Proceedings of the Edinburgh Mathematical Society (2012) **55**, 181–195
DOI:10.1017/S0013091509001813

MULTIPLICITY OF SOLUTIONS TO THE WEIGHTED CRITICAL QUASILINEAR PROBLEMS

SIHUA LIANG[1] AND JIHUI ZHANG[2*]

[1] *Department of Mathematics, Changchun Normal University, Changchun 130032, Jilin, People's Republic of China* (liangsihua@163.com)

[2] *Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing, Jiangsu 210046, People's Republic of China*

(Received 17 December 2009)

Abstract We consider a class of critical quasilinear problems

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{q-2}u}{|x|^{bq}} + \lambda f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $0 \in \Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain and $1 < p < N$, $a < N/p$, $a \leqslant b < a+1$, λ is a positive parameter, $0 \leqslant \mu < \bar{\mu} \equiv ((N-p)/p-a)^p$, $q = q^*(a,b) \equiv Np/[N-pd]$ and $d \equiv a+1-b$. Infinitely many small solutions are obtained by using a version of the symmetric Mountain Pass Theorem and a variant of the concentration-compactness principle. We deal with a problem that extends some results involving singularities not only in the nonlinearities but also in the operator.

Keywords: degenerate quasilinear equation; p-Laplacian operator; variational methods; concentration-compactness principle

2010 *Mathematics subject classification:* Primary 35J60; 35B33; 35J65

1. Introduction

In this paper, we are concerned with the following quasilinear elliptic problem:

$$\left.\begin{aligned} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} &= \frac{|u|^{q-2}u}{|x|^{bq}} + \lambda f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $0 \in \Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain and $1 < p < N$, $a < N/p$, $a \leqslant b < a+1$, λ is a positive parameter, $0 \leqslant \mu < \bar{\mu} \equiv ((N-p)/p-a)^p$, $q = p^*(a,b) \equiv Np/[N-pd]$ is the critical Hardy–Sobolev exponent and $d \equiv a+1-b$. Note that $p^*(0,0) = p^* \equiv Np/(N-p)$. With the help of the symmetric Mountain Pass Lemma due to Kajikiya [**18**] and the

* Present address: Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210046, People's Republic of China.

concentration-compactness principle, we prove that there are infinitely many small weak solutions for equations (1.1) with the general nonlinearities $f(x,u)$ in the dual of the weighted Sobolev space $D_a^{1,p}(\Omega)$.

Throughout this paper, we use the Sobolev space $D_a^{1,p}(\Omega)$, defined as the completion of the space $C_0^\infty(\Omega)$ endowed with the norm

$$\|u\|_D = \left[\int_\Omega |x|^{-ap}|\nabla u|^p \,\mathrm{d}x\right]^{1/p}.$$

A well-known result by Caffarelli *et al.* [**6**] guarantees that the Euler–Lagrange energy functional $I\colon D_a^{1,p}(\Omega) \mapsto \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_\Omega \left[|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right]\mathrm{d}x - \frac{1}{q}\int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega F(x,u)\,\mathrm{d}x \tag{1.2}$$

is well defined, where

$$F(x,u) \equiv \int_0^u f(x,t)\,\mathrm{d}t.$$

By the standard elliptic regularity argument, we deduce that $I \in C^1(D_a^{1,p}(\Omega),\mathbb{R})$ and a weak solution u of problem (1.1) is precisely a critical point of the functional I, that is, $I'(u) = 0$, where

$$\langle I'(u), v\rangle = \int_\Omega \left[|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla v - \mu\frac{|u|^{p-2}uv}{|x|^{p(a+1)}}\right]\mathrm{d}x - \int_\Omega \frac{|u|^{q-2}uv}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega f(x,u)v\,\mathrm{d}x \tag{1.3}$$

holds for all $v \in D_a^{1,p}(\Omega)$.

Problem (1.1) is related to the Caffarelli–Kohn–Nirenberg inequality [**6**]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{bq}}\right)^{p/q} \leqslant \frac{1}{C}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \,\mathrm{d}x \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N), \tag{1.4}$$

which is also called the weighted Hardy–Sobolev inequality, where C is some positive constant. For the sharp constants and extremal functions, see [**16, 27**]. If $b = a+1$, then $q = p^*(a, a+1) = p$ and the following Hardy inequality holds [**6, 24**]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}}\right) \leqslant \frac{1}{\bar{\mu}}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \,\mathrm{d}x \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N), \tag{1.5}$$

where $\bar{\mu} = ((N-p)/p - a)^p$ is the best Hardy constant.

In this paper, we use the following norm:

$$\|u\| := \left[\int_\Omega \left(|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right)\mathrm{d}x\right]^{1/p}.$$

By (1.5) this is equivalent to the usual norm $\|u\|_D$ of the space $D_a^{1,p}(\Omega)$ if $\mu < \bar{\mu}$. According to (1.4) and (1.5), we can define the following best constants for $\mu < \bar{\mu}$:

$$\Lambda_{a,b} := \inf_{u \in D_a^{1,p}(\Omega)\setminus\{0\}} \|u\|^p \left(\int_\Omega \frac{|u|^p}{|x|^{qb}} \, \mathrm{d}x \right)^{-1},$$

$$S_{a,b} := \inf_{u \in D_a^{1,p}(\Omega)\setminus\{0\}} \|u\|^p \left(\int_\Omega \frac{|u|^q}{|x|^{qb}} \, \mathrm{d}x \right)^{q/p}.$$

Thus, $\Lambda_{a,b} > 0$ and $S_{a,b} > 0$.

