lambda)\psi'(1, \lambda) + P_0(\lambda)\varphi'(1, \lambda)] - Q_1(\lambda)[Q_0(\lambda)\psi(1, \lambda) + P_0(\lambda)\varphi(1, \lambda)]. \tag{2.3}$$

Then  $D(\lambda)$  is an entire function of order  $\frac{1}{2}$  since  $\varphi, \varphi', \psi$  and  $\psi'$  are of order  $\frac{1}{2}$  [11, Appendix]. The eigenvalues of (1.1), (1.2) are the solutions of  $D(\lambda) = 0$ , here the order,  $\nu(\lambda)$ , of the zero of  $D$  at  $\lambda$  coincides with the algebraic multiplicity of  $\lambda$  as an eigenvalue of (1.1), (1.2) (see [15, § 2.3]).

For an entire function  $f$ , let  $\mu(f; \lambda)$  denote the order of  $\lambda$  as a root of  $f = 0$ , with  $\mu(f; \lambda) = 0$  if  $f(\lambda) \neq 0$ . Thus, as above, we have

$$\nu(\lambda) = \mu(D; \lambda), \tag{2.4}$$

and we also define

$$\zeta_j(\lambda) = \min\{\mu(P_j; \lambda), \mu(Q_j; \lambda)\}. \tag{2.5}$$

In general, we shall use  $\nu(\lambda)$  to denote the algebraic multiplicity of  $\lambda$  for arbitrary  $\lambda \in \mathbb{C}$  with  $\nu(\lambda) = 0$  indicating that  $\lambda$  is not an eigenvalue of (1.1), (1.2). The spectrum of (1.1), (1.2) then consists of all eigenvalues repeated according to algebraic multiplicity.

In what follows, it will be convenient to regard chains of length 0 as non-existent. For example, if the Jordan structure at  $\lambda$  consists of two chains of lengths 0 and  $m > 0$ , it is to be understood that  $\lambda$  has only one chain (of length  $m$ ).

Our first result compares the Jordan structure at  $\lambda = \xi$  of two problems, one of which is partially reduced at  $\xi$  in the sense that  $P_j$  and  $Q_j$  do not have  $\xi$  as a common root, while the other has  $P_j(\lambda)$  and  $Q_j(\lambda)$  replaced by  $(\lambda - \xi)^{k_j} P_j(\lambda)$  and  $(\lambda - \xi)^{k_j} Q_j(\lambda)$ .

**Proposition 2.1.** *Let  $\xi \in \mathbb{C}$  be such that  $|P_j(\xi)| + |Q_j(\xi)| \neq 0, j = 0, 1$ . If  $\nu(\xi) = m$ , then the boundary-value problem with equation (1.1) and boundary conditions*

$$(\lambda - \xi)^{k_j} P_j(\lambda)y'(j) = (\lambda - \xi)^{k_j} Q_j(\lambda)y(j), \quad j = 0, 1, \tag{2.6}$$

where  $k_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, k_0 + k_1 \geq 1$ , has  $\xi$  as an eigenvalue of algebraic multiplicity  $m + k_0 + k_1$ . The Jordan structure at  $\xi$  for (1.1), (2.6) consists of two chains of lengths  $m + \max\{k_0, k_1\}$  and  $\min\{k_0, k_1\}$ . The chain of greater length commences with a solution of  $\tau y = \xi y$  that satisfies (1.1) for  $j = 0$  (respectively,  $j = 1$ ) when  $k_0 \leq k_1$  (respectively,  $k_0 \geq k_1$ ).

**Proof.** Without loss of generality we assume  $0 \leq k_0 \leq k_1$ . Let

$$y_1^{[0]}(x) = P_0(\xi)\varphi(x, \xi) + Q_0(\xi)\psi(x, \xi)$$

such that  $y_1^{[0]}$  satisfies (1.2) for  $\lambda = \xi, j = 0$ . Let  $y_0^{[0]}$  be another solution of  $\tau y = \xi y$  linearly independent of  $y_1^{[0]}$  and put  $z^{[0]} = Ay_1^{[0]} + By_0^{[0]}$ . Now choose a particular solution  $z^{[r]}$  to  $\tau z^{[r]} = \xi z^{[r]} + z^{[r-1]}, r = 1, 2, \dots$  (for example, using variation of parameters with  $y_0^{[0]}, y_1^{[0]}$  as a fundamental pair of solutions to  $\tau y = \xi y$ ).

We shall show that the  $z^{[r]}$  form a Jordan chain for (1.1), (2.6) of length  $m + k_1$  if  $B = 0$ , and of length  $k_0$  otherwise. To this end, we examine the boundary conditions

$$0 = \sum_{i=0}^r \frac{1}{i!} \frac{\partial^i}{\partial \lambda^i} [(\lambda - \xi)^{k_j} [P_j(\lambda)z^{[r-i]'}(j) - Q_j(\lambda)z^{[r-i]}(j)]]_{\lambda=\xi}, \tag{2.7}$$

for  $j = 0, 1$  and  $r = 0, 1, \dots$ . For  $r = 0, \dots, k_0 - 1$ , (2.7) is automatically satisfied, while for  $r = k_0, j = 0$ , (2.7) becomes

$$P_0(\xi)z^{[0]'}(0) - Q_0(\xi)z^{[0]}(0) = 0,$$

which holds only if  $B = 0$ . Hence, when  $B \neq 0$ , the chain  $z^{[0]}, \dots, z^{[k_0-1]}$  cannot be extended without violating (2.7).

