Canad. J. Math. Vol. **69** (5), 2017 pp. 1143–1168
http://dx.doi.org/10.4153/CJM-2016-013-7
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The Seven Dimensional Perfect Delaunay Polytopes and Delaunay Simplices

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Abstract. For a lattice L of $\mathbb{R}^n$, a sphere $S(c,r)$ of center c and radius r is called *empty* if for any $v \in L$ we have $\|v - c\| \geq r$. Then the set $S(c,r) \cap L$ is the vertex set of a *Delaunay polytope* $P = \operatorname{conv}(S(c,r) \cap L)$. A Delaunay polytope is called *perfect* if any affine transformation ϕ such that $\phi(P)$ is a Delaunay polytope is necessarily an isometry of the space composed with an homothety.

Perfect Delaunay polytopes are remarkable structures that exist only if $n = 1$ or $n \geq 6$, and they have shown up recently in covering maxima studies. Here we give a general algorithm for their enumeration that relies on the Erdahl cone. We apply this algorithm in dimension seven, which allows us to find that there are only two perfect Delaunay polytopes: 3_{21}, which is a Delaunay polytope in the root lattice E_7, and the Erdahl Rybnikov polytope.

We then use this classification in order to get the list of all types of Delaunay simplices in dimension seven and found that there are eleven types.

1 Introduction

A lattice L is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \subset \mathbb{R}^n$ with $(v_1, \ldots, v_n)$ being independent. For such L a sphere $S(c,r)$ of center c and radius r is called *empty* if for any $v \in L$ we have $\|v - c\| \geq r$. A polytope P is called a *Delaunay polytope* if it is full-dimensional and if the vertex-set of P is $S(c,r) \cap L$ with $S(c,r)$ an empty sphere. A Delaunay polytope is called *perfect* if any affine transformation ϕ such that $\phi(P)$ is a Delaunay polytope is necessarily an isometry of the space composed with an homothety.

In [11,23] it was proved that for dimension $n \leq 5$ the only possible perfect Delaunay polytope is the interval $[0,1]$. Also in [11] it was proved that the Gosset's polytopes 2_{21} and 3_{21}, which are Delaunay polytopes of E_6 and E_7, are perfect. From the construction of infinite sequences in [14,19,24,27] we know that for any dimension $n \geq 6$ there exist perfect Delaunay polytopes. In [14] for any $n \geq 6$ we define a Delaunay polytope ED_n of a lattice LD_n. The lattice LD_n is formed by lamination over the root lattice D_{n-1}, and we proved in [22] that ED_n is the unique Delaunay polytope of maximum circumradius of LD_n and compute its covering density.

In [8] we proved that $ED_6 = 2_{21}$ is the unique perfect Delaunay polytope in dimension 6. This work uses a new approach in order to prove the following theorem.

Received by the editors January 13, 2016; revised March 23, 2016.
Published electronically June 21, 2016.
The author gratefully acknowledges support from the Alexander von Humboldt foundation.
AMS subject classification: **11H06**, 11H31.
Keywords: Delaunay polytope, enumeration, polyhedral methods.

https://doi.org/10.4153/CJM-2016-013-7 Published online by Cambridge University Press

Theorem 1.1 *The 7-dimensional perfect Delaunay polytopes are the Gosset polytope $ED_7 = 3_{21}$ and the Erdahl and Rybnikov polytope ER_7* [24, 25].

Proof See Section 9. ■

Perfect Delaunay polytopes are of importance for the theory of Covering Maxima. A *covering maximum* is a lattice L such that its covering density is reduced if it is perturbed. In [22] it was proved that a lattice L is a covering maximum if and only if the Delaunay polytopes of maximum circumradius are perfect and *eutactic* (see [22] for the definition). This characterization echoes Voronoi's theorem [44] for the characterization of lattices of maximum density in terms of perfection and eutacticity. In [22] we proved that LD_n is one such covering maxima. Based on Theorem 1.1 and partial enumerations in dimensions 8, 9, and 10 we state the following conjecture.

Conjecture 1.2 *For each $n \geq 6$, the lattice LD_n defined in* [14] *has maximal covering density among all covering maxima.*

The Minkowski conjecture [34, p. 18] on the product of inhomogeneous forms has inspired a lot of research. Recently, it has been proved for $n \leq 8$ in [29–32] by computational methods based on Korkine–Zolotarev reduction theory. Other theoretical approaches have been attempted in [37, 42] by Dynamical System Theory. In particular, the following theorem was proved in [42, Corollary 1.3].

Theorem 1.3 *If Conjecture 1.2 holds for a dimension $n \geq 1$, then Minkowski's conjecture holds for dimension n.*

As a consequence of the work of this paper, we have that Minkowski's conjecture is correct in dimension 7, thereby confirming [29].

We prove Theorem 1.1 by using the *Erdahl cone*, which is defined as the set of polynomial functions f of degree at most 2 such that $f(x) \geq 0$ for $x \in \mathbb{Z}^n$. We used this cone in [22] for the study of covering maxima. We have then to do a kind of dual description computation with the problem that the number of defining inequalities is infinite, we have no local polyhedrality result as in the perfect form case (see [41] for details), and we are interested in only a subset of the extreme rays (see Theorem 3.3 for the list of possible kinds of extreme rays of the Erdahl cone).

In [8] we used a different approach, *i.e.*, hypermetrics that allowed us to find all the 6-dimensional perfect Delaunay polytopes. But this approach relied on previous work [2, 38] on 6-dimensional Delaunay simplices that we could not extend easily to dimension 7. Thus, it appears that the only way to classify the perfect 7-dimensional Delaunay polytopes is to use the Erdahl cone. Moreover, we are able to use this classification in order to get the classification of Delaunay simplices.

Theorem 1.4 *Up to arithmetic equivalence there are eleven types of seven-dimensional Delaunay simplices. The full list is given in Table 1.*

Proof See Section 10. ■

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i	Representative S_i	$\mathrm{vol}(S_i)$	$\lvert\mathrm{Stab}(S_i)\rvert$	Nb interval
1	$e_0,\dots,e_6,e_7$	1	40320	127
2	$e_0,\dots,e_6,(1,1,1,1,1,1,2)$	2	40320	63
3	$e_0,\dots,e_6,(0,0,1,1,1,1,2)$	2	1440	63
4	$e_0,\dots,e_6,(1,2,2,2,2,2,3)$	3	5040	42
5	$e_0,\dots,e_6,(1,1,1,2,2,2,3)$	3	1152	42
6	$e_0,\dots,e_6,(0,1,1,2,2,2,3)$	3	240	41
7	$e_0,\dots,e_6,(1,1,1,1,1,3,4)$	4	1440	27
8	$e_0,\dots,e_6,(1,1,1,1,2,2,4)$	4	240	31
9	$e_0,\dots,e_6,(1,1,1,2,2,3,4)$	4	144	31
10	$e_0,\dots,e_6,(1,1,3,3,3,4,5)$	5	72	24
11	$e_0,\dots,e_6,(1,1,1,1,2,3,5)$	5	48	24

Table 1: Representative of Delaunay simplices in dimension 7. $e_1,\dots,e_7$ is the standard basis of $\mathbb{Z}^7$ and $e_0 = 0$. $\mathrm{vol}(S)$ is $n!$ times the Euclidean volume of S. $\lvert\mathrm{Stab}(S)\rvert$ is the size of the lattice automorphism group preserving S. "Nb interval" is the number of Delaunay polyhedra of the type $\{0,1\}\times\mathbb{Z}^6$ in which S_i is contained.

In contrast to perfect Delaunay polytopes, the lattices simplices of this list (except the trivial simplex) had not been discovered before. A similar study was undertaken in [33] for the set of shortest vectors of lattices. In view of this work it seems reasonable to think that the classification of Delaunay simplices is possible in dimension 8. Of equal importance, the classification of perfect Delaunay polytopes in dimension 8 could be done, and a conjectural list of the 27 known possibilities is available in [15].

In Section 2 a generalization of delaunay polytopes, *i.e.,* Delaunay polyhedra, are considered; their basic structure and relation to the Erdahl cone are introduced here. The facial structure of the Erdahl cone is reviewed in Section 3, in particular, not all extreme rays of the Erdahl cone are related to Delaunay polyhedra [23]. We also explain how the hypercube $[0,1]^n$ corresponds to the cut polytope in the Erdahl cone. Section 4 is not used in later sections. In it we construct a retraction of the Erdahl cone on the faces defined by Delaunay polyhedra. In Section 5 we establish the link between the Erdahl cone and the classic L-type theory. In Section 6 we do the same for the hypermetric cone. In Section 7 we give the connectivity and finiteness results on which our enumeration algorithm relies. Then we present in Section 8 our enumeration method, which is modelled on the Voronoi algorithm for perfect forms [35] and on the adjacency decomposition method [5]. In Section 9 we give the results obtained in the classification of 7-dimensional perfect Delaunay polyhedra. In Section 10 we use this classification to classify the 7-dimensional types of Delaunay simplices.

2 Delaunay Polyhedra

Denote by $E_2(n)$ the vector space of polynomials of degree at most 2 on $\mathbb{R}^n$ and by $\mathrm{AGL}_n(\mathbb{Z})$ the group of affine integral transformations on $\mathbb{Z}^n$. The *Erdahl cone* is defined as

$$\mathrm{Erdahl}(n) = \{f \in E_2(n) \text{ such that } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}.$$

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It is a convex cone of dimension $(n+1)(n+2)/2$ on which the group $\mathrm{AGL}_n(\mathbb{Z})$ acts. All defining inequalities $f(x) \geq 0$ are equivalent under $\mathrm{AGL}_n(\mathbb{Z})$, and therefore $\mathrm{Erdahl}(n)$ is not polyhedral.

We denote by $\cdot$ the standard scalar product on $\mathbb{R}^n$ defined by $x \cdot y = x^T y$. For a symmetric matrix A and $x \in \mathbb{R}^n$ we define $A[x] = x^T A x$. We write any $f \in E_2(n)$ in the form

$$f(x) = \mathrm{Cst}(f) + 2\,\mathrm{Lin}(f) \cdot x + \mathrm{Quad}(f)[x]$$

with $\mathrm{Cst}(f) \in \mathbb{R}$, $\mathrm{Lin}(f) \in \mathbb{R}^n$ and $\mathrm{Quad}(f)$ a $n \times n$ symmetric matrix. We define S^n to be the set of symmetric matrices, $S^n_{>0}$ the set of positive definite matrices, and $S^n_{\geq 0}$ the set of positive semidefinite matrices. We also define $\mathrm{Erdahl}_{>0}(n)$ to be the set of $f \in \mathrm{Erdahl}(n)$ with $\mathrm{Quad}(f) \in S^n_{>0}$. If $f \in \mathrm{Erdahl}(n)$, then $\mathrm{Quad}(f) \in S^n_{\geq 0}$.

