ON DIVISION NEAR-RINGS

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The following results (**9**, Exercise 26, p. 10; **1**, Theorem 9.2; **8**, Theorem III. 1.11) are known.

(A) *Let R be a ring with more than one element. Then R is a division ring if and only if for every $a \neq 0$ in R, there exists a unique b in R such that $aba = a$.*

(B) *Let R be a near-ring which contains a right identity $e \neq 0$. Then R is a division near-ring if and only if it contains no proper R-subgroups.*

(C) *Let R be a finite near-ring with identity. Then R is a division near-ring if and only if the R-module R^+ is simple.*

In this paper we will show that (A) can be generalized to distributively generated near-rings. We also will extend (B) and (C) to a larger class of near-rings. In particular, the works of Heatherly (**5**) and Clay and Malone (**2**) on near-rings definable on finite simple groups are extended by showing that their results are corollaries of our theorems.

1. Definitions. A near-ring R is a system with two binary operations, addition and multiplication such that:

(i) The elements of R form a group R^+ under addition,
(ii) The elements of R form a multiplicative semigroup,
(iii) $x(y + z) = xy + xz$, for all $x, y, z \in R$,
(iv) $0 \cdot x = 0$, where 0 is the additive identity of R^+ and for all $x \in R$.

In particular, if R contains a multiplicative semigroup S whose elements generate R^+ and satisfy

(v) $(x + y)s = xs + ys$ for all $x, y \in R$ and $s \in S$,

we say that R is a distributively generated (d.g.) near-ring.

The most natural example of a near-ring is given by the set R of identity-preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added by adding images and multiplication is iteration, then the system $(R, +, .)$ is a near-ring. If S is a multiplicative semigroup of endomorphisms of G and R' is the sub-near-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in (**4**).

A near-ring module M is a system consisting of an additive group M, a near-ring R, and a mapping f: $(m, r) \to mr$ of $M \times R$ into M such that

(i) $m(r + s) = mr + ms$ for all $m \in M$ and all $r, s \in R$,
(ii) $m(rs) = (mr)s$ for all $m \in M$ and all $r, s \in R$.

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Let R be the near-ring of mappings associated with an additive group G. Then G can be considered as an R-module.

An R-homomorphism is a mapping f of an R-module M into an R-module M' such that $(m + h)f = mf + hf$ and $(mf)r = (mr)f$, where m and h are in M and $r \in R$. The submodules of an R-module M are defined to be kernels of R-homomorphisms of M.

The kernel K of an R-homomorphsim f of an R-module M into an R-module M' is an additive normal subgroup of M. Also for all $m \in M$, $k \in K$, and $r \in R$, we have $((m + k)r - mr)f = (mf + kf)r - (mf)r = 0 \in M'$.

A subgroup H of an R-module M is called an R-subgroup if

$$HR = \{hr: h \in H, r \in R\} \subseteq H.$$

The R-subgroups of the R-module R^+ are called R-subgroups of the near-ring R. A submodule of an R-module M is an R-subgroup. However, the converse is not true. An example is given in (**1**, p. 14). A module is simple if it has no proper submodules.

2. Division near-rings.

(2.1) *Definition.* A near-ring R that contains more than one element is said to be a division near-ring if and only if the set R' of non-zero elements is a multiplicative group.

Division near-rings were first considered by Dickson (**3**). In 1936, Zassenhaus (**11**) showed that the additive group of a finite division near-ring is commutative. Four years later, Neumann (**10**) extended the result to arbitrary division near-rings. It is known that every finite division near-ring is planar. Zemmer (**12**) exhibited an example of an infinite division near-ring which is not planar. Beidleman (**1**) and Maxon (**8**) each presented a characterization of division near-rings. In the following we extend those results. First, we state the following theorem for easy reference.

(2.2) THEOREM. *The additive group of a division near-ring is abelian.*

It was shown (**6**) that in a division near-ring the additive inverse of the multiplicative identity commutes multiplicatively with all elements. However, this is not true in general.

An element a of a near-ring R is right-distributive if $(b + c)a = ba + ca$ for all $b, c \in R$. An element x of R is anti-right-distributive if $(y + z)x = zx + yx$ for all $y, z \in R$. It now follows at once that an element a is right-distributive if and only if $(-a)$ is anti-right-distributive. In particular, any element of a d.g. near-ring is a finite sum of right- and anti-right-distributive elements.