Assume, then, that  $B = 0$  and, without loss of generality,  $A = 1$  such that  $z^{[r]} = y_1^{[r]}$ . For  $r \geq k_0$ , the right-hand side of (2.7) becomes

$$\begin{aligned} & \sum_{i=0}^r \frac{1}{i!} \sum_{p=0}^i \binom{i}{p} \left[ \frac{\partial^{i-p}}{\partial \lambda^{i-p}} (\lambda - \xi)^{k_j} \frac{\partial^p}{\partial \lambda^p} [P_j(\lambda)y_1^{[r-i]'}(j) - Q_j(\lambda)y_1^{[r-i]}(j)] \right]_{\lambda=\xi} \\ &= \sum_{i=k_j}^r \frac{1}{i!} \binom{i}{i-k_j} k_j! [P_j^{(i-k_j)}(\xi)y_1^{[r-i]'}(j) - Q_j^{(i-k_j)}(\xi)y_1^{[r-i]}(j)] \\ &= \sum_{i=k_j}^r \frac{1}{(i-k_j)!} [P_j^{(i-k_j)}(\xi)y_1^{[r-i]'}(j) - Q_j^{(i-k_j)}(\xi)y_1^{[r-i]}(j)] \\ &= \sum_{i=0}^{r-k_j} \frac{1}{i!} [P_j^{(i)}(\xi)y_1^{[(r-k_j)-i]'}(j) - Q_j^{(i)}(\xi)y_1^{[(r-k_j)-i]}(j)]. \end{aligned} \tag{2.8}$$

By the construction of the functions  $y_1^{[r]}$ , (2.8) vanishes for  $j = 0$  and  $r = k_0, \dots$ . Recall that  $y_1^{[0]}, \dots, y_1^{[m-1]}$  is a Jordan chain at  $\xi$  for (1.1), (1.2) and hence for  $j = 1$ , (2.8) vanishes for  $r - k_1 = 0, \dots, m - 1$ . Thus, relative to (1.1), (1.5),  $\xi$  has a Jordan chain, commencing with  $y_1^{[0]}$ , of length at least  $m + k_1$ . If it were possible to extend this chain, then we could find  $y_1^{[m+k_1]}$  satisfying (2.7) for  $j = 1$  and  $r = m + k_1$ . However, (2.8) would then show that, relative to (1.1), (1.2),  $\nu(\xi) > m$ : a contradiction.  $\square$

The next result shows that the two problems (1.1), (1.2) and (1.1), (2.6) have Jordan structures which differ only at  $\xi$ .

**Proposition 2.2.** *Suppose that for (1.1), (1.2),  $\xi$  has Jordan chains*

$$C_s : y_s^{[0]}, \dots, y_s^{[m_s-1]}, \quad m_s \geq 0, \quad s = 0, 1.$$

Let  $f_j$  be entire functions with  $f_j(\xi) \neq 0$ ,  $j = 0, 1$ . Then  $C_0$  and  $C_1$  are also Jordan chains for (1.1) with boundary conditions

$$f_j(\lambda)P_j(\lambda)y'(j) = f_j(\lambda)Q_j(\lambda)y(j), \quad j = 0, 1.$$

**Proof.** Since  $\tau y_s^{[0]} = \xi y_0^{[0]}$  and  $\tau y_s^{[r]} = \xi y_s^{[r]} + y_s^{[r-1]}$  for  $r = 1, \dots, m_s - 1$ , it remains to show that, for  $s = 0, 1$ ,

$$\sum_{i=0}^r \frac{1}{i!} \frac{\partial^i}{\partial \lambda^i} [f_j(\lambda)[P_j(\lambda)y_s^{[r-i]'}(j) - Q_j(\lambda)y_s^{[r-i]}(j)]]_{\lambda=\xi} \tag{2.9}$$

vanishes for  $r = 0, \dots, m_s - 1$ ,  $j = 0, 1$ , and that, for any solution  $y_s^{[m_s]}$  of

$$\tau y_s^{[m_s]} = \xi y_s^{[m_s]} + y_s^{[m_s-1]},$$

(2.9) does not vanish for at least one value of  $j = 0, 1$ . Now (2.9) can be expressed as

$$\begin{aligned} & \sum_{i=0}^r \frac{1}{i!} \frac{\partial^i}{\partial \lambda^i} [f_j(\lambda)[P_j(\lambda)y_s^{[r-i]'}(j) - Q_j(\lambda)y_s^{[r-i]}(j)]]_{\lambda=\xi} \\ &= \sum_{i=0}^r \frac{1}{i!} \sum_{k=i}^i \binom{i}{k} f_j^{(k)}(\xi) [P_j^{(i-k)}(\xi)y_s^{[r-i]'}(j) - Q_j^{(i-k)}(\xi)y_s^{[r-i]}(j)] \\ &= \sum_{k=0}^r \frac{f_j^{(k)}(\xi)}{k!} \sum_{i=k}^r \left[ \frac{P_j^{(i-k)}(\xi)}{(i-k)!} y_s^{[r-i]'}(j) - \frac{Q_j^{(i-k)}(\xi)}{(i-k)!} y_s^{[r-i]}(j) \right] \\ &= \sum_{k=0}^r \frac{f_j^{(k)}(\xi)}{k!} \left[ \sum_{n=0}^{r-k} \left( \frac{P_j^{(n)}(\xi)}{n!} y_s^{[(r-k)-n]'}(j) - \frac{Q_j^{(n)}(\xi)}{n!} y_s^{[(r-k)-n]}(j) \right) \right]. \tag{2.10} \end{aligned}$$