For $p = 2$, $\mu = 0$ and $a = b = 0$, such a problem has been studied extensively since the publication of the celebrated paper by Brézis and Nirenberg [**5**]. For a singular potential, the existence of infinitely many small solutions which converge to zero was obtained by He and Zou [**15**]. Yang and Shen [**30**] studied the critical singular equation involving the Caffarelli–Kohn–Nirenberg inequalities with the special case $f(x, u)$ by using the Lyusternik–Schnirelman Category Theory and obtained the existence of at least $\mathrm{cat}(\Omega)$ positive solutions. Some existence results for problem (1.1) with $\lambda = 1$, were obtained by Huang and Wu [**17**] using variational methods and analysis techniques. In [**28**], Terracini proved some results about the existence, uniqueness and qualitative behaviour of positive solutions to a class of equations with a singular coefficient and a critical exponent by using variational arguments and the moving-plane method. For other results we refer the reader to [**7**–**9**, **12**].

For $p \neq 2$ and $a = b = 0$, in [**14**], Ghoussoub and Yuan considered problem (1.1) by establishing Palais–Smale-type conditions around appropriately chosen dual sets and obtained the existence of infinitely many non-trivial solutions on a bounded domain. On the other hand, very little is known about singular problems with Hardy–Sobolev critical exponents (the case $p \neq 2$) [**10**, **25**] under the general nonlinearities $f(x, u)$. Also, a similar problem in the case of the p-Laplacian, but with the special case $\lambda f(x, u) = u^{p^*-1}$ was analysed in [**13**]. In [**23**], Musina studied existence and multiplicity results for a weighted p-Laplace equation involving Hardy potentials and critical nonlinearities; a survey on the most recent results and new existence and multiplicity results are given therein. Chen and Li [**11**] obtained the existence of infinitely many solutions by using the minimax procedure in the case $\mu = 0$ and $f(x, u) = k(x)|u|^{r-2}u$, $1 < r < Np/(N - p)$. In all the above-mentioned works, information on the sequence of solutions is not given.

For $p \neq 2$ and $a \neq 0$, $b \neq 0$, by using variational methods, the existence of positive solutions to the problem (1.1) with special case $f(x, u)$ was proved by Kang [**19**], who obtained the properties of the extremal functions by which the best Hardy–Sobolev constant was achieved. By using a version of the concentration-compactness lemma due to Lions, the Krasnosel′skii genus and the symmetric Mountain Pass Theorem due to Rabinowitz, multiplicity results were established in [**1**]. For other results we refer the reader to [**2**, **3**, **29**]. These references, however, do not give any further information on the sequence of solutions.

Recently, Kajikiya [**18**] established a critical point theorem related to the Symmetric Mountain Pass Lemma and applied it to a sublinear elliptic equation. To the best of our knowledge, there are no such results on singular quasilinear elliptic problems (1.1).

Motivated by the reasons above, the aim of this paper is to show the existence of infinitely many solutions for problem (1.1), and there exists a sequence of infinitely many arbitrarily small solutions converging to zero by using a new version of the Symmetric Mountain Pass Lemma due to Kajikiya [**18**]. In order to use the Symmetric Mountain Pass Lemma, the main difficulty in solving this problem is the lack of compactness, which can be illustrated by the fact that the embedding of $D_a^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is no longer compact. Hence, the concentration-compactness principle is used here to overcome this difficulty.

Theorem 1.1. *Suppose that* $f(x,u)$ *satisfies the following conditions:*

(H_1) $f(x,u) \in C(\Omega \times R, R)$, $f(x,-u) = -f(x,u)$ *for all* $u \in R$;

(H_2) $\lim_{|u|\to\infty} f(x,u)/|u|^q = 0$ *uniformly for* $x \in \Omega$;

(H_3) $\lim_{|u|\to 0^+} f(x,u)/u^{p-1} = \infty$ *uniformly for* $x \in \Omega$.

Then problem (1.1) has a sequence of non-trivial solutions $\{u_n\}$ *and* $u_n \to 0$ *as* $n \to \infty$.

Remark 1.2. Without the symmetry condition (i.e. $f(x,-u) = -f(x,u)$), we can obtain at least one non-trivial solution by using the method in this paper.

Remark 1.3. When $p = 2, \mu = 0$ and $a = b = 0$, Li and Zou [**20**] proved the existence of infinitely many solutions for (1.1) under conditions (H_1)–(H_3) and

(H_4) $\frac{1}{2}f(x,u)u - F(x,u) \geqslant a - b|u|^{2^*}$ for almost every $x \in \Omega$ and $u \in R$, $b \geqslant 0, a \leqslant 0$.