A *sublattice* of $\mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$. An *affine sublattice* is one of the form $x_0 + L$ with $x_0 \in \mathbb{Z}^n$ and L a sublattice of $\mathbb{Z}^n$. A lattice $L \subset \mathbb{Z}^n$ is called *saturated* if $(L \otimes \mathbb{R}) \cap \mathbb{Z}^n = L$. If L_1 and L_2 are two sublattices of $\mathbb{Z}^n$, we write $L_1 \oplus_{\mathbb{Z}} L_2 = \mathbb{Z}^n$ if $L_1 \cap L_2 = \{0\}$ and $L_1 \cup L_2$ generates $\mathbb{Z}^n$ over $\mathbb{Z}$. In that case both L_1 and L_2 are saturated.

Definition 2.1 Let us fix $n \geq 1$ and define the following.

(i) A *Delaunay polyhedron* D is a set of the form $D = P_{L'}(D) + L(D) \subset \mathbb{Z}^n$ with
- $\mathrm{conv}(P_{L'}(D))$ a Delaunay polytope of an affine sublattice L' of $\mathbb{Z}^n$,
- $L(D)$ a sublattice of $\mathbb{Z}^n$ and
- $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$.

(ii) A *Delaunay simplex set* S is a Delaunay polyhedron with $|S| = n+1$.

(iii) A *repartitioning set* R is a Delaunay polyhedron with $|S| = n+2$.

The *isotropy lattice* $L(D)$ is uniquely determined by D, and its dimension is called the *degeneracy rank* of D denoted $\mathrm{degrk}(D)$. Note that D is the vertex set of a convex body only when $L(D) = 0$. Also, a Delaunay polyhedron is full-dimensional, *i.e.,* the smallest affine saturated lattice containing D is $\mathbb{Z}^n$ itself.

The set $P_{L'}(D)$ is included in L' and depends on L'. For any two choices L'_1 and L'_2 there exist a bijective affine map $\phi\colon L'_1 \to L'_2$ with $\phi(P_{L'_1}(D)) = P_{L'_2}(D)$. When we consider properties that do not depend on the integral representation, we drop the lattice and write $P(D)$.

For $f \in \mathrm{Erdahl}(n)$ we write

$$Z(f) = \left\{ x \in \mathbb{Z}^n \text{ such that } f(x) = 0 \right\}.$$

In the classical geometry of numbers, the essential tool is the quadratic form Q instead of the quadratic function. The following establish a direct link between them.

Definition 2.2 For a Delaunay simplex set $S \subset \mathbb{Z}^n$ and $Q \in S^n$, there exists a unique function $f \in E_2(n)$ such that

- $f(x) = 0$ for $x \in S$,
- $Q = \mathrm{Quad}(f)$.

This function is denoted $f_{S,Q}$ and depends linearly on Q.

The key reason for using Delaunay polyhedra is the following theorem.

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Theorem 2.3 (i) *If D is a Delaunay polyhedron, then there exist a function $f \in \mathrm{Erdahl}(n)$ such that $D = Z(f)$.*
(ii) *If $f \in \mathrm{Erdahl}(n)$, then either $Z(f)$ is empty or there exist a k-dimensional saturated affine lattice $L \subset \mathbb{Z}^n$ such that $Z(f)$ is a Delaunay polyhedron of L.*

Proof (i) Let us take a Delaunay polyhedron $D = P_{L'}(D) + L(D)$ with $P_{L'}(D)$ being a Delaunay polyhedron of a lattice L' with $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. Let us denote by $S(c, r)$ the sphere around $P_{L'}(D)$ and write $L' = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$. The function

$$\begin{array}{rcl} f' : \mathbb{Z}^n & \longrightarrow & \mathbb{R} \\ x = (x_1, \dots, x_n) & \longmapsto & \left\| \sum_{i=1}^n x_i v_i - c \right\|^2 - r^2 \end{array}$$

belongs to $\mathrm{Erdahl}(k)$. More precisely, $f'(x) = 0$ if and only if $\sum_{i=1}^k x_i v_i \in P_{L'}(D)$. For $x \in \mathbb{Z}^n$ we write $x = x_1 + z$ with $x_1 \in L'$ and $z \in L(D)$ and write $f(x) = f'(x_1)$. It is easy to prove that $f \in \mathrm{Erdahl}(n)$ and $Z(f) = D$.

(ii) This is [23, Corollary 2.5]. ■

We define the *rational closure* $S^n_{\mathrm{rat},\geq 0}$ to be the set of positive semidefinite forms whose kernel is defined by rational equalities.

Corollary 2.4 *If $f \in \mathrm{Erdahl}(n)$ is such that $Z(f)$ is a Delaunay polyhedron, then* $\mathrm{Quad}(f) \in S^n_{\mathrm{rat},\geq 0}$.

Proof Let us write $D = Z(f)$ and take a lattice $L' \subset \mathbb{Z}^n$ with $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. We write any $x \in \mathbb{Z}^n$ as $x = x_1 + z$ with $x_1 \in L'$ and $z \in L(D)$. There is a quadratic function f_1 on L' such that $f(x) = f_1(x_1)$ and $Z(f_1) = P_{L'}(D)$. Thus, $\mathrm{Quad}(f_1)$ is positive definite, and since $L(D)$ is an integral lattice, the matrix $\mathrm{Quad}(f)$ belongs to $S^n_{\mathrm{rat},\geq 0}$. ■

Given a set $V \subset \mathbb{Z}^n$ we will need to be able to test whether or not it is a Delaunay polyhedron. Algorithm 1 does this iteratively for a finite point set by solving larger and larger linear programs until a conclusion is reached. The algorithm can be easily adapted to the case of a point set of the form $R + L$ with L a lattice and R finite. The corresponding algorithm for perfect form is given in [33, Algorithm 1].

If D is a n-dimensional Delaunay polyhedron, then we define

$$\mathrm{Aut}(D) = \{\phi \in \mathrm{AGL}_n(\mathbb{Z}) : \phi(D) = D\}.$$

When using Algorithm 1 it is best to impose that the sought function f is invariant under $\mathrm{Aut}(D)$, since it simplifies the search and a Delaunay polyhedron admits an invariant function (see Corollary 2.6).

Before stating our result on the description of $\mathrm{Aut}(D)$, we review the notion of *semidirect product*. Given a group G, we call G a semidirect product and write $G = N \rtimes H$ if N is a normal subgroup of G, H a subgroup $G = NH$ and $N \cap H = \{e\}$.

Theorem 2.5 *If D is a n-dimensional Delaunay polyhedron of degeneracy degree d, then we have the isomorphism*

$$\mathrm{Aut}(D) = \left\langle (\mathbb{Z}^d)^{1+n-d} \rtimes \mathrm{GL}_d(\mathbb{Z}) \right\rangle \rtimes \mathrm{Aut}(P(D)).$$

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Data: A finite set $V \subset \mathbb{Z}^n$ of full affine rank
Result: A quadratic function $f \in \mathrm{Erdahl}_{>0}(n)$ such that $Z(f) = V$ if it exists and **false** otherwise
$S_{\mathrm{vert}} \leftarrow V$.
$S_{\mathrm{vect}} \leftarrow \varnothing$.
repeat
  $v \leftarrow$ random element of $\mathbb{Z}^n$
  $S_{\mathrm{vert}} \leftarrow S_{\mathrm{vert}} \cup \{v\}$
until *The set* $\{ev_v$ *for* $\in S_{\mathrm{vert}}\}$ *has rank* $(n+1)(n+2)/2$;
while *no solution has been reached* **do**
  Form the linear program

$$\begin{aligned} \text{minimize}_{f \in E_2(n)} \quad & \mathrm{Tr}(\mathrm{Quad}(f)) \\ \text{subject to} \quad & f(v) = 0 \text{ for } v \in V \\ & f(v) \geq 1 \text{ for } v \in S_{\mathrm{vert}} - V \\ & \mathrm{Quad}(f)[v] \geq 1 \text{ for } v \in S_{\mathrm{vect}} \end{aligned}$$

  if *The linear program is infeasible* **then**
    return **false**
  end
  $f_0 \leftarrow$ a rational optimal solution.
  if $f_0 \in \mathrm{Erdahl}(n)$ *and* $Z(f_0) = V$ **then**
    return f_0
  end
  if $\mathrm{Quad}(f_0) \notin S^n_{>0}$ **then**
    Find a vector $v \in \mathbb{Z}^n$ with $Q(f_0)[v] \leq 0$
    $S_{\mathrm{vect}} \leftarrow S_{\mathrm{vect}} \cup \{v\}$
  end
  if $\mathrm{Quad}(f_0) \in S^n_{>0}$ *and* $f_0 \notin \mathrm{Erdahl}(n)$ **then**
    Find a vector $v \in \mathbb{Z}^n$ with $f_0(v) < 0$
    $S_{\mathrm{vert}} \leftarrow S_{\mathrm{vert}} \cup \{v\}$
  end
  if $f_0 \in \mathrm{Erdahl}(n)$ *and* $Z(f_0) \neq V$ **then**
    Find a vector $v \in Z(f_0) - V$.
    $S_{\mathrm{vert}} \leftarrow S_{\mathrm{vert}} \cup \{v\}$
  end
end

Algorithm 1: Testing Delaunay realizability of a finite set of points

Proof Let us take a basis $v = (v_i)_{1 \leq i \leq d}$ of $L(D)$. Any automorphism of D will send v to another basis of $L(D)$, and this determines a component $\mathrm{GL}_d(\mathbb{Z})$ of the automorphism group. Let us write L' for an affine sublattice of $\mathbb{Z}^n$ such that $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. Such affine sublattice are determined by $1 + n - d$ vectors in $L(D)$, and this determines the component $(\mathbb{Z}^d)^{1+n-d}$ of the automorphism group. The last component comes from the fact that the automorphisms have to preserve the polytope $\mathrm{conv}(P(D))$. ■

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We denote by $\mathrm{Aff}(D)$ the normal subgroup $(\mathbb{Z}^d)^{1+n-d} \rtimes \mathrm{GL}_d(\mathbb{Z})$ given in the above theorem.

Corollary 2.6 *A Delaunay polyhedron D admits a function $f \in \mathrm{Erdahl}(n)$ with $Z(f) = D$ that is invariant under $\mathrm{Aut}(D)$.*

Proof Let us write $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. A vector $x \in \mathbb{Z}^n$ is decomposed as $x = x_1 + z$ with $x_1 \in L'$ and $z \in L(D)$. Any function f with $Z(f) = D$ must have $\mathrm{Quad}(f)[z] = 0$ and $\mathrm{Lin}(f) \cdot z = 0$. Therefore, there exist a function f_1 on L' such that $f(x) = f_1(x_1)$. So, f is invariant under $\mathrm{Aff}(D)$. The group acting on f_1 is $\mathrm{Aut}(P_{L'}(D))$, which is finite. The function

$$f_1'(x_1) = \sum_{u \in \mathrm{Aut}(P_{L'}(D))} f(u(x_1))$$

is $\mathrm{Aut}(P_{L'}(D))$ invariant. Thus, we get a function $f'(x) = f_1'(x_1)$ invariant under $\mathrm{Aut}(D)$. ∎

If D and D' are two Delaunay polyhedra such that $D \subset D'$, then we define the stabilizer group

$$\begin{aligned}\mathrm{Stab}(D, D') &= \left\{ \phi \in \mathrm{AGL}_n(\mathbb{Z}) : \phi(D) = D \text{ and } \phi(D') = D' \right\}\\ &= \mathrm{Aut}(D) \cap \mathrm{Aut}(D').\end{aligned}$$

We have the following results.