For  $r = 0, \dots, m_s - 1$ , the internal summation in (2.10) vanishes since  $C_s$  is a Jordan chain for (1.1), (1.2) of length  $m_s$ . If  $y_s^{[m_s]}$  is found with the properties described, then,

for  $r = m_s$ , (2.10) reduces to

$$f_j(\xi) \sum_{n=0}^{m_s} \left( \frac{P_j^{(n)}(\xi)}{n!} y_s^{[m_s-n]'}(j) - \frac{Q_j^{(n)}(\xi)}{n!} y_s^{[m_s-n]}(j) \right).$$

This cannot vanish for both  $j = 0$  and  $j = 1$ , since  $f_j(\xi) \neq 0$ , by hypothesis, and  $C_s$  has length  $m_s$ .  $\square$

The combination of the two propositions above leads to the main result of this section, in which we compare the problem (1.1), (1.2) with its reduced version (1.1), (1.5).

**Theorem 2.3.** *For the reduced problem (1.1), (1.5), let  $\hat{\nu}(\lambda)$  denote the algebraic multiplicity of  $\lambda$ . Then, in the notation of (2.4) and (2.5),*

$$\nu(\lambda) = \hat{\nu}(\lambda) + \zeta_0(\lambda) + \zeta_1(\lambda).$$

All eigenvalues of (1.1), (1.5) are geometrically simple and relative to (1.1), (1.2), the Jordan structure at  $\lambda$  consists of two chains with lengths  $\min\{\zeta_0(\lambda), \zeta_1(\lambda)\}$  and  $\hat{\nu}(\lambda) + \max\{\zeta_0(\lambda), \zeta_1(\lambda)\}$ . The chain of greater length must have as its zeroth element a solution of (1.1) which satisfies (1.5) for  $j = 0$  (respectively,  $j = 1$ ) if  $\zeta_0(\lambda) \leq \zeta_1(\lambda)$ , (respectively,  $\zeta_0(\lambda) \geq \zeta_1(\lambda)$ ).

**Proof.** For (1.1), (1.5) let  $\hat{D}(\lambda)$  denote the expression corresponding to  $D(\lambda)$  for (1.1), (1.2). Then, clearly,

$$\mu(D; \lambda) = \mu(\hat{D}; \lambda) + \zeta_0(\lambda) + \zeta_1(\lambda).$$

Since  $\nu(\lambda) = \mu(D; \lambda)$  and  $\hat{\nu}(\lambda) = \mu(\hat{D}; \lambda)$ , the first claim is immediate. The remainder of the theorem follows from Propositions 2.1 and 2.2.  $\square$

### 3. Spectral asymptotics

In this section we produce asymptotic expressions for the eigenvalues,  $\lambda_n$ , of (1.1), (1.2). This extends the results of [4], where only reduced problems were considered, and albeit with only one  $\lambda$ -independent boundary condition. The eigenvalues of (1.1), (1.2) will be labelled  $\lambda_n$ ,  $n \geq 0$ , repeated according to algebraic multiplicity and listed by increasing real parts.

We write

$$P_j(\lambda) = a_0^j \lambda^{d_j} + \cdots + a_{d_j}^j, \quad (3.1)$$

$$Q_j(\lambda) = b_0^j \lambda^{d_j} + \cdots + b_{d_j}^j, \quad (3.2)$$

where  $d_j = \max\{\deg(P_j), \deg(Q_j)\}$ , such that  $|a_0^j| + |b_0^j| \neq 0$ ,  $j = 0, 1$ .

**Theorem 3.1.** For sufficiently large  $n$ ,  $\lambda_n$  is real and is given by

$$\lambda_n = \begin{cases} \hat{n}^2\pi^2 - 2\frac{b_0^1}{a_0^1} + 2\frac{b_0^0}{a_0^0} + \int_0^1 q + O\left(\frac{1}{n}\right), & a_0^0 \neq 0 \neq a_0^1, \\ (\hat{n} + \frac{1}{2})^2\pi^2 + 2\frac{a_1^1}{b_0^1} + 2\frac{b_0^0}{a_0^0} + \int_0^1 q + O\left(\frac{1}{n}\right), & a_0^0 \neq 0 = a_0^1, \\ (\hat{n} + \frac{1}{2})^2\pi^2 - 2\frac{b_0^1}{a_0^1} - 2\frac{a_1^0}{b_0^0} + \int_0^1 q + O\left(\frac{1}{n}\right), & a_0^0 = 0 \neq a_0^1, \\ (\hat{n} + 1)^2\pi^2 + 2\frac{a_1^1}{b_0^1} - 2\frac{a_1^0}{b_0^0} + \int_0^1 q + O\left(\frac{1}{n}\right), & a_0^0 = 0 = a_0^1, \end{cases}$$

where  $\hat{n} = n - d_0 - d_1$ .