But they did not give any further information on the sequence of solutions. When $p = 2$ and $a = b = 0$, He and Zou [**15**] obtained the existence of infinitely many small solutions for (1.1) under conditions (H_1)–(H_3). Here we shall prove that this sequence of solutions for (1.1) may converge to zero.

Remark 1.4. We should point out that Theorem 1.1 is different from the previous results of [**1, 15, 19, 20**] in three main directions:

1. $p \neq 2$, $a \neq 0$ and $b \neq 0$;
2. the nonlinearity $f(x,u)$ does not satisfy condition (H_4) as in [**20**];
3. we can obtain a sequence of non-trivial solutions $\{u_n\}$ and $u_n \to 0$ as $n \to \infty$.

Definition 1.5. A C^1 functional I on a Banach space X satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ for short) if every sequence $\{u_n\}$ satisfying

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0$$

contains a convergent subsequence.

Under assumption (H_2), we have

$$f(x,u)u = o\left(\frac{|u|^q}{|x|^{bq}}\right),$$

$$F(x,u) = o\left(\frac{|u|^q}{|x|^{bq}}\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon), b(\varepsilon) > 0$ such that

$$|f(x,u)u| \leqslant a(\varepsilon) + \varepsilon\frac{|u|^q}{|x|^{bq}}, \tag{1.6}$$

$$|F(x,u)| \leqslant b(\varepsilon) + \varepsilon\frac{|u|^q}{|x|^{bq}}. \tag{1.7}$$

Hence,

$$\left|F(x,u) - \frac{1}{p}f(x,u)u\right| \leqslant c(\varepsilon) + \varepsilon\frac{|u|^q}{|x|^{bq}}, \tag{1.8}$$

for some $c(\varepsilon) > 0$.

The remainder of the paper is organized as follows. In § 2 we shall prove that the corresponding energy functional satisfies $(\mathrm{PS})_c$. In § 3 we shall prove our main results.

2. Preliminary lemmas

In this section, we first give a concentration-compactness principle which is a weighted version of the concentration-compactness principle in [**29**]. Denote by $\mathcal{M}^+$ the cone of positive finite Radon measure. Since the proof of the following result is similar to that of Lions [**21**, **22**] and is an adaptation of a lemma by Smets [**26**], we just state it here without proof.

Lemma 2.1. *Let $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leqslant b \leqslant a+1$, $q = p^*(a,b) = Np/(N-dp)$, $d = 1+a-b \in [0,1]$, and let $\mathcal{M}^+(\mathbb{R}^N)$ be the space of positive bounded measures on $\mathbb{R}^N$. Suppose that $\{u_n\} \subset D_a^{1,p}(\mathbb{R}^N)$ is a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } D_a^{1,p}(\mathbb{R}^N),\\ ||x|^{-a}|\nabla u_n||^p &\rightharpoonup \zeta \quad \text{in } \mathcal{M}^+(\mathbb{R}^N),\\ ||x|^{-b}|u_n||^q &\rightharpoonup \nu \quad \text{in } \mathcal{M}^+(\mathbb{R}^N),\\ u_n &\to u \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Then the following conclusions hold.

1. *There exists some at most countable set J, a family $\{x_j : j \in J\}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j : j \in J\}$ of positive numbers such that*

$$\nu = ||x|^{-b}|u||^q + \sum_{j\in J}\nu_j\delta_{x_j}, \tag{2.1}$$

where δ_x is the Dirac unitary mass concentrated at $x \in \mathbb{R}^N$.

2. *The following inequality holds:*

$$\zeta \geqslant ||x|^{-a}|\nabla u||^p + \sum_{j\in J}\zeta_j\delta_{x_j}, \tag{2.2}$$

for some family $\{\zeta_j : j \in J\}$ *satisfying*

$$S_{a,b}(\nu_j)^{p/q} \leqslant \zeta_j \quad \text{for all } j \in J. \tag{2.3}$$

In particular, $\sum_{j\in J}(\nu_j)^{p/q} < \infty$.

Lemma 2.2. *Assume condition* (H_2) *holds. Then for any* $\lambda > 0$*, the functional* I *satisfies the local (PS)*$_c$ *in*

$$c \in \left(-\infty,\ \frac{d}{N}S_{a,b}^{N/pd} - \lambda c\left(\frac{d}{2N\lambda}\right)|\Omega|\right)$$

in the following sense: if

$$I(u_n) \to c < \frac{d}{N}S_{a,b}^{N/pd} - \lambda c\left(\frac{d}{2N\lambda}\right)|\Omega|$$

and $I'(u_n) \to 0$ *for some sequence in* $D_a^{1,p}(\Omega)$*, then* $\{u_n\}$ *contains a subsequence converging strongly in* $D_a^{1,p}(\Omega)$.