Theorem 2.7 *Let D and D' be two Delaunay polyhedra satisfying $D \subset D'$.*

(i) *We have $L(D) \subset L(D')$ and $\mathrm{Aff}(D) \subset \mathrm{Aff}(D')$.*

(ii) *There exist a finite group $G_1 \subset \mathrm{Aut}(P(D))$ such that*

$$\mathrm{Stab}(D, D') = \mathrm{Aff}(D) \rtimes G_1.$$

In particular, $\mathrm{Aff}(D)$ is a finite index subgroup of $\mathrm{Stab}(D, D')$

Proof We have $L(D) \subset L(D')$. Let us take a $\mathbb{Z}$-basis $\{e_1, \dots, e_d\}$ of $L(D)$ and complement it by adding $\{e_{d+1}, \dots, e_{d'}\}$ to a basis of $L(D')$. We can find $f_1, \dots, f_{n-d'}$ so that the e_i and f_j form a basis of $\mathbb{Z}^n$. The group $\mathrm{Aff}(D')$ is generated by translations along $f_1, \dots, f_{n-d'}$ and $\mathrm{GL}_{d'}(\mathbb{Z})$. The group generated by translations along $\{e_{d+1}, \dots, e_{d'}\}$ and $\mathrm{GL}_d(\mathbb{Z})$ directly embeds into $\mathrm{GL}_{d'}(\mathbb{Z})$, and this determines the group inclusion. So (i) holds.

By (i) we have the inclusion $\mathrm{Aff}(D) \subset \mathrm{Stab}(D, D')$. Let us choose a lattice L' with $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. Then if $u \in \mathrm{Stab}(D, D') \subset \mathrm{Aut}(D)$, we can find an unique element $n \in \mathrm{Aff}(D)$ such that un^{-1} stabilizes L'. Thus, un^{-1} belongs to $\mathrm{Aut}(P_{L'}(D))$. The image defines the group G_1. The finite index property follows. ∎

3 Facial Structure of the Erdahl Cone

The standard scalar product on S^n is $\langle A, B\rangle = \mathrm{Tr}(AB)$. We equip $\mathrm{Erdahl}(n)$ with the inner product

$$(f, g) = \mathrm{Cst}(f)\,\mathrm{Cst}(g) + 2\,\mathrm{Lin}(f) \cdot \mathrm{Lin}(g) + \langle \mathrm{Quad}(f), \mathrm{Quad}(g)\rangle$$

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and for each $x \in \mathbb{Z}^n$ we define the evaluation function ev_x by $ev_x(y) = (1 + x \cdot y)^2$ such that $(f, ev_x) = f(x)$.

A *convex cone* $\mathcal{C}$ is defined as a set invariant under addition and multiplication by positive scalars. Then $\mathcal{C}$ is called *full-dimensional* if the only vector space containing it is $\mathbb{R}^m$; $\mathcal{C}$ is called *pointed* if no linear subspace of positive dimension is contained in it. Let $\mathcal{C}$ be a full-dimensional pointed convex polyhedral cone in $\mathbb{R}^m$. Given $f \in (\mathbb{R}^m)^*$, the inequality $f(x) \geq 0$ is said to be *valid* for $\mathcal{C}$ if it holds for all $x \in \mathcal{C}$. A *face* of $\mathcal{C}$ is a pointed polyhedral cone $\{x \in \mathcal{C} : f(x) = 0\}$, where $f(x) \geq 0$ is a valid inequality.

A face of dimension 1 is called an *extreme ray* of $\mathcal{C}$; a face of dimension $m - 1$ is called a *facet* of $\mathcal{C}$. The set of faces of $\mathcal{C}$ forms a partially ordered set under inclusion. We write $F \triangleleft G$ if $F \subset G$ and $\dim F = \dim G - 1$. Two extreme rays of $\mathcal{C}$ are said to be *adjacent* if they generate a two-dimensional face of $\mathcal{C}$. Two facets of $\mathcal{C}$ are said to be *adjacent* if their intersection has dimension $m - 2$. Any $(m - 2)$-dimensional face of $\mathcal{C}$ is called a *ridge*, and it is the intersection of exactly two facets of $\mathcal{C}$.

By the Farkas–Minkowski–Weyl Theorem (see *e.g.*, [40, Corollary 7.1a]), a convex cone $\mathcal{C}$ is *polyhedral* if and only it is defined either by a finite set of *generators* $\{v_1, \dots, v_N\} \subseteq \mathbb{R}^m$ or by a finite set of linear functionals $\{f_1, \dots, f_M\} \subseteq (\mathbb{R}^m)^*$:

$$\mathcal{C} = \Big\{ \sum_{i=1}^{N} \lambda_i v_i : \lambda_i \geq 0 \Big\} = \big\{ x \in \mathbb{R}^m : f_i(x) \geq 0 \big\}.$$

Every minimal set of generators $\{v_1, \dots, v_{N'}\}$ defining a polyhedral cone $\mathcal{C}$ has the property

$$\{\mathbb{R}_+ v_1, \dots, \mathbb{R}_+ v_{N'}\} = \{e : e \text{ extreme ray of } \mathcal{C}\}.$$

Every minimal set of linear functionals $\{f_1, \dots, f_{M'}\}$ defining $\mathcal{C}$ has the property that $\{F_1, \dots, F_{M'}\}$ with $F_i = \{x \in \mathcal{C} : f_i(x) = 0\}$ is the set of facets of $\mathcal{C}$. The problem of transforming a minimal set of generators into a minimal set of linear functionals (or vice versa) is called the *dual description problem*.

In our work, we have to deal with Delaunay polyhedra with an infinite number of vertices, and we cannot apply the Farkas–Minkowski–Weyl theorem to them nor of course existing dual-description software [1, 26].

Definition 3.1 Let $D \subset \mathbb{Z}^n$ be a Delaunay polyhedron.

(i) We define the vector space

$$\operatorname{Space}(D) = \{f \in E_2(n) \text{ such that } f(x) = 0 \text{ for } x \in D\}.$$

(ii) The dimension of $\operatorname{Space}(D)$ is called the *perfection rank* $\operatorname{rankperf}(D)$ and D is *perfect* if $\operatorname{rankperf}(D) = 1$.

Proposition 3.2 *Let $D \subset \mathbb{Z}^n$ be a Delaunay polyhedron.*

(i) $\operatorname{Space}(D) \cap \operatorname{Erdahl}(n)$ *is a face of* $\operatorname{Erdahl}(n)$ *of dimension* $\dim \operatorname{Space}(D)$.
(ii) *If D is perfect, then* $\operatorname{Space}(D) \cap \operatorname{Erdahl}(n)$ *is an extreme ray of* $\operatorname{Erdahl}(n)$.

Proof Let $p = \dim \operatorname{Space}(D)$ and let $g_1, \dots, g_p$ be a basis of $\operatorname{Space}(D)$. Since D is a Delaunay polyhedron, there exists a function $f \in \operatorname{Erdahl}(n)$ such that $D = Z(f)$. For each $1 \leq i \leq p$ there exist $\lambda_i > 0$ such that $\lambda_i f + g_i \in \operatorname{Erdahl}(n)$, and so (i) follows. (ii) follows directly from (i). ∎

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For a perfect Delaunay polyhedron D, we denote by f_D a generator of the extreme ray $\mathrm{Space}(D) \cap \mathrm{Erdahl}(n)$.

Theorem 3.3 ([23, Theorem 2.1]) *The generators of extreme rays of* $\mathrm{Erdahl}(n)$ *are*

(i) *the constant function* $f = 1$,
(ii) *the functions of the form* $f_{a,\beta}(x) = (a_1 x_1 + \ldots a_n x_n + \beta)^2$ *with* $(a_1, \ldots, a_n)$ *not collinear to an integral vector,*
(iii) *the functions of the form* f_D *with* D *a perfect Delaunay polyhedron.*

This theorem indicates that the structure of the extreme rays of $\mathrm{Erdahl}(n)$ is more complicated than for a polytope. Since we are interested only in the third class of extreme rays, some reduction will be necessary, and it turns out that we can work out everything with Delaunay polyhedra.

In this paper we will work with both spaces of functions in $E_2(n)$ and with point sets of Delaunay polyhedra.

Definition 3.4 Given a Delaunay polyhedron D,

(i) the cone of admissible functions is defined as

$$\mathrm{Erdahl}_{\mathrm{supp}}(D) = \left\{ g \in E_2(n) : g(x) \geq 0 \text{ for all } x \in D \right\};$$

(ii) the cone of evaluation functions is defined as

$$\mathrm{Erdahl}^*_{\mathrm{supp}}(D) = \left\{ \sum_{x \in D} \lambda_x ev_x \text{ with } \lambda_x \geq 0 \right\}.$$

Theorem 3.5 *Let* D *be a Delaunay polyhedron of perfection rank* r.

(i) $\mathrm{Erdahl}_{\mathrm{supp}}(D)$ *is the product of a pointed convex cone* C_D *with* $\mathrm{Space}(D)$. *The dual of* C_D *is* $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$.
(ii) *Any Delaunay polyhedron* $D' \subset D$ *of perfection rank* $r+1$ *gives a facet of* $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$.
(iii) *If* $L(D) = 0$*, then facets of* $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$ *correspond to Delaunay polyhedra* $D' \subset D$ *of perfection rank* $r+1$.

Proof (i) If f and $-f$ both belong to $\mathrm{Erdahl}_{\mathrm{supp}}(D)$, then $f(x) = 0$ for $x \in D$ and so $f \in \mathrm{Space}(D)$. So, $\mathrm{Erdahl}_{\mathrm{supp}}(D)$ is the sum of $\mathrm{Space}(D)$ and a closed convex cone. The duality result follows from [3, Part IV.5]) for closed full-dimensional convex cones.

(ii) Let $f \in \mathrm{Erdahl}(n)$ such that $Z(f) = D'$. Thus, we have $f(x) = 0$ on D' and $f(x) > 0$ on $D - D'$, and so f defines a facet of $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$.

(iii) If $L(D) = 0$, then for any facet of $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$ we can find a set $D' \subset D$ and a function $f \in E_2(n)$ with $f(x) = 0$ for $x \in D'$ and $f(x) > 0$ for $x \in D - D'$. There exist a function $g \in \mathrm{Erdahl}(n)$ such that $D = Z(g)$. Then we can find $\lambda > 0$ such that $f + \lambda g \in \mathrm{Erdahl}(n)$. Then we have $D' = Z(f + (\lambda + 1)g)$, and so D' is a Delaunay polyhedron. ∎

Theorem 2.1 of [23] shows that if $L(D) \neq 0$, there are other facets of $\mathrm{Erdahl}^*_{\mathrm{supp}}(D)$ than the ones from Delaunay polyhedra.