**Proof.** We rely on the following asymptotic expressions as  $|\lambda| \rightarrow \infty$  for the functions  $\varphi(1, \lambda)$ ,  $\varphi'(1, \lambda)$ ,  $\psi(1, \lambda)$  and  $\psi'(1, \lambda)$  (see [11] for details):

$$\begin{aligned} \varphi(1, \lambda) &= \cos \sqrt{\lambda} + \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} \int_0^1 q + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\lambda}\right), \\ \varphi'(1, \lambda) &= -\sqrt{\lambda} \sin \sqrt{\lambda} + \frac{\cos \sqrt{\lambda}}{2} \int_0^1 q + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\sqrt{\lambda}}\right), \\ \psi(1, \lambda) &= \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} - \frac{\cos \sqrt{\lambda}}{2\lambda} \int_0^1 q + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\lambda^{3/2}}\right), \\ \psi'(1, \lambda) &= \cos \sqrt{\lambda} + \frac{\sin \sqrt{\lambda}}{2\sqrt{\lambda}} \int_0^1 q + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\lambda}\right). \end{aligned}$$

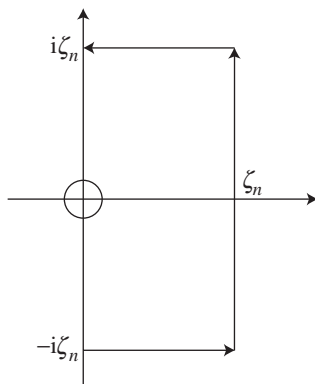
The expressions for  $\psi(1, \lambda)$ ,  $\psi'(1, \lambda)$  given in [11] involve only the leading terms above, but the more accurate asymptotics can readily be obtained using the methods of [11] by which the quoted expressions for  $\varphi(1, \lambda)$  and  $\varphi'(1, \lambda)$  were generated (see also [4, Appendix]).

We may write

$$\begin{aligned} P_j(\lambda) &= \lambda^{d_j} \left( a_0^j + \frac{a_1^j}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right), \\ Q_j(\lambda) &= \lambda^{d_j} \left( b_0^j + \frac{b_1^j}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right). \end{aligned}$$

Consider the cases for which  $a_0^0 \neq 0$ . Then (2.3) and the above expressions lead to

$$\frac{-D(\lambda)}{a_0^0 \lambda^{d_0+d_1}} = \begin{cases} a_0^1 \left[ \sqrt{\lambda} \sin \sqrt{\lambda} + \left( \frac{b_0^1}{a_0^1} - \frac{b_0^0}{a_0^0} - \frac{1}{2} \int_0^1 q \right) \cos \sqrt{\lambda} + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\sqrt{\lambda}}\right) \right], & a_0^1 \neq 0, \\ b_0^1 \left[ \cos \sqrt{\lambda} + \left( \frac{b_0^0}{a_0^0} + \frac{a_1^1}{b_0^1} + \frac{1}{2} \int_0^1 q \right) \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\lambda}\right) \right], & a_0^1 = 0. \end{cases} \tag{3.3}$$

Figure 1.  $\tau_n$  in the  $\xi$ -plane.

When  $a_0^1 \neq 0$ , we put

$$c = \frac{b_0^1}{a_0^1} - \frac{b_0^0}{a_0^0} - \frac{1}{2} \int_0^1 q$$

and write (3.3) as

$$\frac{D(\lambda)}{\lambda^{d_0+d_1}} = -a_0^1 a_0^0 \left[ \sqrt{\lambda} \sin \sqrt{\lambda} + c \cos \sqrt{\lambda} + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\sqrt{\lambda}}\right) \right]. \quad (3.4)$$

For large  $n \in \mathbb{N}$ ,

$$D\left(\left(n + \frac{1}{2}\right)^2 \pi^2\right) = -a_0^1 a_0^0 (n\pi + \frac{1}{2}\pi)^{2(d_0+d_1)+1} \left[ (-1)^n + O\left(\frac{1}{n}\right) \right],$$

which oscillates in sign with  $n$ . Thus, there is  $\mu_n = n\pi + \delta_n$ ,  $|\delta_n| < \frac{1}{2}\pi$  with  $D(\mu_n^2) = 0$ . From (3.4) we obtain

$$(n\pi + \delta_n)(-1)^n \sin \delta_n + c(-1)^n \cos \delta_n = O\left(\frac{1}{n}\right), \quad (3.5)$$

showing that  $\sin \delta_n = O(1/n)$  and, hence,  $\delta_n = \gamma_n/n$ , where  $\gamma_n = O(1)$ . Now (3.5) yields

$$\gamma_n = -\frac{c}{\pi} + O\left(\frac{1}{n}\right),$$

and, hence,

$$\mu_n = n\pi - \frac{c}{n\pi} + O\left(\frac{1}{n^2}\right)$$

which gives the existence of eigenvalues of the desired asymptotic form. It remains to show that all eigenvalues of large modulus are of the form  $\mu_n^2$  and to determine the index of  $\mu_n^2$  in the listing  $\lambda_0, \lambda_1, \dots$ . To this end we use Rouché's theorem.

Let  $R$  be the entire function

$$R(\lambda) = -a_0^1 a_0^0 \lambda^{d_0+d_1} \sqrt{\lambda} \sin \sqrt{\lambda}$$



and let  $\Gamma_n$  be the path in the complex plane

$$\Gamma_n = \{\lambda = \xi^2 \mid \xi \in \tau_n\},$$

where  $\tau_n$  is the path in the  $\xi$ -plane connecting  $-i\zeta_n$  to  $i\zeta_n$  by three line segments, as shown in Figure 1. Here

$$\zeta_n = (n - \frac{1}{2})\pi, \quad n = 1, 2, 3, \dots$$

There is a positive constant  $\kappa$  (independent of  $n$ ) such that, for  $\lambda \in \Gamma_n$ ,

$$|R(\lambda)| \geq \kappa |\sqrt{\lambda}|^{2d_0+2d_1+1} e^{|\text{Im } \sqrt{\lambda}|},$$

and, as

$$D(\lambda) - R(\lambda) = O(\lambda^{d_0+d_1} e^{|\text{Im } \sqrt{\lambda}|}),$$

on  $\Gamma_n$ , for  $n$  large, we have

$$|D(\lambda) - R(\lambda)| < |R(\lambda)|.$$

Rouché’s theorem shows that  $D$  and  $R$  have the same number of zeros enclosed by  $\Gamma_n$ , namely  $n + d_0 + d_1$ . Moreover, the region between  $\Gamma_n$  and  $\Gamma_{n+1}$  contains precisely one zero of  $D$ , namely  $\mu_n^2$ . The upshot is that the  $\mu_n^2$  form all the zeros with large modulus of  $D$ , and that

$$\lambda_{d_0+d_1+n} = \mu_n^2.$$

This completes the proof for the case  $a_0^0 a_0^1 \neq 0$ .