Proof. First we prove that $\{u_n\} \subset D_a^{1,p}(\Omega)$ is bounded in $D_a^{1,p}(\Omega)$.
Let $\{u_n\}$ be a sequence in $D_a^{1,p}(\Omega)$ such that

$$\begin{aligned} I(u_n) &= \frac{1}{p}\int_\Omega \left[|x|^{-ap}|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^{p(a+1)}}\right]\mathrm{d}x \\ &\qquad - \frac{1}{q}\int_\Omega \frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega F(x,u_n)\,\mathrm{d}x \\ &= c + o(1) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \langle I'(u_n), v\rangle &= \int_\Omega \left[|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla v - \mu\frac{|u_n|^{p-2}u_n v}{|x|^{p(a+1)}}\right]\mathrm{d}x \\ &\qquad - \int_\Omega \frac{|u_n|^{q-2}u_n v}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega f(x,u_n)v\,\mathrm{d}x \\ &= o(1)\|v\|. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have

$$\begin{aligned} I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle &= \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega \frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega \left[F(x,u_n) - \frac{1}{p}f(x,u_n)u_n\right]\mathrm{d}x \\ &= c + o(1)\|u_n\|, \end{aligned}$$

i.e.

$$\frac{d}{N}\int_{\Omega}\frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x = \lambda\int_{\Omega}\left[F(x,u_n) - \frac{1}{p}f(x,u_n)u_n\right]\mathrm{d}x + c + o(1)\|u_n\|,$$

where $d = a + 1 - b$. Then by (1.8) we have

$$\left(\frac{d}{N} - \lambda\varepsilon\right)\int_{\Omega}\frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x \leqslant \lambda c(\varepsilon)|\Omega| + c + o(1)\|u_n\|.$$

Setting $\varepsilon = d/2N\lambda$, we get

$$\int_{\Omega}\frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x \leqslant M + o(1)\|u_n\|, \tag{2.6}$$

where $o(1) \to 0$ as $n \to +\infty$ and M is a positive number. On the other hand, by (1.7) and (2.4), we have

$$\begin{aligned} c + o(1)\|u_n\| &= I(u_n) \\ &\geqslant \frac{1}{p}\|u_n\|^p - \lambda b(\varepsilon)|\Omega| - \left[\frac{1}{q} + \lambda\varepsilon\right]\int_{\Omega}\frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x. \end{aligned} \tag{2.7}$$

Thus, (2.6) and (2.7) imply that $\{u_n\}$ is bounded in $D_a^{1,p}(\Omega)$. Therefore, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } D_a^{1,p}(\Omega), \\ u_n &\rightharpoonup u \quad \text{in } L^p(\Omega, |x|^{-p(a+1)}), \\ u_n &\rightharpoonup u \quad \text{in } L^q(\Omega, |x|^{-bq}), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned}$$

From the concentration-compactness principle, there exist non-negative measures ζ, ν and a countable family $\{x_j\} \subset \Omega$ such that

$$\begin{aligned} ||x|^{-b}|u_n||^q &\rightharpoonup \nu = ||x|^{-b}|u||^q + \sum_{j\in J}\nu_j\delta_{x_j}, \\ ||x|^{-a}|\nabla u_n||^p &\rightharpoonup \zeta \geqslant ||x|^{-a}|\nabla u||^p + S_{a,b}\sum_{j\in J}(\zeta_j)^{p/q}\delta_{x_j}. \end{aligned}$$

Now we prove that $u_n \to u$ in $L^q(\Omega, |x|^{-bq})$.

Since $\{u_n\}$ is bounded in $D_a^{1,p}(\Omega)$, we can assume that there exists $\eta \in L^{p'}(\Omega, |x|^{-ap})$ such that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \eta \quad \text{in } L^{p'}(\Omega, |x|^{-ap}),$$

where $1/p + 1/p' = 1$. On the other hand, $|u_n|^{p-2}u_n$ and $|u_n|^{q-2}u_n$ are also bounded in $L^{p'}(\Omega, |x|^{-ap})$ and $L^{q'}(\Omega, |x|^{-bq})$, respectively. Thus, we have

$$\begin{aligned} |u_n|^{p-2}u_n &\rightharpoonup |u|^{p-2}u \quad \text{in } L^{p'}(\Omega, |x|^{-ap}), \\ |u_n|^{q-2}u_n &\rightharpoonup |u|^{q-2}u \quad \text{in } L^{q'}(\Omega, |x|^{-bp}), \end{aligned}$$

https://doi.org/10.1017/S0013091509001813 Published online by Cambridge University Press

where $1/q + 1/q' = 1$. Taking $n \to \infty$ in (2.5), we get

$$\int_\Omega \left[|x|^{-ap}\eta\nabla v - \mu\frac{|u|^{p-2}uv}{|x|^{p(a+1)}}\right] \mathrm{d}x = \int_\Omega \frac{|u|^{q-2}uv}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u)v\,\mathrm{d}x, \tag{2.8}$$

for any $v \in D_a^{1,p}(\Omega)$.