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Proposition 3.6 *If D is a n-dimensional perfect Delaunay polyhedron with degeneracy degree d, then $P(D)$ has at least*

$$\binom{n-d+2}{2} - 1$$

points.

Proof The isotropy lattice $L(D)$ has dimension d, and we can choose a complement lattice L' such that $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. If f is a quadratic function with $Z(f) = D$, then f is determined by its restriction to L'. Hence f belongs to a vector space of dimension $\binom{n-d+2}{2}$, and this gives the minimal number of determining inequalities. ■

A surprising relation has been found between the Erdahl cone of the hypercube $\{0,1\}^n$ and the cut polytope, which is classic polytope of combinatorial optimization [12]. Write $N = \{1,\dots,n\}$; if $S \subset N$, then the cut metric δ_S on N is defined as follows:

$$\delta_S(i,j) = \begin{cases} 1 & \text{if } |\{ij\} \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\delta_S = \delta_{N-S}$, and the *cut polytope* CUTP_n is defined as the convex hull of the cut metrics δ_S. The cone defined by the cut polytope is defined as

$$\mathrm{CCUTP}_n = \Big\{ \sum_{S \subset \{1,\dots,n\}} \lambda_S(1,\delta_S) \text{ with } \lambda_S \geq 0 \Big\}.$$

The facets of the cone CCUTP_n are in one-to-one correspondence with the facets of the polytope CUTP_n.

Theorem 3.7 *The polyhedral cone* $\mathrm{Erdahl}^*_{\mathrm{supp}}(\{0,1\}^n)$ *is linearly equivalent to* CCUTP_{n+1}.

Proof The hypercube $[0,1]^n$ is defined as the convex hull in 2^n of vectors $v = (v_1,\dots,v_n)$ with $v_i \in \{0,1\}$. For every such vector the evaluation function is

$$ev_v(x) = 1 + 2(v \cdot x) + (v \cdot x)^2.$$

Thus, we can associate with ev_v the vector

$$\big(1, (v_i)_{1\leq i\leq n}, (v_i v_j)_{1\leq i\leq j\leq n}\big).$$

Since $v_i^2 = v_i$ for $v_i \in \{0,1\}$ this vector family is linearly equivalent to

$$\big(1, (v_i)_{1\leq i\leq n}, (v_i v_j)_{1\leq i<j\leq n}\big) \text{ for } v \in \{0,1\}^n.$$

With the same v we can associate the vector $\overline{v} = (0, v_1, \dots, v_n)$ and the set

$$S = \{i \in \{0,\dots,n\} \mid \overline{v}_i = 1\} \subset \{0,\dots,n\}.$$

The cut metric δ_S on $\{0,\dots,n\}$ is characterized by $(\delta_S(i,j))_{0\leq i<j\leq n}$ and since $\delta_S(i,j) = (\overline{v}_i - \overline{v}_j)^2$, the family of pairs $(1,\delta_S)$ is linearly equivalent to the family of evaluation map ev_v. ■

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It is interesting to note that the symmetry group of the hypercube $[0,1]^n$ is of size $2^n n!$, but that the symmetry group of the cut polytope CUTP_{n+1} is of size $2^n(n+1)!$ for $n \neq 3$ [10,12].

4 The Delaunay Polyhedra Retract

For $f \in \mathrm{Erdahl}(n)$, we define $\mathrm{Vect}\, Z(f)$ to be the vector space spanned by difference of elements of $Z(f)$. We define $V_f = \mathrm{Vect}\, Z(f) + \mathrm{Ker}\, \mathrm{Quad}(f)$. For a given $f \in \mathrm{Erdahl}(n)$ we call *proper pair* a pair $(g,h) \in E_2(n)^2$ such that $g \in \mathrm{Erdahl}(n)$, $h(x) \geq 0$ for $x \in \mathbb{R}^n$ and $f = g + h$.

Lemma 4.1 *Let* $f \in \mathrm{Erdahl}(n)$.

(i) *For a proper pair* (g,h), $Z(f) \subset Z(g)$ *and* $\mathrm{Ker}\, \mathrm{Quad}(h) \subset V_f$.
(ii) *There exists a proper pair* (g,h) *with* $\mathrm{Ker}\, \mathrm{Quad}(h) = V_f$.

Proof If $x \in Z(f)$, then one sees that necessarily $h(x) = g(x) = 0$. So, $h(x) = 0$ for $x \in \mathrm{conv}(Z(f))$, which implies $\mathrm{Vect}\, Z(f) \subset \mathrm{Ker}\, \mathrm{Quad}(h)$. Also, it is clear that for any vector $v \in \mathrm{Ker}\, \mathrm{Quad}(f)$, we have $\mathrm{Quad}(g)[v] = \mathrm{Quad}(h)[v] = 0$. Hence, (i) holds.

Let us denote by $\mathbb{Z}\, \mathrm{Ker}\, \mathrm{Quad}(f)$ the smallest subspace of $\mathbb{R}^n$ having an integral basis containing $\mathrm{Ker}\, \mathrm{Quad}(f)$. By [23, Decomposition Lemma 3.1] there exist a $g \in \mathrm{Erdahl}(n)$ with $\mathrm{Ker}\, \mathrm{Quad}(g) = \mathbb{Z}\, \mathrm{Ker}\, \mathrm{Quad}(f)$ and a positive semidefinite form Q_1 such that

$$f(x) = g(x) + Q_1[x].$$

Let V (resp. W) denote an integral supplement of $\mathrm{Ker}\, \mathrm{Quad}(f)$ in $\mathbb{Z}\, \mathrm{Ker}\, \mathrm{Quad}(f)$ (resp. $\mathbb{Z}\, \mathrm{Ker}\, \mathrm{Quad}(f)$in $\mathbb{Z}^n$). Denote by $\phi_1, \dots, \phi_m$ some affine functions on $\mathbb{Z}^n$ such that $\phi_i(\mathbb{Z}\, \mathrm{Ker}\, \mathrm{Quad}(f)) = 0$ and

$$\left\{ x \in W \mid \phi_1(x) = \cdots = \phi_m(x) = 0 \right\} = \mathrm{Vect}\, Z(f) \cap W.$$

Then for $\epsilon > 0$ small enough, the function $g_1 - \epsilon \sum_{i=1}^m \phi_i(x)^2$ is still in $\mathrm{Erdahl}(n)$. So, one gets that the pair $(f-h, h)$ with $h(x) = Q_1[x] + \epsilon \sum_{i=1}^m \phi_i(x)^2$ is proper and (ii) is true. ∎

Let us call W an integral supplement of V_f. Denote by $\mathrm{Quad}(f)|_W$ the quadratic form $\mathrm{Quad}(f)$ restricted to W. A proper pair (g,h) is called *extremal* if $\det \mathrm{Quad}(h)|_W$ is maximal among all proper pairs. Lemma 4.1.(i) implies that the notion of being extremal is independent of the chosen subspace W, while Lemma 4.1.(ii) implies that there is at least one form of non-zero determinant.

Theorem 4.2 *Let* $f \in \mathrm{Erdahl}(n)$.

(i) *If* (g,h) *is an extremal proper pair for* f, *then* $Z(f)$ *is a Delaunay polyhedron.*
(ii) *There exists a unique extremal proper pair* (g,h).

Proof Let us take an integral supplement W as above and suppose, to avoid trivialities, that $W \neq \varnothing$. So, by restricting f to W, we can assume that $\mathrm{Quad}(h)$ is positive

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definite:

$$f(x) = g(x) + h(x)$$

with $h(x) \geq 0$ for $x \in \mathbb{R}^n$. This condition on h is equivalent to

$$A_h = \begin{pmatrix} \mathrm{Cst}(h) & \mathrm{Lin}(h)^T \\ \mathrm{Lin}(h) & \mathrm{Quad}(h) \end{pmatrix}$$

being positive definite. Hence we consider the following semidefinite programming problem. Find the $A_h \in S^{n+1}_{\geq 0}$ maximizing $\det \mathrm{Quad}(h)$ and satisfying, for all $x \in \mathbb{Z}^n$,

$$f(x) \geq A_h[(1,x)] = h(x).$$

We also write $g = f - h$.

Suppose that $Z(g)$ is not a Delaunay polyhedron and that $\mathrm{Quad}(h)$ is positive definite. Then $Z(g)$ does not generates $\mathbb{R}^n$ as an affine space and so there exists an affine function ϕ such that $\phi(Z(g)) = 0$. Then there exist $\alpha > 0$ such that the pair $(f - h', h')$ with $h' = h + \alpha\phi^2$ is still proper. Since $\det \mathrm{Quad}(h') > \det \mathrm{Quad}(h)$, the pair is not extremal and (i) holds.

Let us take $N = n(n+1)/2$ points $v_i \in \mathbb{Z}^n$ such that the family $\{(1, v_i)(1, v_i)^T\}_{1\leq i\leq N}$ is of full rank. The inequalities $g(v_i) \geq A[(1, v_i)] \geq 0$ imply that all coefficients of A are bounded. Thus, the problem is actually to minimize the convex function $h \mapsto -\log\det \mathrm{Quad}(h)$ over a compact convex set, hence existence follows.

Since $-\log\det$ is a strictly convex function, we know that if we have two optimal solutions h_1 and h_2 then $\mathrm{Quad}(h_1) = \mathrm{Quad}(h_2)$. Let us denote by $D_1 = Z(g_1)$ and $D_2 = Z(g_2)$ the corresponding Delaunay polyhedra. The function $h_{\mathrm{mid}} = (h_1 + h_2)/2$ is also an optimal solution of the problem. We have $Z(g_{\mathrm{mid}}) = D_1 \cap D_2$. The set $D_1 \cap D_2$ is necessarily a Delaunay polyhedron, since otherwise we could still increase the determinant by the above construction and this would contradict the optimality. But if $Z(f)$ is a Delaunay polyhedron, then the terms $\mathrm{Cst}(f)$ and $\mathrm{Lin}(f)$ are determined by $\mathrm{Quad}(f)$. So, one gets $h_1 = h_2$ and the uniqueness is proved on the restriction to W. But Lemma 4.1(i) implies that once $\mathrm{Quad}(h)$ is known on W then it is known on $\mathbb{Z}^n$. By the condition $h \geq 0$, the linear part is known as well. ■

Note that in the above determinant maximization problem a finite set of inequalities suffices to determine the optimal solution. This follows from the fact that since we are maximizing the determinant we can assume that the lowest eigenvalue of $\mathrm{Quad}(h)$ is bounded away from 0, *i.e.,* that there exist $c > 0$ such that $\mathrm{Quad}(h) \geq cI_n$.