For the case  $a_0^0 \neq 0 = a_0^1$  (such that  $b_0^1 \neq 0$ ), we observe from (3.3) that  $D(n^2\pi^2)$  oscillates in sign with  $n$ , so that  $D(\mu_n^2) = 0$ , where

$$\mu_n = (n + \frac{1}{2})\pi + \delta_n,$$

with  $|\delta_n| < \frac{1}{2}\pi$ . Setting

$$c = \frac{b_0^0}{a_0^0} + \frac{a_1^1}{b_0^1} + \frac{1}{2} \int_0^1 q,$$

we arrive at

$$(-1)^{n+1} \sin \delta_n + \frac{c(-1)^n \cos \delta_n}{(n + \frac{1}{2})\pi + \delta_n} = O\left(\frac{1}{n^2}\right),$$

such that

$$\delta_n = \frac{\gamma_n}{(n + \frac{1}{2})\pi}, \quad \text{where } \gamma_n = O(1).$$

It then follows that  $\gamma_n = c + O(1/n)$  and

$$\mu_n^2 = (n + \frac{1}{2})^2 \pi^2 + 2 \frac{a_1^1}{b_0^1} + 2 \frac{b_0^0}{a_0^0} + \int_0^1 q + O\left(\frac{1}{n}\right).$$

The Rouché argument now uses the entire function

$$R(\lambda) = -b_0^1 a_0^0 \lambda^{d_0+d_1} \cos \sqrt{\lambda}$$

and the path  $\Gamma_n$  with  $\zeta_n = n\pi$ . We leave the remaining details to the reader.

When  $a_0^0 = 0 \neq a_0^1$  (so that  $b_0^0 \neq 0$ ) we consider the problem

$$-Y''(x) + q(1-x)Y(x) = \lambda Y(x), \quad (3.6)$$

$$-P_1(\lambda)Y'(0) = Q_1(\lambda)Y(0), \quad (3.7)$$

$$-P_0(\lambda)Y'(1) = Q_0(\lambda)Y(1). \quad (3.8)$$

Of course, this problem is equivalent to (1.1), (1.2) via  $Y(x) = y(1-x)$ . However, the analysis of the previous case applies, and we readily obtain the claim of the theorem.

For the case  $a_0^0 = 0 = a_0^1$  we have

$$\frac{D(\lambda)}{b_0^0 b_0^1 \lambda^{d_0+d_1-1}} = \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \left( \frac{a_1^0}{b_0^0} - \frac{a_1^1}{b_0^1} + \frac{1}{2} \int_0^1 q \right) + O\left(\frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{\sqrt{\lambda}}\right).$$

The argument now follows lines similar to that for the case  $a_0^0 a_0^1 \neq 0$  and we leave the details to the reader.  $\square$

#### 4. Oscillation theory

To study the oscillation theory associated with (1.1), (1.2), we assume in this section that  $n$  is large enough for  $\lambda_n$  to be a simple real eigenvalue. That this is possible follows from Theorem 3.1. We aim to determine the number of zeros in  $(0, 1)$  of the eigenfunction for  $\lambda_n$ . We shall rely heavily on the Prüfer angle  $\theta$ , for which [1] gives the essential theory. This angle satisfies the first-order equation

$$\theta' = \cos^2 \theta + (\lambda - q) \sin^2 \theta \quad (4.1)$$

and is related to (1.1) via  $\cot \theta = y'/y$ . We impose an initial condition  $\theta(0) = \alpha$ ,  $0 \leq \alpha < \pi$ , on (4.1) and we write  $\theta(x, \lambda, \alpha)$  to display the dependence of  $\theta$  on  $\lambda$  and  $\alpha$ . Evidently,  $\theta$  is increasing in both  $\lambda$  and  $\alpha$ . The oscillation number associated with  $\lambda_n$  is the unique integer  $k_n \geq 0$  for which

$$\theta(1, \lambda_n, \alpha_n) \in (k_n \pi, (k_n + 1)\pi].$$

Here and below, we will use

$$\alpha_n = \cot^{-1} \left( \frac{Q_0}{P_0}(\lambda_n) \right), \quad \beta_n = \cot^{-1} \left( \frac{Q_1}{P_1}(\lambda_n) \right),$$

where we take the principal value of  $\cot^{-1}$  so that  $\alpha_n, \beta_n \in (0, \pi)$ .