Let $v = \phi u_n$ in (2.5), where $\phi \in C_0^\infty(\Omega)$; then it follows that

$$\begin{aligned}\int_\Omega \left[|x|^{-ap}|\nabla u_n|^p\phi - \mu\frac{|u_n|^p\phi}{|x|^{p(a+1)}}\right]\mathrm{d}x + \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\cdot u_n\,\mathrm{d}x \\ = \int_\Omega \frac{|u_n|^q\phi}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u_n)u_n\phi\,\mathrm{d}x.\end{aligned} \tag{2.9}$$

Taking $n \to \infty$ in (2.9) we get

$$\int_\Omega \phi\,\mathrm{d}\zeta - \int_\Omega \mu\frac{|u|^p\phi}{|x|^{p(a+1)}}\,\mathrm{d}x + \int_\Omega |x|^{-ap}\eta\nabla\phi\cdot u\,\mathrm{d}x = \int_\Omega \phi\,\mathrm{d}\nu + \lambda\int_\Omega f(x,u)u\phi\,\mathrm{d}x. \tag{2.10}$$

Let $v = \phi u$ in (2.8); we have

$$\begin{aligned}\int_\Omega |x|^{-ap}\eta\nabla\phi\cdot u\,\mathrm{d}x + \int_\Omega |x|^{-ap}\eta\nabla u\cdot\phi\,\mathrm{d}x - \int_\Omega \mu\frac{|u|^p\phi}{|x|^{p(a+1)}}\,\mathrm{d}x \\ = \int_\Omega \frac{|u|^q\phi}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u)\phi u\,\mathrm{d}x.\end{aligned} \tag{2.11}$$

Thus, (2.10) and (2.11) imply that

$$\int_\Omega \phi\,\mathrm{d}\zeta - \int_\Omega \phi\,\mathrm{d}\nu = \int_\Omega |x|^{-ap}\eta\nabla u\cdot\phi\,\mathrm{d}x - \int_\Omega \frac{|u|^q\phi}{|x|^{bq}}\,\mathrm{d}x,$$

i.e.

$$\int_\Omega \phi\,\mathrm{d}\zeta = \sum_{j\in J}\nu_j\phi(x_j) + \int_\Omega |x|^{-ap}\eta\nabla u\cdot\phi\,\mathrm{d}x. \tag{2.12}$$

It follows from Lemma 2.1 that

$$S_{a,b}(\nu_j)^{p/q} \leqslant \zeta(x_j) = \nu_j.$$

This result implies that

$$\nu_j = 0 \quad \text{or} \quad \nu_j \geqslant S_{a,b}^{N/pd}.$$

If the second case, $\nu_j \geqslant S_{a,b}^{N/pd}$, holds for some $j \in J$, then from (1.8), (2.1), (2.4) and (2.5), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right) \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\lim_{n\to\infty}\int_\Omega \frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\lim_{n\to\infty}\int_\Omega \left[F(x,u_n) - \frac{1}{p}f(x,u_n)u_n\right]\mathrm{d}x \\ &= \frac{d}{N}\int_\Omega \mathrm{d}\nu - \lambda\int_\Omega \left[F(x,u) - \frac{1}{p}f(x,u)u\right]\mathrm{d}x \end{aligned}$$

$$\geqslant \left(\frac{d}{N} - \lambda\varepsilon\right) \int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x + \frac{d}{N} S_{a,b}^{N/pd} - \lambda c(\varepsilon)|\Omega|$$
$$\geqslant \frac{d}{N} S_{a,b}^{N/pd} - \lambda c\left(\frac{d}{2N\lambda}\right)|\Omega|,$$

where $\varepsilon = d/2N\lambda$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in J$ and hence

$$\int_\Omega \frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x \to \int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x.$$

Thus, the Brézis–Lieb Lemma [**4**] implies that $u_n \to u$ in $L^q(\Omega, |x|^{-bq})$.

Next we prove that there exists a convergent subsequence.

To show that $u_n \to u$ in $D_a^{1,p}(\Omega)$, from the Brézis–Lieb Lemma [**4**], it suffices to show that $\nabla u_n \to \nabla u$ a.e. in Ω and $\|u_n\| \to \|u\|$.

On the one hand, we have the inequality

$$|x|^{-ap}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n - \nabla u) \geqslant 0; \tag{2.13}$$

the equality holds if and only if $\nabla u_n = \nabla u$.

On the other hand, let $v = u_n$ and $v = u$ in (2.5), respectively. Then, letting $n \to \infty$, we have

$$\begin{aligned}\|u_n\|^p &= \int_\Omega \left(|x|^{-ap}|\nabla u_n|^p - \mu\frac{|u_n|^p}{|x|^{p(a+1)}}\right)\mathrm{d}x\\ &= \int_\Omega \frac{|u_n|^q}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x + o(1)\|u_n\|\\ &\to \int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u)u\,\mathrm{d}x\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\int_\Omega \left[|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla u - \mu\frac{|u_n|^{p-2}u_n u}{|x|^{p(a+1)}}\right]\mathrm{d}x\\ = \int_\Omega \frac{|u_n|^{q-2}u_n u}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u_n)u\,\mathrm{d}x\\ \to \int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x + \lambda\int_\Omega f(x,u)u\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), it follows that

$$\begin{aligned}&\int_\Omega |x|^{-ap}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_n - \nabla u)\,\mathrm{d}x\\ &\quad= \int_\Omega |x|^{-ap}|\nabla u_n|^p\,\mathrm{d}x - \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u\,\mathrm{d}x\\ &\qquad- \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot(\nabla u_n - \nabla u)\,\mathrm{d}x \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{2.16}$$