For $f \in \mathrm{Erdahl}(n)$, we write $\mathrm{proj}(f) = g$ and $\mathrm{proj}'(f) = h$ with (g, h) the unique extremal pair associated to f. From the unicity of extremal pairs we also get that proj and proj′ commute with the action of $\mathrm{AGL}_n(\mathbb{Z})$.

Conjecture 4.3 *The function* proj *is continuous.*

Let us define $\mathrm{Erdahl}_{dp}(n)$ to be the set of $f \in \mathrm{Erdahl}(n)$ such that $Z(f)$ is a Delaunay polyhedron. The above conjecture would imply that the set $\mathrm{Erdahl}_{dp}(n)$ is simply connected, and this could be of interest for topological applications. However, we were not able to prove the conjecture, and instead we prove connectedness results in later sections that are sufficient for our purposes.

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5 Relation with L-types Theory

In this section we reframe classical L-type theory from [45] (see also [41] for a modern account) in terms of Erdahl cones and state several key lemmas.

Definition 5.1 Let $Q \in S^n_{\mathrm{rat},\geq 0}$. The Delaunay polyhedra tessellation $\mathcal{DPT}(Q)$ defined by Q is the set of Delaunay polyhedra D such that there exist a $f \in \mathrm{Erdahl}(n)$ with

- $Z(f) = D$,
- $\mathrm{Quad}(f) = Q$.

If Q is positive definite, then the Delaunay polyhedra tessellation is the classical Delaunay polytope tessellation; *i.e.,* all Delaunay polyhedra occurring are actually vertex sets of Delaunay polytopes. The number of translation classes of Delaunay polyhedra is always finite. These Delaunay polyhedra tessellations were considered in [20, Section 2.2]. Efficient algorithms for the enumeration of Delaunay polytope tessellations are given in [21].

From this, one can define the L-type which are parameter spaces of Delaunay polytope tessellations.

Definition 5.2 Let us take a Delaunay polyhedra tessellation $\mathcal{T}$. Then the L-type $LT(\mathcal{T})$ is defined as the closure of the set of quadratic forms Q such that $\mathcal{DPT}(Q) = \mathcal{T}$. It is well known (see [41, 45] for proofs) that L-types are polyhedral cones.

A L-type is called *primitive* if it is of maximal dimension; this is equivalent to saying that all its Delaunay polyhedra are Delaunay simplex sets.

The set of all L-types for all possible Delaunay tessellations defines a tessellation of the cone $S^n_{\mathrm{rat},\geq 0}$.

Given two Delaunay polyhedra tessellation $\mathcal{T}$ and $\mathcal{T}'$, we say that $\mathcal{T}'$ *is a refinement of* $\mathcal{T}$ if every Delaunay polyhedron of $\mathcal{T}'$ is included in a single Delaunay polyhedron of $\mathcal{T}$. Then $\mathcal{T}'$ is a *simplicial refinement* if all its Delaunay polyhedra are Delaunay simplex sets.

Proposition 5.3 *Any Delaunay polyhedra tessellation $\mathcal{T}$ admits at least one simplicial refinement.*

Proof Let us denote by $L(\mathcal{T})$ the space $L(D)$ of the Delaunay polyhedra D occurring in the tessellation and by $Q \in S^n_{\mathrm{rat},\geq 0}$ the form realizing it. Let us take a lattice L' such that $L' \oplus_{\mathbb{Z}} L(\mathcal{T}) = \mathbb{Z}^n$. We write $x \in \mathbb{Z}^n$ as $x = x_1 + z$ with $x_1 \in L'$ and $z \in L(\mathcal{T})$. D is a Delaunay polyhedron for the quadratic function $f \in \mathrm{Erdahl}(n)$. Necessarily, f is of the form $f(x) = f_1(x_1)$ with f_1 a quadratic function on L'. Let us denote by $(D_i)_{i \in I}$ the Delaunay polyhedra occurring in the tessellation. Let us take a basis $w_1, \dots, w_m$ of $L(\mathcal{T})$ and define linear forms ϕ_i on $\mathbb{Z}^n$ such that $\phi_i(w_j) = \delta_{ij}$ and $\phi_i(L') = 0$. The quadratic form

$$Q'[x] = Q[x] + \sum_{i=1}^{m} \big(\phi_i(x)\big)^2$$

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is positive definite, and the Delaunay polyhedra tessellation $\mathcal{D}T$ corresponding to Q' is formed by the Delaunay polyhedra

$$D_i + \sum_{k=1}^{m} \{a_k, a_k + 1\} w_k \text{ with } 1 \le i \le r \text{ and } a_k \in \mathbb{Z}.$$

In particular, $\mathcal{T}_2$ is a refinement of $\mathcal{T}$.

Since the L-type domain form a tiling of $S^n_{\mathrm{rat},\ge 0}$, the form Q' belongs to at least one primitive L-type $LT(\mathcal{T}_2)$. This L-type defines a Delaunay polyhedra tessellation by simplices, which is a refinement of $\mathcal{T}_2$ and so of $\mathcal{T}$. ■

Let us take a primitive L-type $\mathcal{T}$. Any facet F of $\mathcal{T}$ is determined by a pair of Delaunay simplex sets S_1 and S_2 in the Delaunay tessellation that determine a repartitioning set. We say that two facet-defining repartitioning sets are in the same class if they define the same facet F of $\mathcal{T}$. If R is a repartitioning set, then $\mathrm{conv}(R)$ admits exactly two triangulations (see [41, Section 4.3.2]). One says that two primitive L-types are adjacent if their intersection is a codimension 1 face in the cone $S^n_{>0}$. When we move from one L-type to another L-type, the Delaunay tessellation is changed and this is done combinatorially by the repartitioning sets. That is, some Delaunay simplex sets are merged into repartitioning sets and the triangulation is changed to the other triangulation, thus yielding another L-type.

Given a Delaunay polyhedron D a Delaunay polyhedra tessellation $\mathcal{T}$ is called *D-proper* if D is the union of the Delaunay polyhedra D' contained in D. We have the following lemma.

Lemma 5.4 *Let D be a Delaunay polyhedron. The graph formed by the primitive L-types whose corresponding Delaunay polyhedra tessellations are primitive and D-proper is connected.*

Proof Let us consider a function $f_D \in \mathrm{Erdahl}(n)$ such that $Z(f_D) = D$. We can consider the triangulations induced by positive definite quadratic forms on D itself. By the theory of regular triangulations (see [7] for an account), this set is connected.

Any triangulation $\mathcal{T}_{\mathrm{part}}$ on D induced by a positive definite quadratic form Q can be extended to a triangulation $\mathcal{T}$ of $\mathbb{Z}^n$. It suffices to replace Q by $Q + \lambda\,\mathrm{Quad}(f_D)$ for λ sufficiently large. The reason is that $\mathrm{Quad}(f_D)$ will not change the Delaunay triangulation for Delaunay simplex sets contained in D.

Now given a primitive L-type LT whose Delaunay polyhedra tessellation $\mathcal{T}$ is D-proper, we denote by $\mathcal{S}$ its set of Delaunay simplex sets included in D. We consider the following cone $\mathcal{C}(\mathcal{S})$:

$$\mathcal{C}(\mathcal{S}) = \left\{ Q \in S^n_{\mathrm{rat},\ge 0} \mid f_{S,Q}(x) \ge 0 \text{ for } S \in \mathcal{S} \text{ and } x \in \mathbb{Z}^n - S \right\}.$$

This cone is convex and is an union of primitive L-types. Thus, this set of L-types is connected. The connectedness follows by combining the above results. ■

Delaunay polytopes are restricted to the set D.

https://doi.org/10.4153/CJM-2016-013-7 Published online by Cambridge University Press

n	1	2	3	4	5	6	7
$\mathrm{PossVol}(n)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,4,5\}$

Table 2: Possible volume of lattice Delaunay simplices. See [2] for the proof for $n \le 6$ and Section 10 for the proof for $n = 7$.

Theorem 5.5 *Consider a Delaunay polyhedron D and two Delaunay simplex sets S and S' in D. Then there exists a sequence $\{S = S_0, S_1, \dots, S_m = S'\}$ of Delaunay simplex sets with $S_i \subset D$ for $0 \le i \le m$ such that $S_i \cup S_{i+1}$ is a repartitioning set for $0 \le i \le m-1$.*

Proof Let us take $f_D \in \mathrm{Erdahl}(n)$ a function such that $Z(f_D) = D$. Take f_S, $f_{S'}$ to be the corresponding functions for S and S'. Denote by $\mathcal{T}$ the Delaunay polyhedra tessellation defined by $\mathrm{Quad}(f_D)$, which obviously has D as one of its components.

When we perturb $\mathrm{Quad}(f_D)$, we are changing the Delaunay tessellation. However, if we take $\epsilon > 0$ small enough, we can ensure that the Delaunay polyhedra tessellations $\mathcal{DPT}(\mathrm{Quad}(f_D + \epsilon f_S))$ and $\mathcal{DPT}(\mathrm{Quad}(f_D + \epsilon f'_S))$ are D-proper. By applying Proposition 5.3 we can find simplicial refinement of those two tessellations, which we name $\mathcal{T}R$ and $\mathcal{T}R'$ and are both D-proper. We call LT and LT' the corresponding primitive L-types.

By Lemma 5.4 there exists a path between LT and LT' that uses only D-proper L-types. By following this path, we can change S into another Delaunay simplex set S_2 in $\mathcal{T}R'$.

Denote by $f_1, \dots, f_r$ the facets of LT'. Every such facet corresponds to a family of repartitioning sets. We say that two Delaunay simplex sets included in D are adjacent if their union is a repartitioning set that gives a facet of LT'. The Delaunay polyhedron D is a coarsening obtained by merging all simplices, so the above defined graph is connected. This means that we can find a path from S_2 to S'. ■

6 Relation with Hypermetric Theory

We define the *volume* $\mathrm{vol}(S)$ of a Delaunay simplex set S to be $n!\,\mathrm{Vol}(\mathrm{conv}(S))$ with Vol the Euclidean volume. This rescaled volume is an integer and satisfies $\mathrm{vol}(S) \le n!$. The possible rescaled volumes $\mathrm{PossVol}(n)$ are given in Table 2 for $n \le 7$ and a super-exponential lower bound on $\max \mathrm{PossVol}(n)$ is proven in [39]. The best known upper bound [12, Proposition 14.2.4] is

$$\max \mathrm{PossVol}(n) \le n! \frac{2^n}{\binom{2n}{n}}. \tag{6.1}$$

Definition 6.1 Let us take two n-dimensional Delaunay polyhedron D, D' with $D \subset D'$. We can define the *generalized hypermetric cone*

$$\mathrm{Hyp}(D, D') = \left\{ f \in E_2(n) \mid f(x) = 0 \text{ if } x \in D \text{ and } f(x) \ge 0 \text{ if } x \in D' \right\}$$

We have the inclusion $\mathrm{Hyp}(D, D') \subset \mathrm{Erdahl}_{\mathrm{supp}}(D')$ and $\mathrm{Hyp}(D, D')$ is a priori defined by an infinity of inequalities.

