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BEZOUT DOMAINS AND RINGS WITH A DISTRIBUTIVE LATTICE OF RIGHT IDEALS

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0. It is the purpose of this paper to discuss a construction of right arithmetical (or right D-domains in **[5]**) domains, i.e., integral domains R for which the lattice of right ideals is distributive (see also [3]). Whereas the commutative rings in this class are precisely the Prüfer domains, not even right and left principal ideal domains are necessarily arithmetical. Among other things we show that a Bezout domain is right arithmetical if and only if all maximal right ideals are two-sided.

Any right ideal of a right noetherian, right arithmetical domain is two-sided. This fact makes it possible to describe the semigroup of right ideals in such a ring in a satisfactory way; [3], [5].

However, very little is known about the corresponding question in the non-noetherian case.

We will construct right arithmetical rings in which the maximal right ideals and their intersections, R and (0) are the only two-sided ideals and where it is still possible to describe the lattice of right ideals in various cases. The construction begins with a left Ore right Bezout domain R and a monomorphism σ of R. We show that R can be localized at those maximal right ideals N_i, i in an index set Λ, of R which are two-sided and for which $\sigma(N_i)$ is contained in N_i.

The intersection

$$R_0 = \bigcap_{i \in \Lambda} R_{N_i},$$

the localization of R at N_i, is a right Bezout, left Ore and right arithmetical ring to which σ can be extended. The quotient ring R_1 of the Ore polynomial ring $R_0[x, \sigma]$ with respect to the Ore set S consisting of all polynomials which have content equal to R_0 can be formed. The right ideals of this ring can be studied via the right ideals of

$$\bar{R}_0 = \cup x^{-n} R_0 x^n,$$

the smallest extension of R_0 which has an automorphism as an extension of σ (see **[8]**). The ring R_1 is a right Bezout, left Ore domain and a right arithmetical ring R_2 can be obtained from it in the same fashion as R_0 was constructed from R. The ring R_2 is neither right noetherian nor are all its

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right ideals two-sided as long as Λ is not empty and there exists an element r in R_0 with $\sigma(r)R_0 \subsetneq rR_0$.

1. We consider right Bezout domains. These are integral domains in which all finitely generated right ideals are principal.

LEMMA 1. *Let R be a right Bezout domain, N a maximal right ideal of R. Then N is a two-sided ideal of R if and only if $S = R\backslash N$ is a right Ore-system.*

Proof. If $S = R\backslash N$ is a right Ore-system then

$$R_N = RS^{-1} = \{as^{-1}; a \text{ in } R, s \text{ in } S\}$$

exists and is a local ring with NR_N as the only maximal right ideal in R_N. This right ideal is therefore a two-sided ideal in R_N and $N = R \cap R_N$ is a two-sided ideal in R.

If conversely N is a two-sided ideal in R and s_1, s_2 are in S then

$$s_1x_1 + n_1 = 1, \quad s_2x_2 + n_2 = 1$$

for some x_i in R and n_i in N. We obtain

$$s_1s_2x_2x_1 + s_1n_2x_1 + n_1 = 1$$

which shows that s_1s_2 is again in S. If s is in S and r is in R we have

$$rR + sR = dR$$

for some d in R and $s = ds_1$, $r = dr_1$ with

$$s_1R + d_1R = R \quad \text{and } s_1 \text{ in } S.$$

Therefore

$$s_1x + r_1y = 1 \quad \text{for some } x, y \text{ in } R \quad \text{and}$$

$$s_1(xs_1 - 1) = -r_1ys_1, \quad r_1(yr_1 - 1) = -s_1xr_1$$

are elements in $s_1R \cap r_1R$. Either y is in N and $yr_1 - 1$ in S or y is in S and hence ys_1 is in S. This shows that S is a right Ore system after multiplying the appropriate equation from the left by d.

Remark. The ring R_N is a right chain ring if it exists.

Proof. The ring R_N is a local right Bezout domain. If a, b are elements in R_N, then

$$aR_N + bR_N = dR_N$$

for some d in R_N and

$$a = da_1, \quad b = db_1, \quad a_1x + b_1y = 1$$

for some a_1, b_1, x, y in R. It follows that at least one of a_1 or b_1 is a unit in R_N and either

$$aR_N = dR_N \supseteq bR_N \quad \text{or} \quad bR_N = dR_N \supseteq aR_N.$$

COROLLARY. *A right Bezout domain R is a right arithmetical ring if and only if all maximal right ideals of R are two-sided.*

This follows from Lemma 1 and [**3**].

A right semifir is a ring in which all right ideals are free as right R-modules with unique rank. This notion is left-right symmetric ([**6**], p. 43), which implies that a right Bezout domain which is also left Ore is also a left Bezout domain.

THEOREM 1. *Let R be a right Bezout, left Ore domain. Let $\{N_i\}$, i in Λ, be a set of maximal right ideals of R that are two-sided ideals. Then $D = \cap R_{N_i}$, i in Λ, is a right Bezout, right arithmetical left Ore domain.*

Proof. We know that the rings R_{N_i}, i in Λ, exist and that the ring D is a ring between R and its field of quotients $K = Q(R)$. We just observed that R is a right and left Bezout domain and this implies that the overring D of R is of the form $D = RM^{-1}$ for a right Ore set M of R ([**1**]). We can see this directly in the following way: An element in D has the form ba^{-1} with a, b in R. We can assume that $Ra + Rb = R$, since otherwise $a = a_1d$, $b = b_1d$, $ba^{-1} = b_1a_1^{-1}$ and $Ra_1 + Rb_1 = R$ if $Ra + Rb = Rd$.

Elements x, y exist therefore in R with

$$xa + yb = 1$$

and $x + yba^{-1} = a^{-1}$ in D follows.