From (2.13) and (2.16) we know that $\nabla u_n \to \nabla u$ a.e. in Ω. On the other hand, we have

$$\begin{aligned} o(1)\|u_n\| &= \|u_n\|^p - \int_\Omega |x|^{-bq}|u_n|^q\,\mathrm{d}x - \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \\ &= \|u_n - u\|^p + \|u\|^p - \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x - \lambda\int_\Omega f(x,u)u\,\mathrm{d}x \\ &= \|u_n - u\|^p + o(1)\|u\|, \end{aligned}$$

since $I'(u) = 0$. Thus, we prove that $\{u_n\}$ converges strongly to u in $D_a^{1,p}(\Omega)$. □

3. Existence of a sequence of arbitrarily small solutions

In this section, we shall prove that there exist infinitely many solutions for problem (1.1) which tend to zero. Let X be a Banach space and define

$$\Sigma := \{A \subset X \setminus \{0\}\colon A \text{ is closed in } X \text{ and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in N\colon \text{there exists } \varphi \in C(A, R^m \setminus \{0\}),\ -\varphi(x) = \varphi(-x)\}.$$

If there is no mapping φ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geqslant k$. We list some properties of the genus [**18**].

Proposition 3.1. *Let A and B be closed symmetric subsets of X that do not contain the origin. Then the following hold.*

1. *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leqslant \gamma(B)$.*
2. *If there exists an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
3. *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geqslant \gamma(A) - \gamma(B)$.*
4. *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk–Ulam Theorem.*
5. *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X\colon \|x - A\| \leqslant \delta\}$.*

The following version of the Symmetric Mountain Pass Lemma is due to Kajikiya [**18**].

Lemma 3.2. *Let E be an infinite-dimensional space and let $I \in C^1(E, R)$ and suppose the following conditions hold.*

(C_1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the local Palais–Smale condition, i.e. for some $\bar{c} > 0$, every sequence $\{u_k\}$ in E satisfying $\lim_{k\to\infty} I(u_k) = c < \bar{c}$ and $\lim_{k\to\infty}\|I'(u_k)\|_{E^*} = 0$ has a convergent subsequence.*

(C_2) *For each $k \in N$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then either (R_1) *or* (R_2) *below holds.*

(R_1) *There exists a sequence* $\{u_k\}$ *such that* $I'(u_k) = 0$, $I(u_k) < 0$ *and* $\{u_k\}$ *converges to zero.*

(R_2) *There exist two sequences* $\{u_k\}$ *and* $\{v_k\}$ *such that* $I'(u_k) = 0$, $I(u_k) < 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k\to\infty} I(v_k) = 0$ *and* $\{v_k\}$ *converges to a non-zero limit.*

Remark 3.3. From Lemma 3.2 we have a sequence $\{u_k\}$ of critical points such that $I(u_k) \leqslant 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

Remark 3.4. In [**18**], the functional $I(u)$ is required to satisfy the Palais–Smale condition. However, if $I(u)$ satisfies the local Palais–Smale condition with the critical value levels $c \leqslant 0$, the results of [**18**, Theorem 1] remain true.

In order to get infinitely many solutions, we need some lemmas. Under the assumptions of Theorem 1.1, we take $\varepsilon = 1/\lambda_1$ (where λ_1 is the first eigenvalue of $L_\mu u := -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu|x|^{-p(a+1)}|u|^{p-2}u$ with Dirichlet boundary condition). Then, by the definition of $S_{a,b}$, (1.7) and Lemma 2.2, for $\lambda \in (0, \lambda_1)$ we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega \left[|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right]\mathrm{d}x - \frac{1}{q}\int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\int_\Omega F(x,u)\,\mathrm{d}x \\ &\geqslant \frac{1}{p}\int_\Omega \left[|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right]\mathrm{d}x - \frac{1+\lambda\varepsilon q}{q}\int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda b(\varepsilon)|\Omega| \\ &\geqslant \frac{1}{p}\|u\|^p - \frac{1+q}{q}S_{a,b}^{-q/p}\|u\|^q - \lambda b\left(\frac{1}{\lambda_1}\right)|\Omega| \\ &= A\|u\|^p - B\|u\|^q - \lambda C, \end{aligned}$$

where

$$A \equiv \frac{1}{p}, \qquad B \equiv \frac{1+q}{q}S_{a,b}^{-q/p}, \qquad C \equiv b\left(\frac{1}{\lambda_1}\right)|\Omega|.$$

Let $Q(t) = At^p - Bt^q - \lambda C$. Then

$$I(u) \geqslant Q(\|u\|).$$

Furthermore, there exists

$$\lambda_* = \min\left\{\lambda_1, \frac{A(q-p)}{qC}\left(\frac{pA}{qB}\right)^{p/(q-p)}\right\} > 0$$

such that, for $\lambda \in (0, \lambda_*)$, $Q(t)$ attains its positive maximum, that is, there exists

$$R_1 \equiv \left(\frac{pA}{qB}\right)^{1/(q-p)}$$

https://doi.org/10.1017/S0013091509001813 Published online by Cambridge University Press

such that

$$e_1 = Q(R_1) = \max_{t \geqslant 0} Q(t) > 0.$$

Therefore, for $e_0 \in (0, e_1)$, we may find $R_0 < R_1$ such that $Q(R_0) = e_0$. Now we define

$$\chi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant R_0, \\ \dfrac{At^p - \lambda C - e_1}{Bt^q}, & t \geqslant R_1, \\ C^\infty, \quad \chi(t) \in [0,1], & R_0 \leqslant t \leqslant R_1. \end{cases}$$