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As a direct application we can express the L-type domains as intersection of generalized hypermetric cones.

Proposition 6.2 *Let $\mathcal{T}$ be a Delaunay polyhedra tessellation. Then we have*

$$LT(\mathcal{T}) = \bigcap_{D\in\mathcal{T}} \operatorname{Quad}\operatorname{Hyp}(D,\mathbb{Z}^n).$$

Proposition 6.3 *The cone* $\operatorname{Hyp}(D,D')$ *is polyhedral.*

Proof Let us take a Delaunay simplex set $S=\{v_0,\dots,v_n\}\subset D$, which exists by Proposition 5.3. If we prove the polyhedrality of $\operatorname{Hyp}(S,D')$, then $\operatorname{Hyp}(D,D')$ is polyhedral as well, since it is obtained from $\operatorname{Hyp}(S,D')$ by adding equalities $f(x)=0$ for $x\in D-S$.

Suppose that a $v\in D'$ defines a relevant inequality. Then there exists a function f such that $f(x)=0$ for $x\in S\cup\{v\}$ and $f(x)>0$ for $x\in D'-S\cup\{v\}$. Since D' is a Delaunay polyhedron, there exists a function g such that $g(x)=0$ for $x\in D'$ and $g(x)>0$ for $x\in\mathbb{Z}^n-D'$. Then we can find $\lambda>0$ such that $f(x)+\lambda g(x)>0$ for $x\in\mathbb{Z}^n-S\cup\{v\}$. As a consequence, the polytope $\operatorname{conv}(S\cup\{v\})$ is a Delaunay polytope. This implies that for any $i\in\{0,\dots,n\}$, the Delaunay simplex set $S_{v,i}=\{v,v_0,\dots,v_{i-1},v_{i+1},\dots,v_n\}$ has $\operatorname{vol}(S_{v,i})\le n!$ (see the proof of [12, Theorem 14.2.1]). Hence, the coefficients of v are bounded by a bound depending only on S, and this proves that $\operatorname{Hyp}(S,D')$ is polyhedral. ∎

For a Delaunay simplex set S of volume 1, the cone $\operatorname{Hyp}(S,\mathbb{Z}^n)$ is called the *hypermetric cone* and is studied in [12]. For other simplices, they are called *Baranovski cone* in [41]. The facets of the Baranovski cones are determined up to dimension 6 in [38]. There is a correspondence between facets of $\operatorname{Hyp}(S,D)$ and repartitioning sets P with $S\subset P\subset D$. That is, the inequality $f(x)\ge 0$ defines a facet of $\operatorname{Hyp}(S,D)$ if and only if $S\cup\{x\}$ is a repartitioning set.

7 Connectivity Results

For a given Delaunay polyhedron D, let us write

$$\mathcal{C}_{r,d}(D)=\left\{D'\subset D \;\middle|\; \begin{array}{l} D' \text{ a Delaunay polyhedron with} \\ \operatorname{rankperf}(D')=r \text{ and } \dim L(D')\le d \end{array}\right\}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are sets of Delaunay polyhedra, then the graph $Gr(\mathcal{A},\mathcal{B})$ is the graph on $\mathcal{A}$ with two Delaunay polyhedra $D_1,D_2\in\mathcal{A}$ adjacent if and only if $D_1\cap D_2\in\mathcal{B}$.

Theorem 7.1 *If D is a Delaunay polyhedron of perfection rank r and degeneracy degree d, then $\mathcal{C}_{r+1,d}(D)$ is decomposed into a finite number of orbits under* $\operatorname{Aut}(D)$.

Proof Without loss of generality, we can write $D=P+\mathbb{Z}^d$. Let us write $D'\subset D$ as $D'=P'+L'$ with $k=\dim L'$. By applying an element of $\operatorname{Aut}(D)$, we can assume that $L'=\mathbb{Z}^k$. So, without loss of generality, we can assume that $L'=0$. Let us take a Delaunay simplex set S in D'; its volume is bounded by $\max\operatorname{PossVol}(n)$. Again by using $\operatorname{Aut}(D)$ we can find a constant C' such that the absolute value of the coordinates

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of S in $P + \mathbb{Z}^d$ are bounded by C'. The polyhedrality of the cones $\mathrm{Hyp}(S, D)$ implies the finiteness. ■

Lemma 7.2 *If $\mathcal{C}$ is a polyhedral cone, F is a face of $\mathcal{C}$ and e, e' are two extreme rays that are not contained in F, then e and e' are connected by a path that does not intersect F.*

Proof By taking the intersection $\mathcal{C} \cap H$ with H a suitable hyperplane, we can transform $\mathcal{C}$ into a polytope P and e, e' into vertices of P. We can find an affine function ϕ such that $\phi(x) \geq 0$ is a valid inequality on P and $\phi(x) = 0$ defines the face $F \cap H$ of P. By maximizing the function ϕ over P and using the simplex algorithm (see [40, 46]), we can find paths $p(v, v_{\mathrm{opt}})$, $p(v', v_{\mathrm{opt}})$ from v, v' to an optimal vertex v_{opt} such that ϕ is monotone on both paths. Since $\phi(v) > 0$ and $\phi(v') > 0$, such paths avoid the face F and joined together give the required path. ■

Theorem 7.3 *If D is a Delaunay polyhedron of perfection rank r and degeneracy degree $d \geq 1$, then*

$$Gr\big(\mathcal{C}_{r+1,d}(D), \mathcal{C}_{r+2,d-1}(D)\big)$$

is connected.

Proof Consider two Delaunay polyhedra D_a and D'_a in D of perfection rank $r+1$. Let us take two Delaunay simplex sets S, S' contained in D_a, D'_a. By using Theorem 5.5 we can find a chain of simplices $(S_i)_{0 \leq i \leq m}$ with $S_i \subset D$, $S_0 = S$ and $S_m = S'$. Denote by $R_i = S_i \cup S_{i+1}$ the repartitioning set. Let e_{-1}, e_m be the extreme rays in $\mathrm{Hyp}(S_0, D)$, $\mathrm{Hyp}(S_m, D)$ corresponding to D_a, D'_a, respectively. For each Delaunay simplex set S_i, we consider the cone $\mathrm{Hyp}(S_i, D)$. The extreme rays correspond to Delaunay polyhedra of rank $r+1$. If a Delaunay polyhedron $D' \subset D$ has degeneracy degree d, then, necessarily, $L(D') = L(D)$. We define the restricted trace function to be

$$\phi(f) = \mathrm{Tr}\big(\mathrm{Quad}(f)|_{L(D)}\big).$$

A function $f \in \mathrm{Erdahl}_{\mathrm{supp}}(D)$ has $Z(f) \cap D$ of degeneracy degree d if and only if $\phi(f) = 0$. The hyperplane $\phi(f) = 0$ determines a face F_i of the cone $\mathrm{Hyp}(S_i, D)$. The intersection is

$$\mathrm{Hyp}(S_i, D) \cap \mathrm{Hyp}(S_{i+1}, D) = \mathrm{Hyp}(R_i, D).$$

Thus, we can find a ray e_i in $\mathrm{Hyp}(R_i, D)$, which is not contained in F_i. Since $\mathrm{Hyp}(S_i, D)$ is polyhedral, by Lemma 7.2 there exists a path from e_{i-1} to e_i in $\mathrm{Hyp}(S_i, D)$ that avoids the face F_i. So, by putting all these paths together, we get the required connectivity result. ■

Lemma 7.4 *If D_1 and D_3 are two Delaunay polyhedra of perfection rank r and $r+2$ with $D_3 \subset D_1$, then there exist exactly two Delaunay polyhedra $D_{2,1}$ and $D_{2,2}$ such that $D_3 \subset D_{2,i} \subset D_1$ with* $\mathrm{rankperf}(D_{2,i}) = r+1$.

Proof Since D_3 is a Delaunay polyhedron, there exists a Delaunay simplex set $S \subset D_3$. The Delaunay polyhedra D_3, D_1 correspond to faces F_3, F_1 of dimension $r+2$, r in the cone $\mathrm{Hyp}(S, \mathbb{Z}^n)$. It is well known from polytope theory [46, Theorem 2.7.(iii)]

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that there are exactly two faces $F_{2,1}$, $F_{2,2}$ containing F_1 and contained in F_3. Those give the corresponding Delaunay polyhedron. ∎

By using this theorem, we are able to compute inductively the Delaunay polyhedra in $\mathbb{Z}^n$. The property with the degeneracy degree ensures that we are able to effectively reduce the complexity of the computation at each step, and thus we are reduced in the end to computation with Delaunay polyhedra of degeneracy 0, *i.e.,* polytopes for which polytopal methods exist.

8 Algorithms

In [5] a general survey is presented of methods for computing dual description of highly symmetric polytopes with many facets. Among the methods presented there, we want to adapt the Recursive Adjacency Decomposition Method to our situation, *i.e.,* to a case with an infinite group and an infinity of defining inequalities.

8.1 Computing $\mathrm{Aut}(D)$

In this subsection we explain the techniques needed to compute $\mathrm{Aut}(D)$, compute $\mathrm{Stab}_{D'}(\mathrm{Aut}(D))$, and split orbits. In the decomposition of Theorem 2.5, the only component that is not clear is $\mathrm{Aut}(D_1)$, *i.e.,* the computation of the automorphism group of a Delaunay polytope. For that purpose the methods of [21] that we are using is the method of isometry groups can be used.

Definition 8.1 Let D be a Delaunay polyhedron. Take a lattice L' with $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. Take a basis $w_1, \dots, w_r$ of L'. Denote by $v_1, \dots, v_m$ the expression of the vertices of $P_{L'}(D)$ in the basis (w_i).
We define the matrix Q by $Q = \sum_{i=1}^m \binom{1}{v_i}(1, v_i^t)$. From there we define the distance function $f_D\colon \mathbb{Z}^n \times \mathbb{Z}^n \mapsto \mathbb{R}$ by

- $f_D(x, x') = \phi(x) Q^{-1} \phi(x')^T$
- with $\phi(x) = (1, u_1, \dots, u_r)$ if $x = u_1 w_1 + \cdots + u_r w_r + z$ and $z \in L(D)$.

The construction of the matrix Q and its inverse above is relatively standard. We used it first in [43], and further work on this was done in [4, 5].

The interest of this construction is that it allows to compute automorphism groups.

Theorem 8.2 *Let D be a Delaunay polyhedron. The following hold.*

(i) *If $u \in \mathrm{Aut}(D)$, then we have $f_D(u(x), u(y)) = f_D(x, y)$ for $x, y \in D$.*
(ii) *If L' is a sublattice such that $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$ and u is a permutation of $P_{L'}(D)$ such that $f_D(u(x), u(y)) = f_D(x, y)$ for $x, y \in D$, then u is induced by an affine rational transformation of $L' \otimes \mathbb{R}$.*

Proof (i) By its construction, if $z \in L(D)$, then we have $f_D(x + z, y) = f_D(x, y)$. Thus if $u \in \mathrm{Aff}(D)$ then u preserves f_D. On the other hand, if $u \in \mathrm{Aut}(P_{L'}(D))$, then we can see by summation that u preserves f_D. The proof is available, for example, in [5, 43].