Let M be the set of units in D that are elements in R, i.e., $M = U(D) \cap R$ with $U(D)$ the group of units of D. Let a be an element in M. Then

$$aD = D \quad \text{and} \quad aR_{N_i} = R_{N_i}$$

and a is in $R \backslash (\cup N_i)$, i in Λ. Conversely, if a is in $R \backslash (\cup N_i)$, i in Λ, then

$$aR_{N_i} = R_{N_i} \quad \text{and} \quad aD = D,$$

a is in M.

We conclude that

$$D = RM^{-1} = \{bm^{-1};\ b \text{ in } R,\ m \text{ in } M\}$$

and M is a right Ore system using the familiar argument: If m is in M, r in R, then $m^{-1}r = r_1m_1^{-1}$ for r_1 in R, m_2 in M and $rm_1 = mr_1$. The ring $D = RM^{-1}$ is a right Bezout domain that is left Ore with N_iD, i in Λ, as its maximal right ideals. These right ideals are two-sided, since

$$N_iD = D \cap N_iR_{N_i}.$$

We need one more result concerning the symmetry of our basic conditions.

LEMMA 2. *If R is a right Bezout, left Ore domain whose maximal right ideals are two-sided then the following hold*:

i) $\sum Ra_i = R$ *if and only if* $\sum a_iR = R$ *for elements* $a_1, \dots, a_n$ *in* R.

ii) *All maximal left ideals of R are two-sided and equal maximal right ideals.*

iii) *R is left arithmetical.*

Proof. If

$$L = \sum Ra_i = R$$

it is impossible that

$$\sum a_iR = K \neq R,$$

since K is contained in a maximal right ideal N in this case and $L = R \subseteq N$ would follow.

Conversely, if

$$\sum a_iR = R \quad \text{and} \quad \sum Ra_i = L \neq R$$

we obtain $L = Rd$ with d not a unit in R and therefore $0 \neq d$ is contained in a maximal right ideal N of R. This implies $a_i = a_i'd$ for elements a_i' in R, $i = 1, \dots, n$, and

$$\sum a_iR = \sum a_i'dR \subseteq N,$$

a contradiction. To prove ii) let L be a maximal left ideal of R.

Then either $LR \neq R$ and LR is contained in some maximal right ideal N or $LR = R$. We have $L \subseteq LR \subseteq N$ for the left ideals L and N in the first case and $L = N$ by the maximality of L.

If $LR = R$ then $\sum a_ir_i = 1$ for a_i in L, r_i in R, $i = 1, \dots, n$. But then

$$R = \sum Ra_i \subseteq L,$$

using i), a contradiction.

iii) follows from ii) and the corollary to Lemma 1.

2. The construction which we will now consider in detail corresponds to the construction of the Kronecker function ring, [7] Section 32 in the commutative case, see also [**4**], [**9**] and [**10**].

Let R_0 be a right Bezout, left Ore domain with a monomorphism σ from R_0 to R_0.

Let $\{N_i\}$, i in Λ, be the set of those maximal right ideals of R_0 which are two-sided and satisfy $\sigma(N_i) \subseteq N_i$. Let

$$S = R_0 \backslash (\cup N_i) \quad i \text{ in } \Lambda.$$

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Using Theorem 1 we can form the ring R_0S^{-1} which is a right Bezout, right arithmetical left Ore domain. The monomorphism σ can be extended from R_0 to R_0S^{-1}, since s in $S = R_0\backslash(\cup N_i)$, i in Λ, implies $\sigma(s)$ in S. To see this, we observe that $sR_0 + N_i = R$ for every i in Λ, and hence

$$sr_i + n_i = 1 \quad \text{for some } r_i \text{ in } R,\ n_i \text{ in } N_i.$$

Applying σ to this equation shows that $\sigma(s)$ is not contained in N_i and therefore $\sigma(s)$ is an element of S.

Replacing R_0 by R_0S^{-1} we can therefore assume that R_0 is a right Bezout, right arithmetical left Ore domain with a monomorphism σ such that $\sigma(N) \subseteq N$ for all maximal right ideals N of R_0.

Next, consider the Ore polynomial ring

$$R = R_0[x, \sigma] = \{\sum a_ix^i;\ a_i \text{ in } R_0\}$$

with $xa = \sigma(a)x$ defining the multiplication.

Since R_0 is left Ore, it follows from Proposition 8.4 in **[6]** that R is left Ore; i.e., for elements $0 \neq f$, g in R there exist elements $0 \neq f_1$, g_1 in R with $f_1g = g_1f$.

We denote with S the set

$$\{\sum a_ix^i \text{ in } R;\ \sum a_iR_0 = R_0\}$$

of all those polynomials f in R which have right content R_0. Here, the right content of an element f in R is the right ideal $c(f)$ of R_0 generated by the coefficients of f.

We want to show that S is a left Ore system of R. To show that S is multiplicatively closed let

$$s_1 = \sum a_ix^i \quad \text{and} \quad s_2 = \sum b_jx^j$$

be elements in S with $s_1s_2 = p(x)$ as their product.

For any maximal right ideal N of R_0 there exists an index i_0 minimal with the property that a_{i_0} is not in N. Similarly, a lowest coefficient b_{j_0} exists with b_{j_0} not in N. Using the fact that $\sigma(r)$ is in N if and only if r is in N for r in R_0 and that N is two-sided it follows that the coefficient of $x^{i_0+j_0}$ in $p(x)$ is not contained in N. Hence, $p(x)$ is in S.

Let $s(x)$ be in S and $f(x)$ in R. It was observed earlier that elements $h(x)$ and $0 \neq g(x)$ exist in R with

$$h(x)s(x) = g(x)f(x).$$

If $h(x)$ is factored as $c \cdot h_1(x)$ with c in R_0, $h_1(x)$ in S and similarly $g(x) = d \cdot g_1(x)$ with d in R_0, $g_1(x)$ in S we conclude that $h_1(x)s(x)$ is again in S and $c = dr$ for some r in R_0. Therefore,

$$rh_1(x)s(x) = g_1(x)f(x) \quad \text{with } g_1(x) \text{ in } S.$$

This shows that S is a left Ore system in R and the ring of quotients

$$R_1 = S^{-1}R = \{s(x)^{-1}f(x); f(x) \text{ in } R, s(x) \text{ in } S\}$$

exists.