Then it is easy to see that $\chi(t) \in [0,1]$ and $\chi(t)$ is C^∞. Let $\psi(u) = \chi(\|u\|)$ and consider the perturbation of $I(u)$:

$$G(u) = \frac{1}{p}\int_\Omega \left[|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right] \mathrm{d}x - \frac{\psi(u)}{q}\int_\Omega \frac{|u|^q}{|x|^{bq}}\,\mathrm{d}x - \lambda\psi(u)\int_\Omega F(x,u)\,\mathrm{d}x. \tag{3.1}$$

Then

$$\begin{aligned} G(u) &\geqslant A\|u\|^p - B\psi(u)\|u\|^q - \lambda C \\ &= \bar{Q}(\|u\|), \end{aligned}$$

where $\bar{Q}(t) = At^p - B\chi(t)t^q - \lambda C$ and

$$\bar{Q}(t) = \begin{cases} Q(t), & 0 \leqslant t \leqslant R_0, \\ e_1, & t \geqslant R_1. \end{cases}$$

From the above arguments, we have the following result.

Lemma 3.5. *Let $G(u)$ be defined as in (3.1). Then*

(i) *$G \in C^1(D_a^{1,p}(\Omega), R)$ and G is even and bounded from below,*

(ii) *if $G(u) < e_0$, then $\bar{Q}(\|u\|) < e_0$ and, consequently, $\|u\| < R_0$ and $I(u) = G(u)$,*

(iii) *there exists λ^* such that, for $\lambda \in (0, \lambda^*)$, G satisfies a local (PS) condition for*

$$c < e_0 \in \left(0, \min\left\{e_1, \frac{d}{N}S_{a,b}^{N/pd} - \lambda c\left(\frac{d}{2N\lambda}\right)|\Omega|\right\}\right).$$

Proof. Items (i) and (ii) are immediate. Item (iii) is a consequence of item (ii) and Lemma 2.2. □

Lemma 3.6. *Assume that* (H_3) *of Theorem 1.1 holds. Then, for any $k \in N$, there exists $\delta = \delta(k) > 0$ such that $\gamma(\{u \in D_a^{1,p}(\Omega) \colon G(u) \leqslant -\delta(k)\} \setminus \{0\}) \geqslant k$.*

https://doi.org/10.1017/S0013091509001813 Published online by Cambridge University Press

Proof. First, by (H_3) of Theorem 1.1, for any fixed $u \in D_a^{1,p}(\Omega)$, $u \neq 0$, we have

$$F(x, \rho u) \geqslant M(\rho)(\rho u)^p \quad \text{with } M(\rho) \to \infty \text{ as } \rho \to 0.$$

Next, given any $k \in N$, let E_k be a k-dimensional subspace of $D_a^{1,p}(\Omega)$. There then exists a constant σ_k such that

$$\|u\| \leqslant \sigma_k |u|_p \quad \text{for all } u \in E_k.$$

Therefore, for any $u \in E_k$ with $\|u\| = 1$ and ρ small enough, we have

$$\begin{aligned}
G(\rho u) &= I(\rho u) \\
&= \frac{\rho^p}{p} \int_\Omega \left[|x|^{-ap} |\nabla u|^p - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right] \mathrm{d}x - \frac{\rho^q}{q} \int_\Omega \frac{|u|^q}{|x|^{bq}} \,\mathrm{d}x - \lambda \int_\Omega F(x, \rho u) \,\mathrm{d}x \\
&\leqslant \frac{\rho^p}{p} \|u\|^p - \frac{\rho^q}{q} \int_\Omega \frac{|u|^q}{|x|^{bq}} \,\mathrm{d}x - \lambda M(\rho) \int_\Omega |u|^p \,\mathrm{d}x \\
&\leqslant \left(\frac{1}{p} - \frac{\lambda M(\rho)}{\sigma_k^p} \right) \rho^p \\
&= -\delta(k) < 0,
\end{aligned}$$

since $\lim_{|\rho| \to 0} M(\rho) = +\infty$. Therefore,

$$\{u \in E_k : \|u\| = \rho\} \subset \{u \in D_a^{1,p}(\Omega) : G(u) \leqslant -\delta(k)\} \setminus \{0\}.$$

This completes the proof. $\square$

Proof of Theorem 1.1. Recall that

$$\Sigma_k = \{A \in D_a^{1,p}(\Omega) \setminus \{0\} : A \text{ is closed and } A = -A,\ \gamma(A) \geqslant k\}$$

and define

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} G(u).$$

By Lemma 3.5 (i) and 3.6, we know that $-\infty < c_k < 0$. Therefore, assumptions (C_1) and (C_2) of Lemma 3.2 are satisfied. This means that G has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 3.5 (ii). $\square$

Acknowledgements. The project is supported by the NSFC (Grant 10871096), the Research Foundation for the 12th Five-Year Plan Period of the Department of Education of Jilin Province, China (Grant [2011] no. 196) and the Natural Science Foundation of Changchun Normal University. The authors thank the anonymous referee for valuable suggestions regarding the original manuscript.

