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Mapping Class Groups of Hyperbolic Surfaces and Automorphism Groups of Graphs

PAUL SCHMUTZ SCHALLER
Section de mathématiques, Université de Genève, CP 240, CH-1211 Geneva 24, Switzerland.
e-mail: paul.schmutz@math.unige.ch

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Abstract. Let M be a hyperbolic surface and $\Gamma(M)$ its extended mapping class group. We show that $\Gamma(M)$ is isomorphic to the automorphism group of the following graph $G(M)$. The set of vertices of $G(M)$ is the set $S(M)$ of nonseparating simple closed geodesics of M. Two vertices u and v of $S(M)$ are related by an edge if u and v intersect exactly once in M. The graph $G(M)$ can be thought of as a combinatorial model for M.

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1. Introduction

Let M be a Riemann surface equipped with a complete metric of constant curvature -1. Let M have genus g and n cusps (and no further boundary components); M is then called a (g, n)-surface; we will always exclude the case $(g, n) = (0, 3)$.

Let $\Sigma(M)$ be the set of the simple closed geodesics of M. Traditionally, one considers the sets of mutually disjoint elements of $\Sigma(M)$ as the most important finite subsets of $\Sigma(M)$. These subsets have many important applications, some of them I am going to describe now. A (maximal) set of $3g - 3 + n$ disjoint elements of $\Sigma(M)$ partitions M into pairs of pants and provides parameters for the Teichmüller space $T(g, n)$ of M (one half of the Fenchel–Nielsen parameters). These $3g - 3 + n$ disjoint elements also serve as parameters for the set $\Sigma(M)$ itself; this has first been discovered by Dehn [2] and was rediscovered by Thurston (see [13]) who used these elements also for defining the geodesic laminations. Further, Harvey [5] has defined the so-called complex of curves where every subset of $\Sigma(M)$ of $k + 1$ disjoint elements is considered as a k-simplex ($k \geqslant 0$). This complex of curves $C(M)$ has many interesting properties (see, for example, [6, 12]); one of them is that the automorphism group of $C(M)$ is isomorphic to the extended mapping class group $\Gamma(M)$ (containing also the isotopy classes of orientation reversing self-homeomorphisms of M). This has been proved by Ivanov [7, 8] for $g \geqslant 2$ and, independently, by Korkmaz [9] and Luo [11] for the remaining cases.

During my work on simple closed geodesics (see [14] for a survey), I came to the conclusion that one has to consider more general finite subsets of $\Sigma(M)$ than those described above. In particular, the subsets are important which appear as set of systoles of a (g, n)-surface (a systole is a shortest simple closed geodesic of a (g, n)-surface). As a consequence, I propose to study the following 'systolic complex of curves' $\mathrm{SC}(M)$. If $n = 0$ or $n = 1$ then the k-simplices are now the sets of $k + 1$ nonseparating elements of $\Sigma(M)$ which mutually intersect at most once. If $n \geqslant 2$, then we allow also those separating elements of $\Sigma(M)$ which separate a pair of pants with two cusps from the rest of the surface; such a separating element is allowed to intersect other elements of the simplex at most twice. Of course, not every k-simplex of $\mathrm{SC}(M)$ will correspond to a set of systoles of a (g, n)-surface, but $\mathrm{SC}(M)$ is the natural combinatorial object which 'contains' all interesting sets of systoles.

As a first test of the properties of $\mathrm{SC}(M)$ one has the following conjecture.

CONJECTURE. *The automorphism group of the systolic complex of curves* $\mathrm{SC}(M)$ *is isomorphic to* $\Gamma(M)$.

I can prove the conjecture for a few cases such as $(g, n) \in \{(1, 1), (1, 2), (2, 0)\}$. The content of this paper is another result which may be viewed as an important and necessary step towards a better understanding of the systolic complex of curves. Namely, I consider here the following graph $\mathrm{G}(M)$. If $g = 1$, the set of vertices of $\mathrm{G}(M)$ is the set $\mathrm{S}(M)$ of the nonseparating elements of $\Sigma(M)$ and two elements of $\mathrm{S}(M)$ are related by a nonoriented edge if they intersect exactly ones. If $g = 0$, then the set of vertices is the set $\mathrm{S}(M)$ of elements of $\Sigma(M)$ which separate a pair of pants with two cusps from the rest of the surface; two elements of $\mathrm{S}(M)$ are related by a nonoriented edge if they intersect exactly twice. Note that $\mathrm{G}(M)$ is a subgraph of the systolic complex of curves $\mathrm{SC}(M)$ when in the latter we only consider 0- and 1-simplices (in the sequel I shall treat $\mathrm{SC}(M)$ and $\mathrm{C}(M)$ as graphs, but continue to call them 'complexes'). The following is the main result of the paper.

THEOREM A. *Let* $M \in T(g, n)$. *Then* $\Gamma(M)$ *is isomorphic to the group* $\mathrm{Aut}(\mathrm{G}(M))$ *of automorphisms of* $\mathrm{G}(M)$, *except in the cases* $(g, n) \in \{(0, 4), (1, 1), (1, 2), (2, 0)\}$.