We have proved the first part of the following theorem.

THEOREM 2. *Let R_0 be a right Bezout, left Ore domain such that all maximal right ideals N_i, i in Λ, are two-sided. Let σ be a monomorphism of R_0* such that *$\sigma(N_i)$ is contained in N_i for every maximal right ideal. Then $R_1 = S^{-1}R$ exists and is a right Bezout left Ore domain, where $R = R_0[x, \sigma]$ is the Ore polynomial ring and S is the set of polynomials with R_0 as their right content.*

Proof. It remains to prove that R_1 is a right Bezout left Ore domain. One can write two arbitrary elements in R_1 with a common denominator in S and it is sufficient to show that

$$f^{-1}(x)g(x)R_1 + f^{-1}(x)h(x)R_1 = I$$

is a principal right ideal.

However, $g(x)R_1 = aR_1$ and $h(x)R_1 = bR_1$ for certain elments a, b in R_0 and $I = f^{-1}(x)dR_1$ if $aR_0 + bR_0 = dR_0$ for some d in R_0.

The fact that R_1 is left Ore follows from the earlier observation that R is left Ore.

We would like to obtain more information about one-sided and two-sided ideals in R_1. It is useful to introduce the subring

$$\bar{R}_0 = \bigcap_{n \geqq 0} R_0^{(n)}$$

of R_1 where

$$R_0^{(n)} = x^{-n}R_0x^n.$$

Since

$$R_0^{(n+1)} = x^{-(n+1)}R_0x^{n+1} \supseteq x^{-(n+1)}\sigma(R_0)x^{n+1} = x^{-n}R_0x^n = R_0^{(n)},$$

it follows that $\bar{R}_0$ is indeed a subring of R_1 and again a right Bezout left Ore domain containing R_0.

By defining

$$\bar{\sigma}(x^{-n}ax^n) = x^{-n}\sigma(a)x^n$$

we see that $\bar{\sigma}$ is an extension of σ and is the restriction to $\bar{R}_0$ of the inner automorphism of R_1 that sends f in R_1 to xfx^{-1}. Both this inner automorphism and its inverse map $\bar{R}_0$ to $\bar{R}_0$ and $\bar{\sigma}$ is therefore an automorphism of $\bar{R}_0$. The ring $\bar{R}_0$ and the element x, which remains algebraically independent over $\bar{R}_0$, are both contained in R_1. Therefore,

$$\bar{R} = \bar{R}_0[x, \bar{\sigma}] = \{\sum \bar{a}_ix^i; \bar{a}_i \text{ in } \bar{R}_0\},$$

the Ore polynomial ring in x over $\bar{R}_0$ with the automorphism $\bar{\sigma}$, is contained in R_1.

We consider the set

$$\bar{S} = \{\sum \bar{a}_i x^i \text{ in } \bar{R};\ \sum \bar{a}_i \bar{R}_0 = \bar{R}_0\}$$

and want to prove that $\bar{S}$ is a right and left Ore system in $\bar{R}$ with

$$\bar{S}^{-1}\bar{R} = \bar{R}\bar{S}^{-1} = R_1.$$

We begin with showing that $\bar{S}$ is multiplicatively closed. If

$$f(x) = \sum \bar{a}_i x^i \quad \text{and} \quad g(x) = \sum \bar{b}_i x^i$$

are elements in $\bar{S}$ then their coefficients are also contained in $R_0^{(n)}$ for a sufficiently large n which can be chosen such that

$$\sum \bar{a}_i R_0^{(n)} = R_0^{(n)} \quad \text{and} \quad \sum \bar{b}_i R_0^{(n)} = R_0^{(n)}.$$

This implies (observe that $\bar{\sigma}(R_0^{(n)})$ is contained in $R_0^{(n)}$) that $f(x)g(x)$ has coefficients in $R_0^{(n)}$ that generate $R_0^{(n)}$ as a right ideal; using the fact that $R_0^{(n)}$ is a right Bezout, left Ore domain whose maximal right ideals are two-sided. We know that $\bar{R}_0$ is a right and left Ore domain and that $\bar{\sigma}$ is an automorphism of $\bar{R}_0$. It follows as in Section 2 that $\bar{R}_0[x, \bar{\sigma}] = \bar{R}$ is a right and left Ore domain. We use this fact to show that $\bar{S}$ is a left Ore system.

Given $f(x)$ in $\bar{S}$, $g(x)$ in $\bar{R}$ then there exist $f_1(x)$, $g_1(x)$ in $\bar{R}$ with

$$f_1(x)g(x) = g_1(x)f(x).$$

We can write

$$f_1(x) = c_1 f_2(x), \quad g_1(x) = d_1 g_2(x)$$

with c_1, d_1 in $\bar{R}_0$ and $f_2(x)$, $g_2(x)$ in $\bar{S}$.

As in the above argument, there exists an n such that the coefficients of $f(x)$, $g(x)$, $f_2(x)$, $g_2(x)$ and c_1, d_1 are elements of $R_0^{(n)}$.

The product $g_2(x)f(x)$ is in $\bar{S}$ and it follows that $d_1 = c_1 d_2$ for an element d_2 in $R_0^{(n)}$. Therefore,

$$f_2(x)g(x) = d_2 g_2(x)f(x)$$

with $f_2(x)$ in $\bar{S}$ shows that $\bar{S}$ is a left Ore system.