When (1.1) is subject to the boundary conditions

$$\frac{y'}{y}(0) = \cot \alpha, \quad \frac{y'}{y}(1) = \cot \beta, \quad 0 \leq \alpha < \pi, \quad 0 < \beta \leq \pi,$$

the resulting eigenvalues  $\lambda_n^{\alpha, \beta}$ ,  $n \geq 0$ , have asymptotic developments

$$\lambda_n^{\alpha, \beta} = (n + \nu)^2 \pi^2 - 2 \cot \beta + 2 \cot \alpha + \int_0^1 q + O\left(\frac{1}{n}\right),$$

where  $\nu = 0$  if  $\alpha \neq 0$  and  $\beta \neq \pi$ ;  $\nu = \frac{1}{2}$  if  $\alpha = 0$  and  $\beta \neq \pi$  or if  $\alpha \neq 0$  and  $\beta = \pi$ ; and  $\nu = 1$  if  $\alpha = 0$  and  $\beta = \pi$ . Moreover,

$$\theta(1, \lambda_n^{\alpha, \beta}, \alpha) = n\pi + \beta.$$

With this background, we present our first result. For convenience we shall continue to use

$$\hat{n} = n - d_0 - d_1.$$

The value of  $k_n$  depends on whether the limits as  $\lambda \rightarrow \infty$  of  $(Q_j/P_j)(\lambda)$ ,  $j = 0, 1$ , are finite,  $+\infty$  or  $-\infty$ . This apparently gives nine cases, although there are only three outcomes.

**Theorem 4.1.** *For large  $n$ , the oscillation number associated with  $\lambda_n$  is  $k_n = \hat{n} + \eta_0 + \eta_1$ , where*

$$\eta_0 = \begin{cases} 1 & \text{if } \lim_{\lambda \rightarrow \infty} \frac{Q_0}{P_0}(\lambda) = -\infty, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta_1 = \begin{cases} 1 & \text{if } \lim_{\lambda \rightarrow \infty} \frac{Q_1}{P_1}(\lambda) = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We shall rely heavily on the relations

$$\begin{aligned} \cot \theta(1, \lambda_n, \alpha_n) &= \frac{Q_1}{P_1}(\lambda_n), \\ \theta(1, \lambda_n, \alpha_n) &= k_n\pi + \beta_n. \end{aligned}$$

(i) Suppose  $Q^j/P^j(\lambda) \rightarrow +\infty$ ,  $j = 0, 1$ , as  $\lambda \rightarrow \infty$ , such that  $\alpha_n \downarrow 0$  and  $\beta_n \downarrow 0$ . By Theorem 3.1,

$$\lambda_n = (\hat{n} + 1)^2\pi^2 + c + \bar{q} + O\left(\frac{1}{n}\right), \tag{4.2}$$

where  $c$  is a constant and we write  $\bar{q} = \int_0^1 q$ . Then, for large  $n$ ,

$$\lambda_{\hat{n}}^{0, \pi/2} < \lambda_n < \lambda_{\hat{n}+1}^{\pi/2, \pi},$$

and so

$$(\hat{n} + \frac{1}{2})\pi = \theta(1, \lambda_{\hat{n}}^{0, \pi/2}, 0) < \theta(1, \lambda_n, \alpha_n) < \theta(1, \lambda_{\hat{n}+1}^{\pi/2, \pi}, \frac{1}{2}\pi) = (\hat{n} + 2)\pi.$$

Since  $\beta_n > 0$ , we have  $k_n = \hat{n} + 1$ .

(ii) Suppose  $Q^0/P^0(\lambda) \rightarrow \text{const.}$  and  $Q^1/P^1(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , such that  $\alpha_n \rightarrow \alpha \neq 0$ ,  $\pi, \beta_n \downarrow 0$ . Then, by Theorem 3.1,

$$\begin{aligned}\lambda_n &= (\hat{n} + \tfrac{1}{2})^2 \pi^2 + c + \bar{q} + O\left(\frac{1}{n}\right) \\ &> \hat{n}^2 \pi^2 + 2 \cot(\alpha - \varepsilon) + \bar{q} + O\left(\frac{1}{n}\right) \quad (\text{for small } \varepsilon > 0) \\ &= \lambda_{\hat{n}}^{\alpha - \varepsilon, \pi/2} + O\left(\frac{1}{n}\right).\end{aligned}\tag{4.3}$$

Hence, for large  $n$ ,

$$\lambda_{\hat{n}}^{\alpha - \varepsilon, \pi/2} < \lambda_n < \lambda_{\hat{n}}^{0, \pi},$$

and so

$$(\hat{n} + \tfrac{1}{2})\pi = \theta(1, \lambda_{\hat{n}}^{\alpha - \varepsilon, \pi/2}, \alpha - \varepsilon) < \theta(1, \lambda_n, \alpha_n) < \theta(1, \lambda_{\hat{n}}^{0, \pi}, 0) + \pi = (\hat{n} + 2)\pi.$$

Since

$$\theta(1, \lambda_n, \alpha_n) = k_n \pi + \beta_n,$$

where  $\beta_n \downarrow 0$ , we have  $k_n = \hat{n} + 1$ .