References

1. R. B. Assunção, P. C. Carrião and O. H. Miyagaki, Multiplicity of solutions for critical singular problems, *Appl. Math. Lett.* **19** (2006), 741–746.
2. R. B. Assunção, P. C. Carrião and O. H. Miyagaki, Critical singular problems via concentration-compactness lemma, *J. Math. Analysis Applic.* **326** (2007), 137–154.
3. R. B. Assunção, P. C. Carrião and O. H. Miyagaki, Subcritical perturbations of a singular quasilinear elliptic equation involving the critical Hardy–Sobolev exponent, *Nonlin. Analysis* **66** (2007), 1351–1364.
4. H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Am. Math. Soc.* **88** (1983), 486–490.
5. H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical exponents, *Commun. Pure Appl. Math.* **34** (1983), 437–477.
6. L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.* **53**(3) (1984), 259–275.
7. D. Cao and P. Han, Solutions for semilinear elliptic equations with critical exponents and Hardy potential, *J. Diff. Eqns* **205** (2004), 521–537.
8. D. Cao and P. Han, Solutions to critical elliptic equations with multi-singular inverse square potentials, *J. Diff. Eqns* **224** (2006), 332–372.
9. D. Cao and S. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, *J. Diff. Eqns* **193** (2003), 424–434.
10. J. Chabrowski, On multiple solutions for the nonhomogeneous p-Laplacian with a critical Sobolev exponent, *Diff. Integ. Eqns* **8** (1995) 705–716.
11. J. Chen and S. Li, On multiple solutions of a singular quasi-linear equation on unbounded domain, *J. Math. Analysis Applic.* **275** (2002), 733–746.
12. A. Ferrero and F. Gazzola, Existence of solutions for singular critical growth semilinear elliptic equations, *J. Diff. Eqns* **177** (2001), 494–522.
13. R. Filippucci, P. Pucci and F. Robert, On a p-Laplace equation with multiple critical nonlinearities, *J. Math. Pures Appl.* **91** (2009), 156–177.
14. N. Ghoussoub and C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, *Trans. Am. Math. Soc.* **352** (2000), 5703–5743.
15. X. M. He and W. M. Zou, Infinitely many arbitrarily small solutions for singular elliptic problems with critical Sobolev-Hardy exponents, *Proc. Edinb. Math. Soc.* **52** (2009), 97–108.
16. T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, *J. Inequal. Applicat.* **1** (1997), 275–292.
17. L. Huang, X. Wu and C. Tang, Existence and multiplicity of solutions for semilinear elliptic equations with critical weighted Hardy–Sobolev exponents, *Nonlin. Analysis* **71** (2009), 1916–1924.
18. R. Kajikiya, A critical-point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Analysis* **225** (2005), 352–370.
19. D. Kang, Positive solutions to the weighted critical quasilinear problems, *Appl. Math. Computat.* **213** (2009), 432–439.
20. S. Li and W. Zou, Remarks on a class of elliptic problems with critical exponents, *Nonlin. Analysis* **32** (1998), 769–774.
21. P. L. Lions, The concentration-compactness principle in the caculus of variations: the limit case, I, *Rev. Mat. Ibero.* **1** (1985), 45–120.
22. P. L. Lions, The concentration-compactness principle in the caculus of variations: the limit case, II, *Rev. Mat. Ibero.* **1** (1985), 145–201.
23. R. Musina, Existence and multiplicity results for a weighted p-Laplace equation involving Hardy potentials and critical nonlinearities, *Rend. Lincei Mat. Appl.* **20** (2009), 127–143.

24. S. Secchi, D. Smets and M. Willem, Remarks on a Hardy–Sobolev inequality, *C. R. Acad. Sci. Paris* **336** (2003), 811–815.
25. E. A. Silva and M. S. Xavier, Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents, *Annales Inst. H. Poincaré Analyse Non Linéaire* **20** (2003), 341–358.
26. D. Smets, A concentration-compactness principle lemma with applications to singular eigenvalue problems, *J. Funct. Analysis* **167** (1999), 463–480.
27. G. Talenti, Best constant in Sobolev inequality, *Annali Mat. Pura Appl.* **110** (1976), 353–372.
28. S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, *Adv. Diff. Eqns* **2** (1996), 241–264.
29. B. Xuan, The solvability of quasilinear Brézis–Nirenberg-type problems with singular weights, *Nonlin. Analysis* **62** (2005), 703–725.
30. M. Yang and Z. Shen, The effect of domain topology on the number of positive solutions for singular elliptic problems involving the Caffarelli–Kohn–Nirenberg inequalities, *J. Math. Analysis Applic.* **334** (2007), 273–288.