We need Lemma 2 to prove that $\bar{S}$ is also a right Ore system. From the fact that $\bar{\sigma}$ is an automorphism we conclude that

$$f(x) = \sum_{i=0}^{k} \bar{a}_i x^i$$

in $\bar{S}$ can also be written as

$$f(x) = \sum x^i \bar{\sigma}^{-i}(\bar{a}_i)$$

and the elements $\{\sigma^{-i}(\bar{a}_i), i = 0, \ldots, k\}$ will still generate $\bar{R}_0$ as a right ideal. This follows by working again in a ring $R_0^{(n)}$ that contains all the $\bar{a}_i$ and $\bar{\sigma}^{-i}(\bar{a}_i)$ and observing the fact that the maximal right ideals of $R_0^{(n)}$ are exactly the right ideals $x^{-n}Nx^n$ where N is a maximal right ideal of R_0. For every such right ideal there exists an i with $\bar{a}_i$ not in $x^{-n}Nx^n$. This implies that $\bar{\sigma}^{-i}(\bar{a}_i)$ is not in $x^{-n}Nx^n$, since $\sigma(r)$ is in N if and only if r is in N for r in R_0. We get

$$\begin{aligned}\bar{S} &= \{\textstyle\sum \bar{a}_i x^i; \sum \bar{a}_i R_0 = \bar{R}_0\}\\ &= \{\textstyle\sum x^i \bar{a}_i; \sum \bar{a}_i \bar{R}_0 = \bar{R}_0\}\\ &= \{\textstyle\sum x^i \bar{a}_i; \sum \bar{R}_0 \bar{a}_i = \bar{R}_0\}\end{aligned}$$

by Lemma 2 and it follows that $\bar{S}$ is a right Ore system because of the symmetry of our assumption and the fact that $\bar{\sigma}$ is an automorphism.

We saw above that $\bar{R}$ is contained in R_1 and for every element $f(x)$ in $\bar{S}$ there exist an n such that $x^n f(x) x^{-n}$ is in R and hence in S. This implies that the inverses of the elements in $\bar{S}$ are in R_1 and

$$R_1 \subseteq \bar{S}^{-1}\bar{R} = \bar{R}\bar{S}^{-1} \subseteq R_1$$

shows the equality of these rings.

We use this to describe the right ideals in R_1.

LEMMA 3. *The right ideals of R_1 are in one-to-one correspondence with the right ideals of $\bar{R}_0$. If I is a right ideal of R_1 then $(I \cap \bar{R}_0)R_1 = I$ and if $\bar{I}_0$ is a right ideal of $\bar{R}_0$ then $\bar{I}_0 R_1 \cap \bar{R}_0 = \bar{I}_0$.*

Proof. A principal right ideal in R_1 has the form

$$f^{-1}(x)g(x)R_1 = T$$

with $f(x)$ in S and $g(x)$ in R. However,

$$f^{-1}(x)g(x) = \bar{h}(x)\bar{s}^{-1}(x)$$

with $\bar{h}(x)$ in $\bar{R}$ and $\bar{s}(x)$ in $\bar{S}$. Further,

$$\bar{h}(x) = \bar{a}\bar{t}(x) \quad \text{for } \bar{a} \text{ in } \bar{R}_0 \text{ and } \bar{t}(x) \text{ in } \bar{S}.$$

This implies $T = \bar{a}R_1$. If $\bar{a}R_1 = \bar{b}R_1$ for elements $\bar{a}$, $\bar{b}$ in $\bar{R}_0$ then

$$\bar{a}\bar{s}_1(x) = \bar{b}\bar{s}_2(x) \quad \text{for elements } \bar{s}_1(x), \bar{s}_2(x) \text{ in } \bar{S}.$$

Since both the principal right ideals $\bar{a}\bar{R}_0$ and $\bar{b}\bar{R}_0$ are the content of the element $\overline{as}_1(x)$ in $\bar{R}$, they are equal and it follows that $\bar{a}R_1 = \bar{b}R_1$ if and only if $\bar{a}\bar{R}_0 = \bar{b}\bar{R}_0$.

With a similar argument, using the content again, one shows that $\bar{a}R_1 \subseteq \bar{b}R_1$ if and only if $\bar{a}\bar{R}_0 \subseteq \bar{b}\bar{R}_0$.

The proof of the lemma follows easily from what has been said, but we consider the proof of the containment

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$$\bar{I}_0R_1 \cap \bar{R}_0 \subseteq \bar{I}_0.$$

Let $\bar{a}$ be in $\bar{I}_0$ and $\bar{c} = \bar{a}\bar{f}(x)\,\bar{s}(x)^{-1}$ be in $\bar{R}_0$ with $\bar{f}(x)\bar{s}(x)^{-1}$ in R_1. Then $\bar{c}\bar{s}(x) = \bar{a}\bar{f}(x)$ and a content argument shows that $\bar{c} = \bar{a}\bar{b}$ for some $\bar{b}$ in $\bar{R}_0$, since $\bar{s}(x)$ is in S.

We can describe the principal right ideals in R_1 even further, see also Theorem 2 in [**4**].

LEMMA 4. *A principal right ideal in R_1 has the form $x^{-n}aR_1$ for some non-negative integer n and a in R_0. Two such right ideals $x^{-n}aR_1$ and $x^{-m}bR_1$ are equal if and only if*

$$\sigma^m(a)R_0 = \sigma^n(b)R_0.$$

The first part of the lemma follows from Lemma 3 and the fact that every element $\bar{a}$ in $\bar{R}_0$ has the form $x^{-n}ax^n$ for a suitable n and an a in R_0. The second part follows if we prove that

$$U(\bar{R}_0) \cap Q(R_0) = U(R_0)$$

where $Q(R_0)$ is the field of quotients of R_0 and $U(R_0)$, $U(\bar{R}_0)$ are the groups of units of R_0 and $\bar{R}_0$ respectively. To see this we use an argument similar to one used in the proof of Theorem 1. The overring $\mathcal{O} = \bar{R}_0 \cap Q(R_0)$ of R_0 is of the form $\mathcal{O} = R_0T^{-1}$ for some Ore system T of R_0.