(iii) Suppose  $Q^0/P^0(\lambda) \rightarrow \text{const.}$  and  $Q^1/P^1(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ , such that  $\alpha_n \rightarrow \alpha \neq 0$ ,  $\pi, \beta_n \uparrow \pi$ . Then (4.3) holds and, for small  $\varepsilon > 0$  and large  $n$ ,

$$\begin{aligned}\lambda_n &< (\hat{n} + 1)^2 \pi^2 + 2 \cot(\alpha + \varepsilon) + (c - 2 \cot(\alpha + \varepsilon) + 2) - 1 + O\left(\frac{1}{n}\right) \\ &= \lambda_{\hat{n}+1}^{\alpha + \varepsilon, \beta} - 1 + O\left(\frac{1}{n}\right),\end{aligned}$$

where  $\beta$  is defined in  $(0, \pi)$  by  $-2 \cot \beta = c - 2 \cot(\alpha + \varepsilon) + 2$ . Hence,

$$\lambda_{\hat{n}-1}^{0, \pi} < \lambda_n < \lambda_{\hat{n}+1}^{\alpha + \varepsilon, \beta},$$

and so

$$\hat{n}\pi = \theta(1, \lambda_{\hat{n}-1}^{0, \pi}, 0) < \theta(1, \lambda_n, \alpha_n) < \theta(1, \lambda_{\hat{n}+1}^{\alpha + \varepsilon, \beta}, \alpha + \varepsilon) = (\hat{n} + 1)\pi + \beta.$$

Since

$$\theta(1, \lambda_n, \alpha_n) = k_n \pi + \beta_n,$$

where  $\beta_n \uparrow \pi$ , we see that  $k_n = \hat{n}$ .

(iv) Suppose  $Q^0/P^0(\lambda) \rightarrow \infty$  and  $Q^1/P^1(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ , such that  $\alpha_n \downarrow 0$ ,  $\beta_n \uparrow \pi$ . By Theorem 3.1,

$$\lambda_n < (\hat{n} + 1)^2 \pi^2 + c + 2 + \bar{q} - 1 + O\left(\frac{1}{n}\right) = \lambda_{\hat{n}+1}^{\alpha, \beta} - 1 + O\left(\frac{1}{n}\right),$$

where  $\alpha$  and  $\beta$  are defined in  $(0, \pi)$  by  $2 \cot \alpha = \frac{1}{2}c + 1$  and  $-2 \cot \beta = \frac{1}{2}c + 1$ . Thus, for large  $n$ ,

$$\lambda_{\hat{n}}^{0, \pi/2} < \lambda_n < \lambda_{\hat{n}+1}^{\alpha, \beta},$$

and so

$$(\hat{n} + \frac{1}{2})\pi = \theta(1, \lambda_{\hat{n}}^{0, \pi/2}, 0) < \theta(1, \lambda_n, \alpha_n) < \theta(1, \lambda_{\hat{n}+1}^{\alpha, \beta}, \alpha) = (\hat{n} + 1)\pi + \beta.$$

Since

$$\theta(1, \lambda_n, \alpha_n) = k_n\pi + \beta_n,$$

where  $\beta_n \uparrow \pi$ , we have  $k_n = \hat{n}$ .

(v) Suppose  $[Q^0/P^0](\lambda) \rightarrow -\infty$  and  $[Q^1/P^1](\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , so that  $\alpha_n \uparrow \pi$  and  $\beta_n \downarrow 0$ . By Theorem 3.1,

$$\lambda_n > (\hat{n} + 1)^2\pi^2 + c - 2 + \bar{q} + 1 + O\left(\frac{1}{n}\right) = \lambda_{\hat{n}+1}^{\alpha, \beta} + 1 + O\left(\frac{1}{n}\right),$$

where  $\alpha$  and  $\beta$  are defined in  $(0, \pi)$  by  $2 \cot \alpha = \frac{1}{2}c - 1$  and  $-2 \cot \beta = \frac{1}{2}c - 1$ . Thus, for large  $n$ ,

$$\lambda_{\hat{n}+1}^{0, \pi/2} > \lambda_n > \lambda_{\hat{n}+1}^{\alpha, \beta},$$

and so

$$(\hat{n} + \frac{5}{2})\pi = \theta(1, \lambda_{\hat{n}+1}^{0, \pi/2}, 0) + \pi > \theta(1, \lambda_n, \alpha_n) > \theta(1, \lambda_{\hat{n}+1}^{\alpha, \beta}, \alpha) = (\hat{n} + 1)\pi + \beta.$$

Since

$$\theta(1, \lambda_n, \alpha_n) = k_n\pi + \beta_n,$$

where  $\beta_n \downarrow 0$ , we have  $k_n = \hat{n} + 2$ .

(vi) Suppose  $Q^j/P^j(\lambda) \rightarrow \text{const.}$ ,  $j = 0, 1$ , as  $\lambda \rightarrow \infty$ , so that  $\alpha_n \rightarrow \alpha \neq 0$ ,  $\pi, \beta_n \rightarrow \beta \neq 0, \pi$ . Then

$$\lambda_n = \hat{n}^2\pi^2 + c + \bar{q} + O\left(\frac{1}{n}\right),$$

and, for small  $\varepsilon > 0$  and large  $n$ ,

$$(\hat{n} - \frac{1}{2})^2 + 2 \cot(\alpha - \varepsilon) + \bar{q} + O\left(\frac{1}{n}\right) < \lambda_n < (\hat{n} + \frac{1}{2})^2 + 2 \cot(\alpha + \varepsilon) + \bar{q} + O\left(\frac{1}{n}\right) < \lambda_n$$

and thus

$$\lambda_{\hat{n}-1}^{\alpha-\varepsilon, \pi} < \lambda_n < \lambda_{\hat{n}}^{\alpha+\varepsilon, \pi},$$

so

$$\hat{n}\pi = \theta(1, \lambda_{\hat{n}-1}^{\alpha-\varepsilon, \pi}, \alpha - \varepsilon) < \theta(1, \lambda_n, \alpha_n) < \theta(1, \lambda_{\hat{n}}^{\alpha+\varepsilon, \pi}, \alpha + \varepsilon) = (\hat{n} + 1)\pi.$$

Hence,  $k_n = \hat{n}$ .

