STRUCTURE THEOREMS OF A MODULE OVER A RING WITH A BILINEAR MAPPING

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Let R be a ring with an identity 1, and R' a ring anti-isomorphic to R. Let V be an R-module as well as an R'-module. We assume that $1a = a$ for all elements a in V and that V satisfies the minimum condition for R-submodules. Elements of R will be denoted by $\alpha, \beta, \ldots,$ and those of V by $a, b, \ldots$ Elements of R' will be $\alpha', \beta', \ldots,$ where α' corresponds to α by the anti-isomorphism. A mapping f of $V \times V$ to R is called a bilinear mapping of V to R if it satisfies the following.

(1) $\quad f(a+b, c) = f(a, c) + f(b, c),$

(2) $\quad f(a, b+c) = f(a, b) + f(a, c),$

(3) $\quad \beta f(a, b) = f(\beta a, b),$

(4) $\quad f(a, b)\alpha = f(a, \alpha' b).$

We also assume that the images $f(V, V)$ generate additively the whole set R. When all these assumptions are satisfied, we say a system (V, R, R', f) is given. In this note, by a system, we always mean the above system. The purpose of this note is to determine the ring theoretic structure of such a system. We shall define simplicity and semi-simplicity of systems and a radical of a system and shall show that a system is semi-simple if and only if its radical is zero. Then structures of a simple system and of a semi-simple system will be clarified. However, the latter result is a special case of the more general result given in [1] by the present author, and the proof will be omitted.

In the following, we denote $f(a, b)c$ by abc for the sake of simplicity. Then the property (3) implies an associative

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law:

$$(3') \qquad ab(cde) = (abc)de.$$

DEFINITION. A system (V, R, R', f) is called simple if for any non zero elements a and b in V there exists the third element c such that $acb \neq 0$. A system is called semi-simple if for any non zero element a in V there exists an element c such that $aca \neq 0$.

For a subset S of V, we set

$$N(S) = \{a \in V \mid avs = 0 \text{ for all } v \text{ in } V \text{ and all } s \text{ in } S\}.$$

DEFINITION. The radical of a system is defined to be the intersection $\bigcap_T N(T)$ where T ranges over all irreducible R-submodules of V.

In order to prove the first result, some concepts and notations will be required. For two elements a and b in V, we consider an R-homomorphism of V to itself such that $v \to vab$ for v in V. This is in fact an R-homomorphism due to (3'). Denote this R-homomorphism by ab. For an R-submodule T of V, we shall designate by VT the additive group generated by R-homomorphisms vt for v in V and t in T. Then we consider some homomorphisms of the additive group VT to T and of T to VT as follows. Take an element a in V, and it will define a homomorphism δ_a of VT to T such that $\delta_a(vt) = avt$. Also, an element b in V will define a homomorphism δ'_b of T to VT such that $\delta'_b(t) = bt$. It is easy to see that $\delta_c \delta'_b \delta_a = \delta_{cba}$ due to (3'), where δ_a operates first, δ'_b second and δ_c last.

THEOREM 1. *A system (V, R, R', f) is semi-simple if and only if the radical of the system is zero.*

Proof. Assume first that the radical is zero, and take any non zero element a in V. Since a is not in the radical, there exists an irreducible R-submodule T of V such that $a \notin N(T)$ We apply the above discussion for this T. $a \notin N(T)$ implies that there exists an element vt in VT such that $\delta_a(vt) = avt \neq 0$

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On the other hand, let b be a non zero element contained in $\delta_a(VT)$. Since T is irreducible, T is generated by a single element, say, b; T = Rb. Then $t = \alpha b$ with an element α in R. Using (4), we can see that $0 \neq avt = av(\alpha b) = a(\alpha' v)b$. If we put $c = \alpha' v$, then $\delta_a \delta'_c \delta_a \neq 0$, i.e. $\delta_{aca} \neq 0$, which implies $aca \neq 0$. Conversely assume that (V, R, R', f) is semi-simple. Furthermore, assume that a non zero element a is in the radical. Take an irreducible R-submodule T contained in Ra. T = Rb as before, and hence $b = \beta a$ with β in R. Since the system is semi-simple, there exists an element c such that $bcb \neq 0$. Then $(\beta a)cb \neq 0$, which implies $acb \neq 0$, i.e., $a \notin N(T)$. This is naturally absurd. Thus the radical should be zero if the system is semi-simple.

Lastly the structures of simple and of semi-simple systems will be given as follows, specializing the result in [1]. To do so, we need one more concept. Taking V for T in the previous discussion of VT, we can define an additive group VV. Moreover, VV is seen to be a ring since (xab)cd = xa(bcd) implies (ab)(cd) = a(bcd). Denote this ring by Q. Then V is a Q-(right) module. Now let D be a division ring and D_n the ring of all matrices of type n x n with components in D. $D_{n,m}$ (or $D_{m,n}$) denotes the set of all matrices of type n x m (or m x n) with components in D. By usual matrix calculation, we can multiply elements of D_n and of $D_{n,m}$ or elements of $D_{n,m}$ and of $D_{m,n}$ in this order.

THEOREM 2. <u>Suppose that</u> V <u>satisfies the minimum condition for</u> Q-<u>submodules as well as for</u> R-<u>submodules. If a system</u> (V, R, R', f) <u>is simple, then there exist an isomorphism</u> φ <u>of</u> R <u>onto</u> D_n <u>as rings and isomorphisms</u> φ_1 <u>and</u> φ_2 <u>of</u> V <u>onto</u> $D_{n,m}$ <u>and onto</u> $D_{m,n}$ <u>respectively as additive groups such that</u> $f(a, b) = \varphi^{-1}(\varphi_1(a)\varphi_2(b))$ <u>and that</u> $\beta a = \varphi_1^{-1}(\varphi(\beta)\varphi_1(a))$.

When $R = R_1 + R_2 + \ldots + R_m$ is a direct ring sum (namely an orthogonal decomposition), we denote the naturally defined projection of f to R_i by f_i. In this case, we also have a direct ring sum $R' = R'_1 + R'_2 + \ldots + R'_m$.

THEOREM 3. Suppose that V satisfies the minimum condition for Q-submodules as well as for R-submodules. If a system (V, R, R', f) is semi-simple, then we have a direct ring sum $R = R_1 + R_2 + \ldots + R_m$ and a direct sum $V = V_1 + V_2 + \ldots V_m$, where $R_i V_j = 0$ unless $i = j$ and $V_i V_j V_k = 0$ unless $i = j = k$, and (V_i, R_i, R'_i, f_i) are simple for $i = 1, 2, \ldots, m$.

REFERENCE

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