Let a, b be elements in R_0 with $aR_0 + bR_0 = R_0$ and $a^{-1}b$ in $U(\bar{R}_0) \cap Q(R_0)$. It follows that a^{-1} is in $\mathcal{O}$ and hence in $U(\bar{R}_0) \cap Q(R_0)$, and a^{-1} is therefore a unit in $R_0^{(n)}$ for a suitable n.

However, the maximal right ideals of $R_0^{(n)}$ are of the form $x^{-n}Nx^n$, N a maximal right ideal of R_0 and unless a is a unit in R_0 and not contained in any maximal right ideal N it is not possible that a is not contained in any $x^{-n}Nx^n$. This implies

$$U(\bar{R}_0) \cap Q(R_0) = U(R_0).$$

If we now assume that $a\bar{R}_0 = b\bar{R}_0$ for a, b in R_0 then $a\bar{u} = b$ for a unit $\bar{u}$ in $\bar{R}_0$. But $\bar{u} = a^{-1}b$ is an element in

$$U(\bar{R}_0) \cap Q(R_0) = U(R_0)$$

and $aR_0 = bR_0$.

The lemma follows if we observe that

$$x^{-n}aR_1 = x^{-m}bR_1$$

if and only if

$$x^maR_1 = \sigma^m(a)R_1 = x^nbR_1 = \sigma^n(b)R_1.$$

We saw in Lemma 3 that a right ideal I of R_1 is determined by the right ideal $\bar{I} = I \cap \bar{R}_0$ of $\bar{R}_0$. Such a right ideal is uniquely determined by the sequence $\{I_{(n)}\}$ of right ideals

$$I \cap R_0^{(n)} = \bar{I} \cap R_0^{(n)} = I_{(n)}$$

of $R_0^{(n)}$. This is obvious since $\bar{I} = \bar{J}$ implies $I_{(n)} = J_{(n)}$ and $I_{(n)} = J_{(n)}$ implies

$$\bar{I} = \cup I_{(n)} = \cup J_{(n)} = \bar{J}.$$

4. We discuss maximal right ideals and two-sided ideals of R_1 in this section.

Let I be a two-sided ideal in R_1. We begin with the observation that the two-sided ideal $\bar{I} = I \cap \bar{R}_0$ of $\bar{R}_0$, satisfies $\bar{\sigma}(\bar{I}) = \bar{I}$, since

$$x\bar{I}x^{-1} \subseteq \bar{I} \quad \text{and} \quad x^{-1}\bar{I}x \subseteq \bar{I}.$$

It follows that

$$I_0 = I_{(0)} = I \cap R_0$$

is a two-sided ideal of R_0 with the property that $\sigma(r)$ is in $I_{(0)}$ if and only if r is in I_0 for r in R_0.

LEMMA 5. *The two-sided ideals I in R_1 are of the form $I = \bar{I}R_1$ with*

$$\bar{I} = \cup x^{-n}I_0x^n,$$

n a non-negative integer, where I_0 is a two-sided ideal in R_0 such that $\sigma(r)$ is in I_0 if and only if r is in I_0. Two such ideals I and J are equal in R_1 if and only if $I_0 = I \cap R_0$ is equal to $J_0 = J \cap R_0$ in R_0.

Proof. Let I be a two-sided ideal in R_1. We know that I is uniquely determined by $\bar{I} = I \cap \bar{R}_0$ and that $I_0 = I \cap R_0$ is a two-sided ideal of R_0 such that $\sigma(r)$ is in I_0 if and only if r is in I_0 for r in R_0.

Consider $I_{(n)} = \bar{I} \cap R^{(n)}$ and we want to show that

$$I_{(n)} = x^{-n}I_0x^n.$$

Since I_0 is in $\bar{I}$, we see that $x^{-n}I_0x^n$ is contained in $I_{(n)}$. Conversely, since $I_{(n)}$ is contained in $R^{(n)}$ it follows that $x^nI_{(n)}x^{-n}$ is contained in $R_0 \cap \bar{I} = I_0$. This shows that

$$I = (\cup x^{-n}I_0x^n)R_1 \quad \text{for } I_0 = I \cap R_0.$$

To finish the proof we now consider any two-sided ideal I_0 in R_0 with the property that $\sigma(r)$ is contained in I_0 if and only if r is in I_0 for r in R_0.

We form $I = \bar{I}R_1$ with

$$\bar{I} = \cup x^{-n}I_0x^n,$$

n runs through the negative integers, and must show that I is a two-sided ideal in R_1 with $I \cap R_0 = I_0$.

We observe that $\bar{I}$ is a two-sided ideal of $\bar{R}_0$ with the property $\bar{\sigma}(\bar{I}) = \bar{I}$. To see this, let $x^{-n}ax^n$, a in I_0, be an element in $\bar{I}$. Then

$$\bar{\sigma}(x^{-n}ax^n) = x^{-n}\sigma(a)x^n$$

is in $\bar{I}$ and

$$x^{-n}ax^n = x^{-(n+1)}\sigma(a)x^{n+1}$$

is in $\bar{\sigma}(\bar{I})$.

We want to show now that I is a two-sided ideal. Let α be an element in $\bar{I}$ and

$$f(x) = \sum \alpha_i x^i$$

be in R. Then

$$f(x)\alpha = \sum \alpha_i\sigma^i(\alpha)x^i = \sum \alpha_i' x^i$$

with α_i' in $\bar{I}$, and $f(x)\alpha$ in I follows.

If $s(x)$ is an element in S, α in $\bar{I}$, then

$$s(x)^{-1}\alpha = \beta s_2(x)s_3^{-1}(x)$$

for β in $\bar{R}_0$, $s_2(x)$, $s_3(x)$ in $\bar{S}$.