The remaining cases can be obtained from cases (i)–(iii) by using a reflection argument and considering the problem (3.6)–(3.8) as in the proof of Theorem 3.1.  $\square$

## 5. Example

We consider the problem

$$-y'' = \lambda y, \quad (5.1)$$

$$\lambda y'(0) = -\lambda^2 y(0), \quad (5.2)$$

$$\lambda^2 y'(1) = \lambda^3 y(1). \quad (5.3)$$

### 5.1.

The corresponding reduced problem has (5.2) and (5.3) replaced by

$$y'(0) = -\lambda y(0), \quad y'(1) = \lambda y(1). \quad (5.4)$$

It is a straightforward exercise to show that (5.1), (5.4) has a sequence of eigenvalues  $\mu_n^2$  with corresponding eigenfunctions  $y_n$ , where

$$\left. \begin{aligned} \mu_0 &= 0, & y_0(x) &= 1; \\ \tan \mu_n &= \frac{2\mu_n}{\mu_n^2 - 1}, & y_n(x) &= \sin(\mu_n x) - \mu_n^{-1} \cos(\mu_n x), \quad n \geq 1. \end{aligned} \right\} \quad (5.5)$$

We note (taking  $\mu_n \geq 0$  for convenience) that  $\mu_n = (n-1)\pi + \varepsilon_n$ , where  $0 < \varepsilon_n = O(1/n)$ . By applying Theorem 3.1 to the reduced problem, we see that these are the only eigenvalues of (5.1), (5.4) and, moreover, that each eigenvalue is simple. Note that this does require proof, since (5.4) does not satisfy the ‘right-definiteness’ condition of [3] (in fact a self-adjoint operator formulation, as in, say [16], would require a Pontryagin space of index 2).

### 5.2.

For the non-reduced problem, Proposition 2.2 shows that the only difference in Jordan structure will occur at  $\lambda = 0$ , and then (5.1) has general solution  $y = Cx + D$ . Thus, we can start a Jordan chain for (5.1)–(5.3) with  $y^{[0]}(x) = Cx + D$ . The equation for  $y^{[1]}$  is  $-y'' = Cx + D$ , which has solution

$$y(x) = -\frac{1}{3!}Cx^3 - \frac{1}{2!}Dx^2 + Ex + F.$$

The boundary conditions to be satisfied are  $y^{[0]'}(0) = 0$ , which requires  $C = 0$ , and a null condition at  $x = 1$ . Thus, unless  $C = 0$ , the chain terminates with  $y^{[0]}$  and so has length 1.

When  $C = 0$ , we can, without loss of generality, take  $D = 1$ , and then

$$y^{[1]}(x) = -\frac{1}{2!}x^2 + Ex + F.$$

The equation for  $y^{[2]}$  is

$$-y'' = -\frac{1}{2}x^2 + Ex + F,$$

which has solution

$$y(x) = \frac{1}{4!}x^4 - \frac{E}{3!}x^3 - \frac{F}{2!}x^2 + Gx + H.$$

The boundary conditions are  $0 = y^{[1]'}(0) + y^{[0]}(0)$ , which forces  $E = -1$ , and  $y^{[0]'}(1) = 0$ , which is automatically satisfied.

Thus, we now have

$$y^{[2]}(x) = \frac{1}{4!}x^4 + \frac{1}{3!}x^3 - \frac{1}{2!}Fx^2 + Gx + H.$$

The equation for  $y^{[3]}$  is  $-y'' = y^{[2]}$  with boundary conditions  $0 = y^{[2]'}(0) + y^{[1]}(0)$ , which forces  $G = -F$ , and  $y^{[1]'}(1) = y^{[0]}(1)$ , which requires  $-2 = 1$ : an impossibility. Hence, this chain terminates with  $y^{[2]}$  and so has length 3.

For (5.1)–(5.3), the eigenvalue  $\lambda = 0$  generates two chains, one of length 3 and the other of length 1, thus  $\lambda = 0$  has algebraic multiplicity 4. This is in accord with the results of §2, since  $\lambda = 0$  is an algebraically simple eigenvalue of (5.1) with boundary conditions (5.4), but in the notation of (2.5),  $\zeta_j(0) = 1$  and  $\zeta_j(1) = 2$ .

### 5.3.

Now we can turn to the results of §§3 and 4. Counting  $\lambda_0$  four times (as above) we have  $\lambda_n = \mu_{n+4}^2$  for  $n \geq 1$  in the notation of Theorem 3.1 ( $P$  and  $Q$  being unreduced) so there is an index shift of 4 in the asymptotics. To illustrate Theorem 4.1, we consider the oscillation of  $y_n$  for large  $n$ . Note that  $\sin(\mu_n x)$  has  $n - 1$  zeros in  $(0, 1)$  and the  $-\mu_n^{-1} \cos(\mu_n x)$  term adds an extra zero of  $y_n$  near  $x = 0$ . Finally,

$$y_n(1) = \cos(\mu_n) \left( \tan \mu_n - \frac{1}{\mu_n} \right) = \frac{\mu_n^2 + 1}{\mu_n(\mu_n^2 - 1)}$$

from (5.5), so no extra zero is added near  $x = 1$ . Thus,  $y_n$  has  $n$  zeros in  $(0, 1)$ , and the index shift for oscillation is 5. This exceeds the shift for asymptotics, a phenomenon which cannot happen for right-definite [3] or  $\lambda$ -independent boundary-condition [12] problems.

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