The following theorem is of fundamental importance.

(2.3) THEOREM. *Let R be a near-ring which contains a right-distributive* *element* $r \neq 0$. *Then* R *is a division near-ring if and only if for each* $a \neq 0$ *in* R, $aR = R$.

Proof. Necessity is quite clear. If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$. For if not, there exist a_e and b_e such that $aa_e = a$ and $bb_e = a_e$. Thus $0 = (ab)b_e = a(bb_e) = aa_e = a$. This is a contradiction. Now let r be a non-zero right-distributive element of R. Then there is an element e in R such that $re = r$. However, $r(er - r) = rer - rr = 0$. From above, we have $er = r$. This means that e is a two-sided identity for r. Since we know from the first part of the proof that the set of non-zero elements is closed under multiplication and multiplication is associative, it remains to prove that e is a right identity for the non-zero elements of R and every non-zero element of R has a right inverse. Let $x \neq 0$ be an element of R. Then $(xe - x)r = xer - xr = xr - xr = 0$. Since $r \neq 0$, we have $xe = x$. Also $xR = R$ implies that there is an x' in R such that $xx' = e$. Thus we have shown that the near-ring R is a division near-ring.

(2.4) THEOREM. *Let* R *be a near-ring with a non-zero right-distributive element* w *and for every* $x \neq 0$ *in* R, *there exists a* y *in* R, *possibly depending on* x, *such that* $xy \neq 0$. *Then* R *is a division near-ring if and only if* R *has no proper* R-*subgroups*.

Proof. For each $x \neq 0$ in R, xR is an R-subgroup of R. Since there exists a y in R such that $xy \neq 0$, and R has no proper R-subgroups, we conclude that $xR = R$. Thus by (2.3), R is a division near-ring.

Since any right identity of a near-ring is right-distributive, we have the following result.

(2.5) COROLLARY (Beidleman). *Let* R *be a near-ring that contains a right identity* $e \neq 0$. *Then* R *is a division near-ring if and only if* R *has no proper* R-*subgroups*.

In order to see that (2.4) is indeed a generalization of (2.5), we now exhibit a near-ring which has a non-zero right-distributive element and for each $x \neq 0$, there exists a y such that $xy \neq 0$. Furthermore, this near-ring has no right identities. Let $R = \{0, 1, 2, 3\}$ with addition and multiplication as defined below. Then it can be verified that this near-ring has the properties stated above.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	1	2	3

(2.6) THEOREM. *Let R be a finite near-ring that contains a right-distributive element $w \neq 0$ and for each $x \neq 0$ in R, there is a y in R such that $xy \neq 0$. Then R is a division near-ring if and only if the R-module R^+ is simple.*

Proof. For each $x \neq 0$ in R, define $T(x) = \{r \in R: xr = 0\}$. It is easily checked that $T(x)$ is a submodule of R^+. Since there is a y in R such that $xy \neq 0$, it follows that $T(x) = 0$. This shows that the set of non-zero elements of R is closed under multiplication. Consider the map $f_x: R \rightarrow xR$ defined by $(a)f_x = xa$ for all $a \in R$. From above, f_x is clearly a one-to-one map. Since R is finite, we conclude that $xR = R$. By (2.3), R is a division near-ring.

(2.7) COROLLARY (Maxon). *Let R be a finite near-ring with identity. Then R is a division near-ring if and only if the R-module R^+ is simple.*

Clay and Malone (**2**) have shown that a near-ring with identity on a finite simple group is a field. More recently, Heatherly (**5**) has extended this result and we show now that his theorem is a corollary of (2.6).

(2.8) COROLLARY (Heatherly). *If $(R, +)$ is a finite simple group and if $(R, +, .)$ is a near-ring with a non-zero right-distributive element r, then either $ab = 0$ for each $a, b \in R$ or $(R, +, .)$ is a field.*

Proof. Suppose that $ab \neq 0$ for some a and b. Let $T_a = \{x \in R: ax = 0\}$. This is a normal subgroup of $(R, +)$. Since $b \neq 0$, we have $T_a = 0$. Consider $rT = \{x \in R: xr = 0\}$. Since r is right-distributive, it follows that rT is a normal subgroup of $(R, +)$. Thus $rT = 0$. Now suppose that $c \neq 0$ is any element in R. Again consider $T_c = \{x \in R: cx = 0\}$. Since $cr \neq 0$, we have $T_c = 0$. Thus we have shown that if $x \neq 0$ in R, then $xy \neq 0$ for any $y \neq 0$ in R.