We obtain

$$\alpha s_3(x) = s(x)\beta s_2(x).$$

It follows from this equation that

$$\gamma_i\sigma^i(\beta) = \alpha\omega_i \quad \text{for } i = 0, \dots, n$$

if

$$s(x) = \sum_{i=0}^{n} \gamma_i x^i$$

for certain ω_i in $\bar{R}_0$, since $\alpha\bar{R}_0$ is the content of $\alpha s_3(x)$ as well as of $s(x)\beta$.

From this we conclude that $\bar{\sigma}^{-i}(\gamma_i)\beta$ is in $\bar{I}$ for all i and the elements

$$\{\sigma^{-1}(\gamma_i);\ i = 0, \dots, n\}$$

generate $\bar{R}_0$ as a right as well as a left ideal. We use $R_0^{(n)}$ for a suitable n, Lemma 2 and the fact that $s(x)$ is in S for these arguments. Hence, there exist elements u_i in $\bar{R}_0$ with

$$\sum u_i\bar{\sigma}^{-i}(\gamma_i) = 1 \quad \text{and}$$

$$\beta = \sum u_i\bar{\sigma}^{-i}(\gamma_i)\beta \quad \text{is in } \bar{I}.$$

This proves that I as defined above is a two-sided ideal in R_1.

It remains to prove that $I \cap R_0 = I_0$. We pick an element r in $I \cap R_0$ and write

$$r = \sum \alpha_i f_i(x)s^{-1}(x)$$

where the α_i's are elements in $\bar{I}$, the $f_i(x)$ are in $\bar{R}$ and $s(x)$ is in $\bar{S}$.
Hence,

$$rs(x) = \sum \alpha_i' x^i$$

for certain elements α_i' in $\bar{I}$. For

$$s(x) = \sum \gamma_i x^i$$

we obtain $r\gamma_i = \alpha_i'$ in $\bar{I}$ and elements u_i exist in $\bar{R}_0$ with

$$\sum \gamma_i u_i = 1$$

and $r = \sum \alpha_i' u_i$ in $\bar{I}$ follows.

THEOREM 3. *Let R_0, σ and R_1 be as in Theorem 2 and write $M_i = \bar{N}_i R_1$, i in Λ, with*

$$\bar{N}_i = \cup x^{-n} N_i x^n$$

where the N_i are the maximal right ideals in R_0. The M_i, i in Λ, are maximal right ideals in R_1, they are two-sided ideals in R_1 and every maximal right ideal M of R_1, with $\bar{\sigma}(M \cap \bar{R}_0) \subseteq M$ is a member of the set $\{M_i;\ i \text{ in } \Lambda\}$.

Proof. The first two statements follow from Lemma 3 and Lemma 5 and the comment made at the end of Section 3.

To prove the last statement we write

$$\bar{M} = M \cap \bar{R}_0 \quad \text{and} \quad M_{(n)} = M \cap R_0^{(n)}.$$

Claim:

$$M_{(n)} = x^{-n} N_{i_n} x^n = N_{i_n}^{(n)}$$

for a suitable N_{i_n}.

If this is not true we have

$$M_{(n)} \subsetneq x^{-n} N x^n = N^{(n)}$$

for a suitable maximal right ideal N of R_0.

An element γ exists in $N^{(n)} \backslash M_{(n)}$ with

$$\gamma R_0^{(n)} + M_{(n)} \subseteq N^{(n)},$$

but

$$\gamma \bar{R}_0 + \bar{M} = \bar{R}_0.$$

It follows from the last equation that there exists an index t and elements α in $R_0^{(t)}$, μ in $M_{(t)}$ with $\gamma\alpha + \mu = 1$.

We can assume that $t = n + 1$ and obtain

$$1 = \bar{\sigma}(1) = \bar{\sigma}(\gamma)\bar{\sigma}(\alpha) + \bar{\sigma}(\mu) \subseteq N^{(n)} + M_{(n)} \subseteq N^{(n)},$$

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a contradiction. This proves our first claim:

$$M_{(n)} = N_{i_n}^{(n)}.$$

It remains to show that $i_n = i_m$ for all n, m. We assume

$$M_{(n)} = N_1^{(n)} \quad \text{and} \quad M_{(n+1)} = N_2^{(n+1)}.$$

However,

$$\bar{\sigma}(M_{(n+1)}) = \bar{\sigma}N_2^{(n+1)} = N_2^{(n)} \subseteq N_1^{(n)}$$

which is impossible because $N_1 \neq N_2$ in R_0. Hence,

$$\bar{M} = \cup N^{(n)} = \cup x^{-n}Nx^n$$

for a certain maximal right ideal N of R_0.

COROLLARY 1. *All maximal right ideals M of R_1 that are two-sided are of the form $M = (\cup N^{(n)})R_1$, N a maximal right ideal in R_0.*

We only need to observe that xMx^{-1} is contained in M if M is two-sided. This implies $\bar{\sigma}(\bar{M}) \subseteq M$.

COROLLARY 2. *Let R_0 have the additional property that $\sigma(aR_0) \subseteq aR_0$ for every a in R_0. Then every maximal right ideal in R_1 is two-sided.*

Proof. We must show that $\bar{\sigma}(M \cap \bar{R}_0) \subseteq M$ for every maximal right ideal M of R_1. Let $x^{-n}mx^n$ be an element in $M \cap \bar{R}_0$ with m in R_0. We have

$$\bar{\sigma}(x^{-n}mx^n) = x^{-n}\sigma(m)x^n = x^{-n}mx^nx^{-n}rx^n$$

if $\sigma(m) = mr$ for r in R_0.

COROLLARY 3. *Let R_0 be a principal right and left ideal domain whose maximal right ideals are two-sided. Then every maximal right ideal in R_1 is two-sided.*

This follows immediately from Corollary 2 if we observe that every element $a \neq 0$, not a unit, can be written as $a = p_1 \ldots p_n$ with p_iR_0 maximal right ideals with $R_0p_i \subseteq p_iR_0$.