(ii) The reverse implication is also available from [5, 43]. ∎

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Let us denote by $\mathrm{Aut}_{\mathbb{Q}}(P(D))$ the group of rational transformations preserving $P(D)$. By Theorem 8.2.(ii) we have $\mathrm{Aut}(P(D)) = \mathrm{AGL}_r(\mathbb{Z}) \cap \mathrm{Aut}_{\mathbb{Q}}(P(D))$ with r the dimension of L'. The computation of $\mathrm{Aut}_{\mathbb{Q}}(P(D))$ is done efficiently by using known partition backtracking software such as [36]; see [4, 5] for more details. In the cases considered in this paper, the number of vertices is quite small and this computation is very easy.

A Delaunay polytope is called *generating* if difference between its vertices generate $\mathbb{Z}^n$. If a Delaunay polytope is non-generating, then it is actually a Delaunay polytopes for more than one lattice. If $P(D)$ is generating, then we have $\mathrm{Aut}(P(D)) = \mathrm{Aut}_{\mathbb{Q}}(P(D))$, and we are done. Otherwise, we can apply some of the strategies listed in [4, Section 3.1]. Here, the situation is particularly simple, and the simplest strategy of iterating over the group elements and keeping the integral ones works very well. Also note that the above methods with only slight modifications work for testing the equivalence of Delaunay polyhedra.

8.2 Computing Stabilizers

We now give methods for computing stabilizers of Delaunay polyhedra, more precisely of the transformations preserving two polyhedra $D \subset D'$, which occurs in our computations.

Let us select a lattice L' such that $L' \oplus_{\mathbb{Z}} L(D) = \mathbb{Z}^n$. Denote by G_1 the group occurring in Theorem 2.7.

Let us define the function $f_{D,D'}$ on $P_{L'}(D)$ by $f_{D,D'}(x, y) = \big(f_D(x, y), f_{D'}(x, y)\big)$. By Theorem 8.2, the elements of the group G_1 must preserve the function $f_{D,D'}$. Thus, we can use the partition backtrack algorithm of Section 8.1 to get the group $\mathrm{Aut}(f_{D,D'})$ of permutations preserving $f_{D,D'}$.

Then we obtain the group G_1 by keeping only the elements that are in $\mathrm{AGL}_n(\mathbb{Z})$ and preserve D'. This is possible since the group $\mathrm{Aut}(f_{D,D'})$ is finite and of moderate size in most cases. All the algorithms above have equivalents for testing equivalence, and, of course, what has been done for pairs $D \subset D'$ of Delaunay polyhedra can be extended to triples $D \subset D' \subset D''$.

8.3 Splitting Orbits

Suppose that we have an orbit Gx of an element x under a group G. For a subgroup $H \subset G$ we wish to decompose Gx into orbits Hx_i. Such an orbit splitting decomposition $Gx = \bigcup_{i=1}^{m} Hx_i$ with $x_i = g_i x$ is equivalent to a double coset decomposition $G = \bigcup_{i=1}^{m} Hg_i \,\mathrm{Stab}_G(x)$.

In the case of interest to us, we have $G = \mathrm{Aut}(D)$, $H = \mathrm{Aut}(D) \cap \mathrm{Aut}(D')$ for D, D' Delaunay polyhedra with $D \subset D'$ and x a Delaunay polyhedron included in D. Since a priori $\mathrm{Aut}(D)$ is infinite, we cannot apply standard tools from computer algebra software such as `GAP` [28]. By the finiteness result Theorem 4.2.(ii) we can find a coset decomposition $G = \bigcup_{i=1}^{m} Hg_i$. However, it is not a double coset decomposition; *i.e.*, we can have $Hg_i \neq Hg_j$ but still have $Hg_i x = Hg_j x$. Therefore, we need to eliminate duplicate in order to do the orbit splitting.

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8.4 The Flipping Algorithm

Suppose that $D_1 \subset D_2 \subset D_3$ are Delaunay polyhedra having

$$\text{rankperf } D_1 = 1 + \text{rankperf } D_2 = 2 + \text{rankperf } D_3.$$

By Lemma 7.4 we know that there exists a unique Delaunay polyhedron D_2' having $D_1 \subset D_2' \subset D_3$ and $D_2 \neq D_2'$.

We can find functions $f_i \in \text{Erdahl}(n)$ such that $Z(f_i) = D_i$. We can also assume that $C_{D_i} = \mathbb{R} f_i \oplus C_{D_{i+1}}$ for $i = 1, 2$. We need to find $f_2' \in \text{Erdahl}(n)$ such that $Z(f_2') = D_2'$.

If $L(D_3) = 0$, then D_3 is a polytope; *i.e.*, it has a finite number of vertices, and the algorithm is called the *gift wrapping* procedure [5, 6, 43]. If $L(D_3) \neq 0$, we have to modify the algorithm in order to take care of the fact that we have an infinity of vertices by writing an iterative algorithm. This is quite similar to the flipping in the Voronoi algorithm [41].

Data: Delaunay polyhedra D_1, D_2, D_3 with $D_i = Z(f_i)$, $f_i \in \text{Erdahl}(n)$, $D_1 \subset D_2 \subset D_3$ and

$$\text{rankperf } D_1 - 2 = \text{rankperf } D_2 - 1 = \text{rankperf } D_3.$$

Result: $f_2' \in \text{Erdahl}(n)$, Delaunay polyhedra $D_2' = Z(f_2')$, $D_2' \neq D_2$ and $D_1 \subset D_2' \subset D_3$

$\mathcal{V} \leftarrow \emptyset$
repeat
  $v \leftarrow$ random element of $Z(f_3)$
  $\mathcal{V} \leftarrow \mathcal{V} \cup \{v\}$
  $\mathcal{L} \leftarrow (f_1(w), f_2(w))$ for $w \in \mathcal{V}$
until $\mathcal{L}$ *has rank* 2;
repeat
  $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} \leftarrow$ generators of extreme rays of $\mathcal{L}$.
  $f_2' \leftarrow \{\alpha_1 f_1 + \beta_1 f_2, \alpha_2 f_1 + \beta_2 f_2\} - \mathbb{R} f_2$.
  if *there is a* $v \in Z(f_3)$ *with* $f_2'(v) < 0$ **then**
    $\mathcal{V} \leftarrow \mathcal{V} \cup \{v\}$.
    $\mathcal{L} \leftarrow (f_1(w), f_2(w))$ for $w \in \mathcal{V}$
  end
until $f_2' \geq 0$ *on* $Z(f_3)$;
repeat
  $f_2' \leftarrow f_2' + f_3$
until $f_2' \geq 0$ *on* $\mathbb{Z}^n$;
$f_2' \leftarrow f_2' + f_3$

Algorithm 2: Flipping algorithm

The non-negativity test for f_2' on $\mathbb{Z}^n$ is done by solving a closest vector problem. The non-negativity test on D_3 is done by decomposing it into $\{v_1 + L(D_3)\} \cup \cdots \cup \{v_m + L(D_3)\}$. The non-negativity is tested by m closest vector problems. The final operation on f_2' is done to ensure that $f_2'(x) > 0$ if $x \notin Z(f_3)$.

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8.5 The Recursive Adjacency Decomposition Method

Given a Delaunay polyhedron D of perfection rank r the algorithm of this section will give the orbits of Delaunay polyhedra $D' \subset D$ of perfection rank $r+1$. If $\mathrm{degrk}(D) = 0$, then the computation of the orbits of Delaunay polyhedra can be achieved by Theorem 3.5(iii).

If $\mathrm{degrk}(D) > 0$, we have to proceed differently. By Theorem 7.3 we can limit ourselves to Delaunay polyhedra with $\mathrm{degrk}(D) \leq \mathrm{degrk}(D) - 1$. The algorithm takes one initial Delaunay polyhedron of perfection rank $r+1$ and computes the adjacent Delaunay polyhedron of perfection rank $r+1$. If an obtained Delaunay polyhedron is not equivalent to an existing one, then we insert it into the list. We iterate until all orbits have been treated. The computation of the adjacent Delaunay polyhedra adjacent to a Delaunay polyhedron D' requires the computation of orbits the Delaunay polyhedra contained in D'. Thus, we have a recursive call to the algorithm. Fortunately, the degeneracy degree diminish by at least 1 so there is no infinite recursion.

The mapping from $\mathcal{F}_1$ to $\mathcal{F}_2$ is done using the orbit splitting procedure. Finding D'' from D_2, D', and D is done using the flipping procedure.

There is a degree of choice in the initial Delaunay polyhedron D_{init}. The standard choice is if $D = P(D) + L(D)$ with $L(D) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{\mathrm{degrk}(D)}$ to take D_{init} a Delaunay polyhedron of the form

$$D_{\mathrm{init}} = P(D) + \{0, v_1\} + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_{\mathrm{degrk}(D)}.$$

The schematic of the algorithm is shown in Algorithm 3.

9 Perfect Delaunay Polytopes in Dimension 7

In the enumeration of an inhomogeneous perfect form in dimension 7, we need to describe the Delaunay polytopes that will occur. The list of perfect Delaunay polyhedra in dimension 7 is thus the following:

(a) $\{0,1\} \times \mathbb{Z}^6$,
(b) $2_{21} \times \mathbb{Z}$ with 2_{21} the Schläfli polytope,
(c) 3_{21} the Gosset polytope [12],
(d) ER_7 the polytopes discovered by Erdahl and Rybnikov [24, 25].

The geometry of the Schläfli and Gosset polytopes are described in more detail in [12, 15].

An *affine basis* of a n-dimensional Delaunay polytope D is a family of $n+1$ vertices $v_0, \ldots, v_n$ such that for any vertex v of D there exist $\lambda_i \in \mathbb{Z}$ such that $v = \sum_{i=0}^{n} \lambda_i v_i$. The perfect Delaunay polytopes of dimension 7 have an affine basis, but it is possible that in higher dimension there are perfect Delaunay polytopes without an affine basis. It is known that in dimension at least 12, there are Delaunay polytopes with no affine basis [18]. Also, the perfect Delaunay polytopes of dimension 7 are generating. Note that in [19] we found some non-generating perfect Delaunay polytopes for $n \geq 13$. We have $\mathrm{rankperf}(\{0,1\}^n) = n$ (see, for example, [12, 13]).