If $(R, +)$ has a proper submodule, then $(R, +)$ has a proper normal subgroup, contrary to assumption. By (2.6), $(R, +, .)$ is a division near-ring. By (2.2), $(R, +)$ is commutative. Let $M = \{x \in R: (a + b)x = ax + bx, \text{ for all } a, b \in R\}$. It is easily shown that M is a normal subgroup of $(R, +)$. Since $r \neq 0$ is in M, we conclude that $M = R$. Thus $(R, +, .)$ is a finite division ring and hence a field.

3. Distributively generated near-rings.

In (**6**) we extended several results in ring theory to d.g. near-rings. In the following we generalize another result. It is not too difficult to show that a ring R with more than one element is a division ring if and only if for every $a \neq 0$ in R, there exists a unique b in R such that $aba = a$.

(3.1) THEOREM. *Let R be a d.g. near-ring with more than one element. Then R is a division ring if and only if for each $a \neq 0$ in R, there exists a unique b in R such that $aba = a$.*

Proof. Suppose that $a \neq 0$ and $c \neq 0$, then $ac \neq 0$. For if not, let $a = a_1 + a_2 + \ldots + a_n$, where each a_i is either right-distributive or anti-right-distributive. Then $a(b + c)a = (ab + ac)(a_1 + a_2 + \ldots + a_n) =$

$(ab + ac)a_1 + (ab + ac)a_2 + \ldots + (ab + ac)a_n = aba_1 + aba_2 + \ldots + aba_n = aba$. This contradicts the fact that b is unique. Thus the set of non-zero elements of R is closed under multiplication. For each $a \in R$, $aba = a$ implies $a(bab - b) = 0$. Thus $bab = b$. Let $r \neq 0$ be a right-distributive element of R. Then there exists a w such that $rwr = r$. Thus $r(wrr - r) = 0$ and this together with $rwr = r$ imply that $wr = e$ is a two-sided identity for r. Let d be any element in R. Then $(de - d)r = der - dr = 0$ and $r(ed - d) = red - rd = 0$. Thus $de = ed = d$. By hypothesis, there exists a d' such that $dd'd = d = de$. Thus $d(d'd - e) = 0$ and this implies that $d'd = e$. Hence $d'(dd' - d'd) = 0$ and $dd' = d'd = e$. Thus every $d \neq 0$ in R has a right inverse and R is therefore a division near-ring. By (2.2), R^+ is abelian. It now follows (**4**, p. 93) that R is a division ring.

Since the additive group of a division near-ring is abelian and a d.g. near-ring R is a ring if R^+ is commutative, the following corollaries of (2.3) are generalizations of some well-known theorems in ring theory.

(3.2) COROLLARY (**6**, Theorem 3.4). *A d.g. near-ring D with more than one element is a division ring if and only if for all non-zero a in D, $aD = D$.*

(3.3) COROLLARY. *Let F be a finite d.g. near-ring with the property that $ab \neq 0$ if $a \neq 0$ and $b \neq 0$. Then F is a field.*

(3.4) COROLLARY (**6**, Corollary 3.5). *A d.g. near-ring D with a right identity $e \neq 0$ is a division ring if and only if it has no proper D-subgroups.*

Remark. If we do not require the near-rings to be distributively generated, then any division near-ring satisfies the hypotheses of (3.1) and (3.4). Since there exist division near-rings which are not division rings (**11**), we conclude that (3.1) and (3.4) cannot be extended to arbitrary near-rings. Let G be an additive group with at least three elements. For each non-zero $g \in G$, define $gx = x$ for all $x \in G$ and $0 \cdot g = 0$ for all $g \in G$. Then $(G, +, .)$ is a near-ring (**7**). Thus neither (3.2) nor (3.3) can be extended to arbitrary near-rings.

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