LEMMA 6. *Let the notation and assumptions be as in Theorem 3. We assume further that the index set Λ is finite, i.e., R_0 has only finitely many maximal right ideals. Then the M_i, i in Λ, are exactly the maximal right ideals of R_1.*

Proof. Let M be any maximal right ideal in R_1 and $\bar{M} = \bar{R} \cap M$. We must show that $\bar{M} = \bar{N}_i$ for a suitable i in Λ. If

$$M_{(n)} = M \cap R_0^{(n)} = N_i^{(n)} = x^{-n}N_ix^n$$

for a certain i and n, then

$$M_{(m)} = N_i^{(m)} \quad \text{for } m \geqq n,$$

otherwise

$$N_i^{(n)} \subseteq M_{(m)} \subseteq N_j^{(m)},$$

a contradiction for $i \neq j$. Hence, we must show that $M_{(n)} = N_i^{(n)}$ for a certain i and n.

Let $M_{(n)} \subsetneqq N_i^{(n)}$ for a certain n. Then there exists c in $N_i^{(n)}$, not in $M_{(n)}$ and c is not in $\bar{M}$. Hence, there exists an index $m > n$ with

$$c\alpha_m + \mu_m = 1$$

with α_m in $R_0^{(m)}$ and μ_m in $M_{(m)}$. It follows that for all $s \geqq m$ the inequality

$$M_{(s)} \subseteq N_i^{(s)}$$

is impossible since otherwise

$$1 = c\alpha_m + \mu_m \in N_i^{(n)} + M_{(s)} \subseteq N_i^{(s)}.$$

We repeat the above argument for indices $s > m$ and obtain after a finite number of steps the equality $M_{(t)} = N_j^{(t)}$ for a certain j in Λ and a certain positive integer t.

5. We illustrate the results of the earlier sections with some examples. Consider the field $F_p = \mathbf{Z}/p\mathbf{Z}$ with p elements, p a prime, and the polynomial ring $R_0 = F_p[t]$ in one indeterminate t over F_p. This ring has a monomorphism σ defined by

$$\sigma(f(t)) = f(t)^p.$$

The ring R_0 together with σ satisfies the condition of Theorem 2. The ring R_1 exists and its only two-sided ideals $\neq R_1$, (0) are the ideals

$$(\cup x^{-n} p_1(t) \dots p_s(t) R_0 x^n) R_1$$

where $\{p_1(t), \dots, p_s(t)\}$ is any finite set of distinct irreducible elements in R_0, using Lemma 5. The maximal right ideals of R_1 are exactly the ideals

$$(\cup x^{-n} p(t) R_0 x^n) R_1$$

where $p(t)$ is irreducible in R_0, illustrating Theorem 3 and corollary. These maximal right ideals are two-sided ideals and they are not finitely generated as right ideals of R_1.

We will now consider the case $p = 2$ in order to compute the set W of all principal right ideals of R_1, and

$$\widetilde{H}(R_1) = \{\widetilde{r};\ 0 \neq r \text{ in } R_1, \widetilde{r}\,(aR_1) = raR_1 \text{ for } aR \text{ in } W\},$$

the generalized semigroup of divisibility of R_1. The elements of $\widetilde{H}(R_1)$

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are the mappings $\tilde{r}$ from W to W with $\tilde{r}(aR_1) = raR_1$ for $r \neq 0 \neq a$ in R_1. The operation in $\tilde{H}(R_1)$ is defined by $\tilde{r}\tilde{r}' = \widetilde{rr'}$.

It follows immediately from Lemma 4 that

$$W = \{x^{-n}ax^nR_1 \text{ for } 0 \neq a \text{ in } R_0 = F_2[t]\}$$

with the equality

$$x^{-n}ax^nR_1 = x^{-m}bx^mR_1$$

holding if and only if

$$\sigma^m(a)R_0 = \sigma^n(b)R_0.$$

We order the set $\{p_i(t);\ i \text{ in } \Lambda\}$, of irreducible polynomials of R_0 and write $p_1(t) = t$, $p_2(t) = t^2 + t + 1, p_3(t), \dots, p_i(t), \dots$, etc. With each principal right ideal

$$x^{-n}p_1(t)^{m_1} \dots p_s(t)^{m_s}x^nR_1$$

we associate the element

$$\left(\frac{m_1}{2^n}, \frac{m_2}{2^n}, \dots \frac{m_s}{2^n}, 0, 0, \dots\right)$$

in the direct sum $W' = \sum L_i$, $i = 1, 2, 3, \dots$ where

$$L = L_i = \left\{\frac{m}{2^n};\ m, n \text{ non-negative integers}\right\} \text{ for all } i.$$

It follows from the condition for equality that every principal right ideal of R_1 is uniquely determined by its associated element in W'. We point out that the set W of principal right ideals of R_1 can not be made into a semigroup by using multiplication of right ideals as operation; as it is possible in the commutative and right invariant case.

We must now study the mapping $\tilde{r}$ for an element r on W. We will interpret such a mapping as a mapping $\hat{r}$ from W' to W'. The elements r in R_1 have the form

$$r = (\sum a_i(t)x^i)^{-1}(\sum b_j(t)x^i)$$

with $a_i(t)$, $b_j(t)$ in R_0. It appears to be the easiest to explain this by an example. Let

$$r = t(t^2 + t + 1)x^2 + t^4(t^2 + t + 1)^2x + t^6(t^2 + t + 1)^5.$$

If we compute

$$rp_1^{z_1} \dots p_s^{z_s}$$

with

$$z_i = \frac{m_i}{2^{n_i}} \quad \text{in } L \quad \text{and} \quad p_i^{z_i} = x^{-n_i} p_i^{m_i} x^{n_i},$$

we obtain

$$\begin{aligned} &p_1^{1+4z_1} p_2^{1+4z_2} p_3^{4z_3} \dots p_5^{4z_s} x^2 \\ &+ p_1^{4+2z_1} p_2^{2+2z_2} p_3^{2z_3} \dots p_s^{2z_3} x \\ &+ p_1^{6+z_1} p_2^{5+z_2} p_3^{z_3} \dots p_s^{z_s}. \end{aligned}$$