In terms of computation, the overwhelming majority of the time is spent computing the rank 2 faces of $\{0,1\} \times \mathbb{Z}^6$. By the recursive approach chosen, the method requires the computation of the facets of $\mathrm{Erdahl}^*_{\mathrm{supp}}(\{0,1\}^7)$, and so by Theorem 3.7

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Data: Delaunay polyhedron D of perfection rank r
Result: Set $\mathcal{F} = \{D_1, \dots, D_m\}$ of Delaunay polyhedron of perfection rank $r+1$ inequivalent under $\operatorname{Aut} D$

```
if degrk(D) = 0 then
    U ← orbits of facets of Erdahl*_supp(D).
    F ← orbits of Delaunay polyhedron from Theorem 3.5.(iii).
else
    T ← {D_init} with D_init ∈ Erdahl(n), rankperf Z(D_init) = r + 1 and
    degrk(D_init) = degrk(D) − 1.
    F ← ∅.
    while there is a D' ∈ T with degrk(D') ≤ degrk(D) − 1 do
        F ← F ∪ {D'}.
        T ← T ∖ {D'}.
        F_1 ← orbits of Delaunay polyhedra of perfection rank r + 2 in D' under
        Aut D'.
        F_2 ← orbits of Delaunay polyhedra of perfection rank r + 2 in D' under
        Stab(D', D).
        for D_2 ∈ F_2 do
            find Delaunay polyhedron D'' ⊂ D of perfection rank r + 1 with
            D_2 = D' ∩ D''.
            if D'' is not equivalent under Aut D to an element of F ∪ T then
                if degrk(D'') = degrk(D) then
                    F ← F ∪ {f'}
                else
                    T ← T ∪ {f'}
                end
            end
        end
    end
end
```

Algorithm 3: Enumeration of inequivalent sub Delaunay polyhedra

of the facets of CUTP_8. We actually computed the list of orbits of facets of CUTP_8 (and some other graph cut polytopes) in [9]. In dimension 8 the partial enumeration algorithm of [16] found 27 perfect Delaunay polytopes, and it is likely that the list is complete. But to prove its completeness by using the method of this work would require the determination of all facets of CUTP_9, and this is very difficult [6]. In [16] a partial enumeration of perfect Delaunay polytopes was done with only Delaunay polyhedra with $L(D) = 0$ being considered. The two perfect Delaunay polytopes of dimension 7 were determined in this paper, and our enumeration proves that the list is complete.

The implementation is available from [17] and uses the `GAP` computer algebra system [28].

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10 Classification of Delaunay Simplices in Dimension 7

Formula (6.1) gives 187 as an upper bound on the volume of Delaunay simplex sets. With this upper bound we can devise an algorithm for enumeration of Delaunay simplex sets, which will unfortunately prove inefficient.

Remark 10.1 Suppose we have a list of types of Delaunay simplex sets in dimension $n-1$. If $S = \{v_0, \dots, v_{n-1}, v_n\}$ is a Delaunay simplex set of dimension n, then $\{v_0, \dots, v_{n-1}\}$ is a $n-1$ dimensional Delaunay simplex sets of the lattice $v_1 - v_0, \dots, v_{n-1} - v_0$.

Suppose we have an $n-1$ dimensional Delaunay simplex set $v_0 = 0$, $v_i \in \mathbb{Z}^{n-1}$ for $1 \le i \le n-1$ of volume v. We write the n-dimensional simplex as $v'_0 = (v_0, 0)$, $v'_1 = (v_1, 0), \dots, v'_{n-1} = (v_{n-1}, 0)$, and $v'_n = (x_1, \dots, x_{n-1}, x_n)$ with $x_n > 0$. The volume of the simplex defined by (v'_i) is vx_n.

For a fixed x_n the number of possibilities for $(x_1, \dots, x_{n-1})$ is x_n^{n-1}/v.

The Delaunay simplices of dimension 6 were classified in [2], and so we could apply the algorithm of Remark 10.1. Unfortunately the number of possibilities to be applied is very large, of the order of 187^6, on which we have to apply Algorithm 1.

Therefore, we need a different method.

Lemma 10.2 *Let S be a Delaunay simplex set that is not contained in any perfect Delaunay polyhedron different from $\{0,1\} \times \mathbb{Z}^{n-1}$. The possibilities are the following:*

(i) *For $n \le 4$ the Delaunay simplices of volume* 1*.*
(ii) *For $n = 5$ the Delaunay simplex of volume* 1 *or* 2*.*
(iii) *For $n = 6$ or 7 there are no possibilities.*

Proof Let us take a Delaunay simplex set $S = \{v_0, \dots, v_n\}$. We can assume that v_0 is located at the origin by using translation if necessary. For each $1 \le i \le n$ let us define ℓ_i the linear form on $\mathbb{R}^n$ such that $\ell_i(v_i) = 1$ and $\ell_i(v_j) = 0$ for $i \ne i$. Any Delaunay polyhedron D isomorphic to $\{0,1\} \times \mathbb{Z}^{n-1}$ and such that $S \subset D$ corresponds to a linear form ℓ on $\mathbb{R}^n$ such that $\ell(v_i) \in \{0,1\}$ and

$$D = \{x \in \mathbb{Z}^n \mid \ell(x) = 0 \text{ or } 1\}.$$

The linear form ℓ is then called *admissible*, and the corresponding quadratic function is $q_\ell(x) = \ell(x)(\ell(x) - 1)$.

Let us denote by $S \subset \{1, \dots, n\}$ the set of points i such that $\ell(x) = 1$. Clearly, one can write $\ell = \sum_{i \in S} \ell_i$. A function ℓ is admissible if and only if ℓ is integral valued on $\mathbb{Z}^n$. If it is not integral valued, then there exists a $v \in \mathbb{Z}^n$ such that $0 < \ell(v) < 1$ which implies that $q_\ell(v) < 0$ which is not allowed. If it is integral valued, then $D = Z(q_\ell)$ is equivalent to $\{0,1\} \times \mathbb{Z}^{n-1}$. In the sequel, for a set $S \subset \{1, \dots, n\}$, we write

$$\ell_S = \sum_{i \in S} \ell_i \quad \text{and} \quad v_S = \sum_{i \in S} v_i.$$

Let us define

$$\mathcal{S} = \left\{ S \subset \{1, \dots, n\} \mid \ell_S \text{ is integral valued} \right\}.$$

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Let us denote by $\mathbb{Z}\mathcal{S}$ the $\mathbb{Z}$-span of the elements v_S for $S \in \mathcal{S}$. This defines a lattice $\mathcal{L}$ of $\mathbb{Z}^n$. Since S is contained only in Delaunay polyhedra isomorphic to $\{0,1\} \times \mathbb{Z}^{n-1}$ the set of the function q_{ℓ_S} is full-dimensional. This implies that $|\mathcal{S}| \geq n(n+1)/2$ and that the lattice $\mathcal{L}$ is actually full-dimensional. Denote its index by h.

The set of the function ℓ_S is also full-dimensional in $(\mathbb{Z}^n)^*$. Its index is

$$h/\operatorname{vol}(S) \geq 1.$$

We are interested in the point sets of the form $\{0,1\}^n \cap L$ with L an affine subspace of $\mathbb{Z}^n$. By direct enumeration we obtain the full list of 3363 orbits of such points for $n = 7$. By selecting the point sets whose cone of functions q_{ℓ_S} is full-dimensional, we get an upper bound of 3 on the index h and so an upper bound of 3 on the possible volumes of such simplices.

With volume at most 3 we can apply the algorithm implied by Remark 10.1. Each facet of such a Delaunay simplex is also a Delaunay simplex of one dimension lower. Therefore, we can use previous enumeration result to get a list of 796 possible candidates of 7-dimensional Delaunay simplices. We then use Algorithm 1 for checking which of them are indeed Delaunay simplices. This gives 6 cases (the ones of Table 1 of volume at most 3). Each one of them is also contained in a Delaunay polytope ER_7 and so there is no such Delaunay simplices in dimension 7.

Dimension $n \leq 6$ follows from known results. ∎

Proof of Theorem 1.4 If S is a Delaunay simplex set, then $\operatorname{Hyp}(S, \mathbb{Z}^7)$ is a full-dimensional polyhedral cone, *i.e.*, defined by a finite number of inequalities and having a finite number of extreme rays. Any such extreme ray corresponds to a perfect Delaunay polyhedron D. We have $|S| = 8$, and S defines a face of the cone $\operatorname{Erdahl}^*_{\mathrm{supp}}(D)$.

By Lemma 10.2, S has to be contained in a Delaunay polyhedron of type 3_{21}, ER_7, or $2_{21} \times \mathbb{Z}$.

The perfect Delaunay polyhedron ER_7 has 35 vertices. This matches the lower bound given by Proposition 3.6. As a consequence, any 8-element subset of ER_7 defines a face of $\operatorname{Erdahl}^*_{\mathrm{supp}}(ER_7)$. The automorphism group of ER_7 has size 1440, and by using it one can get easily the 9434 orbits of 8-element subsets of ER_7. Actually, all eleven types of simplices occur this way.

The Gosset polytope 3_{21} has 56 vertices, and the automorphism group is equal to the Weyl group of the root lattice E_7. We found 521 orbits of 8-element sets in 3_{21}; 474 of them correspond to faces of $\operatorname{Erdahl}^*_{\mathrm{supp}}(3_{21})$.

For the perfect Delaunay polyhedron $2_{21} \times \mathbb{Z}$ we have to proceed differently, since the number of points to be considered is infinite. We have to enumerate the possible 8-point subsets of $2_{21} \times \mathbb{Z}$ of volume at most 187 (Formula (6.1)) up to the action of $\operatorname{Aut}(2_{21} \times \mathbb{Z})$. The 8 points are expressed in the form $v_i = (w_i, h_i)$ with $w_i \in 2_{21}$ and $h_i \in \mathbb{Z}$. The set of points $(w_i)_{1 \leq i \leq 8}$ must define a 6-dimensional affine space. Thus, 7 of them, say $(w_i)_{1 \leq i \leq 7}$, must be sufficient to define a 6-dimensional Delaunay simplex set S_{Sch}. An exhaustive enumeration on the 27 vertices of 2_{21} gives 31 types up to isomorphism. The volume $\operatorname{vol}(S_{\mathrm{Sch}})$ can be 1, 2, or 3. If the volume is 1, then we can use an element of $\operatorname{Aff}(2_{21} \times \mathbb{Z})$ and obtain $h_i = 0$ for $1 \leq i \leq 7$. For higher volumes, the situation is more complicated, but by using $\operatorname{Aff}(2_{21} \times \mathbb{Z})$ and linear algebra we can reduce to $\operatorname{vol}(S_{\mathrm{Sch}})$ possibilities, *i.e.*, 2 or 3. For the last point (v_8, h_8), we have 27

https://doi.org/10.4153/CJM-2016-013-7 Published online by Cambridge University Press

possibilities for v_8 and a finite number for h_8 due to the upper bound of 187. We then apply Algorithm 1 to test realizability of the finite list of possible cases. This gives us the eleven possible simplices. ∎

Acknowledgments The author thanks Viacheslav Grishukhin and Achill Schürmann for useful discussion on this work.

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https://doi.org/10.4153/CJM-2016-013-7 Published online by Cambridge University Press