This element will generate the principal right ideal in R_1 that corresponds to the following element in W':

$$\begin{aligned} &\hat{r}(z_1, \dots, z_s, 0, \dots) = (\min\{1 + 4z_1, 4 + 2z_1, 6 + z_1\}, \\ &\min\{1 + 4z_2, 2 + 2z_2, 5 + z_2\}, z_3, z_4, z_5, \dots, z_s, 0, \dots) \\ &= (\phi_1(z_1), \phi_2(z_2), \phi_3(z_3), \dots). \end{aligned}$$

The first component $\phi_1(z_1)$ is therefore equal to the following:

$$\phi_1(z_1) = \begin{cases} 1 + 4z_1 \text{ for } 0 \leqq z_1 \leqq \dfrac{3}{2} \\ 4 + 2z_1 \text{ for } \dfrac{3}{2} \leqq z_1 \leqq 2 \\ 6 + z_1 \quad \text{for } 2 \leqq z_1. \end{cases}$$

Similarly, one obtains the function ϕ_2 defined on L_2 through

$$\phi_2(z_2) = \begin{cases} 1 + 4z_2 \text{ for } 0 \leqq z_2 \leqq \dfrac{1}{2} \\ 2 + 2z_2 \text{ for } \dfrac{1}{2} \leqq z_2 \leqq 3 \\ 5 + z_2 \quad \text{for } 3 \leqq z_2. \end{cases}$$

Finally we have $\phi_i(z_i) = z_i$ for all $i > 2$. The element $\hat{r}$ is completely described by the element $(\phi_1, \phi_2, \phi_3, \dots, \phi_i, \dots)$ operating on W' and we write

$$\hat{r} = (\phi_1, \phi_2, \dots).$$

The elements ϕ_i can be represented by graphs consisting of finitely many linear pieces described by equations of the form $g(z) = b + 2^m z$ for z in L_i, b in $\pm L_i$. An operation on this set of elements $(\phi_1, \phi_2, \dots)$ is defined through the operation $\tilde{r}_1\tilde{r}_2 = \widetilde{r_1 r_2}$ as the component wise composition of mappings, i.e.,

$$(\phi_1, \phi_2, \dots)^*(\phi_1', \phi_2', \dots) = (\phi_1 \circ \phi_1', \phi_2 \circ \phi_2', \dots)$$

where $\phi_i \circ \phi_i'$ is the composition of mappings on L_i.

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An element $\hat{r} = (\phi_1, \phi_2, \dots)$ has an inverse if $\phi_i(0) = 0$ for every i, i.e., if the graph of every ϕ_i goes through the origin.

The inverse of such an element is equal to

$$\hat{r}^{-1} = (\phi_1^{-1}, \sigma_2^{-1}, \dots)$$

and the graph of ϕ_i^{-1} is the reflection of the graph of ϕ_i on the graph of $f(z_i) = z_i$. If $\phi_i(z) = a + 2^m z$ for $c_1 \leqq z \leqq c_2$ then

$$\phi_i^{-1}(z) = -a2^{-m} + 2^{-m}z \quad \text{for } \phi_i(c_1) \leqq z \leqq \phi_1(c_2).$$

In this final example let $F = \mathbf{Q}[t]$, the polynomial ring in one indeterminate over the field $\mathbf{Q}$ of rational numbers, and let σ be defined by

$$\sigma\left(\sum q_i t^i\right) = \sum q_i t^{2i}.$$

We can not use the pair F, σ as a pair for R_0, σ in Theorem 2. The image $\sigma((t + 1)F)$ of the maximal right ideal $(t + 1)F$ is not contained in $(t + 1)F$. It is obvious that the maximal right ideal $N_0 = tF$ satisfies the condition $\sigma(N_0) \subseteq N_0$ and it follows from [**2**] that the maximal right ideals $p_n(t)F = N_n$ also satisfy $\sigma(N_n) \subseteq N_n$ where $p_n(t)F$ is the n^{th} cyclotomic polynomial and n is odd. We form $R_0 = FM^{-1}$ with $M = F\backslash(\cup N_i)$ where $i = 0$ or odd. The monomorphism σ can be extended to R_0 and we can now apply Theorem 2 to obtain a ring R_1. It follows as in the previous examples that the ideals

$(\cup \bar{x}^n N_i x^n)R_1$ for $i = 0$ or odd positive are the maximal right ideals of R_1.

The set of principal right ideals of R_1 corresponds to the set

$$W' = \{ (z_0, z_1, z_3, z_5, \dots) \}$$

where z_0 is in the set

$$\left\{ \frac{n}{2^m}, \ n, m \text{ non-negative integers} \right\}$$

but where the remaining z_i's are just non-negative integers, almost all $z_i = 0$. To see this we point out that

$$x^{-1}p_n(t)x = p_n(t)x^{-1}p_n(-t)^{-1}xc_n, \ c_n \neq 0 \text{ in } Q,$$

where $x^{-1}p_n(-t)x$ is a unit in R_1, since $p_n(-t)$ is a unit in R_0; its roots are the negatives of the primitive n^{th} roots of unity.

The elements in the semi group $\widetilde{H}(R_1)$ correspond to elements of the form $(\phi_0, \phi_1, \phi_3, \phi_5, \dots)$ where the graph of ϕ_0 is again piecewise linear with the pieces defined by equations of the form

$$\phi_0(z_0) = a + 2^m z.$$

The functions ϕ_i, i positive odd, are all equal to the identity except for finitely many which are of the form

$$\phi_i(z_i) = a_i + z_i$$

for some integer a_i.

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