If $(g, n) \in \{(1, 1), (1, 2), (2, 0)\}$, *then* $\mathrm{Aut}(\mathrm{G}(M))$ *is isomorphic to* $\Gamma(M)/Z_2$ *(since these surfaces are all hyperelliptic).*

If $(g, n) = (0, 4)$, *then* $\mathrm{Aut}(\mathrm{G}(M))$ *is isomorphic to* $\Gamma(M)/H$ *with* $H \simeq Z_2 \oplus Z_2$ *(since these surfaces have three hyperelliptic involutions).*

In all cases, $\mathrm{G}(M)$ *is connected.*

We therefore have the somewhat surprising result that the two automorphism groups $\mathrm{Aut}(\mathrm{C}(M))$ and $\mathrm{Aut}(\mathrm{G}(M))$ are isomorphic despite the fact that the graphs $\mathrm{C}(M)$ and $\mathrm{G}(M)$ are quite different. For example the maximal order of complete subgraphs is different; it is $3g - 3 + n$ in the case of $\mathrm{C}(M)$ while in the case of $\mathrm{G}(M)$ it is

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$2g+1$ if $g \geqslant 1$ (independent of n) and it is $n-1$ if $g=0$. Note that G(M) is vertex-transitive and edge-transitive which both is not the case for C(M). Further, Theorem A also holds if $(g, n) \in \{(0, 4), (1, 1), (1, 2)\}$ while for these cases Aut(C(M)) is not isomorphic to $\Gamma(M)/H$ (H being the subgroup generated by the hyperelliptic involutions); see the references cited above.

On the other hand, both C(M) and G(M) are related to SC(M), introduced above, and from this point of view it is less surprising that Aut(C(M)) and Aut(G(M)) are isomorphic.

In order to prove Theorem A, I introduce the following notation (which here is explained for the case $g \geqslant 1$). Let u and v be two nonseparating simple closed geodesics of M which are not disjoint. Then u induces a partition of v into a number of connected components which I call *components of v with respect to u*. It will be sufficient to study these components. There are three topological possibilities for such a component v_1 (of v with respect to u). The first possibility is that v_1 separates $M \setminus u$ ($M \setminus u$ is the surface obtained by cutting M along u). If v_1 does not separate $M \setminus u$, then v_1 either starts and ends on the same copy of u in $M \setminus u$ (v_1 is 'one-sided') or relates the two copies of u in $M \setminus u$ (v_1 is 'two-sided').

The paper is organized as follows. Section 2 contains the proof of Theorem A for $g \geqslant 1$. Section 3 contains the proof of Theorem A for $g = 0$. In Section 4, I briefly discuss some other natural subgraphs of the systolic complex of curves which also have the same automorphism group as G(M).

2. Proof of the Main Theorem if $g \geqslant 1$

DEFINITION. (i) A *surface* is a Riemann surface equipped with a metric of constant curvature -1. A (g, n)-surface is a surface of genus g with n cusps (and no further boundary components). The case $(g, n) = (0, 3)$ is excluded in this paper.

(ii) A *boundary component* of a surface is, by definition, a simple closed geodesic (also called boundary geodesic) or a cusp.

(iii) Let M be a (g, n)-surface. An embedded subsurface $M' \subset M$ is called a $(g', n') -$ *subsurface* if M' has genus g' and n' boundary components.

(iv) A *pair of pants* is a surface of genus zero with three boundary components.

(v) Let M be a (g, n)-surface. By $\Gamma(M)$ is denoted the *extended* mapping class group of M (which also contains the isotopy classes of orientation reversing self-homeomorphisms of M).

Remark. Let M be a (g, n)-surface and let u be a nonseparating simple closed geodesic of M. I shall often use the surface which is the closure of $M' = M \setminus u$ (the closure of M' has two copies of u among the boundary components). By abuse of notation I shall not make a difference between M' and its closure. In the same spirit, I shall also say that M (or an embedded subsurface of M) 'contains' its cusps.

DEFINITION. Let M be a (g, n)-surface, $g \geqslant 1$.

(i) Let $\mathrm{S}(M)$ denote the set of nonseparating simple closed geodesics of M.
(ii) Let $u, v \in \mathrm{S}(M)$. Then $i(u, v)$ denotes the number of intersection points of u and v. If $u = v$, then $i(u, v) = 0$. If $i(u, v) = 0$ and $u \neq v$, then u and v are called *disjoint*. The same definition also applies if u, v are simple closed geodesic of M which are not in $\mathrm{S}(M)$.
(iii) If $i(u, v) = 1$, then I say that u and v are *orthogonal* and write $u \perp v$.

Remark. The relation 'orthogonal' (or $\perp$) defined above is symmetric, but neither reflexive nor transitive; the name of this relation has been introduced by F. Luo [10].

DEFINITION.

(i) Let M be a (g, n)-surface. $\mathrm{G}(M)$ denotes the following *graph*. $\mathrm{S}(M)$ is the set of vertices of $\mathrm{G}(M)$ and

$$\{(u, v) \in \mathrm{S}(M) \times \mathrm{S}(M) : u \perp v\}$$

is the set of (nonoriented) edges. Instead of $\mathrm{G}(M)$, I also use the notation $\mathrm{G}(g, n)$.
(ii) Let $F = \{u_1, u_2, \ldots, u_k\} \subset \mathrm{S}(M)$, $k \geqslant 1$. Define

$$N(F) := N(u_1, \ldots, u_k) := \{x \in \mathrm{S}(M) : x \perp u_i,\ \forall i = 1, \ldots, k\}.$$

(iii) $\mathrm{Aut}(\mathrm{G}(M))$ denotes the automorphism group of $\mathrm{G}(M)$.

Remark. Let M be a (g, n)-surface. Note that $\Gamma(M)$ and $\mathrm{G}(M)$ depend only on g and n and not on the particular (g, n)-surface M.

DEFINITION. Let M be a (g, n)-surface, $g \geqslant 1$, let $u \in \mathrm{S}(M)$. Let $v \in \mathrm{S}(M)$ such that $i(u, v) \geqslant 2$. Let v_1 be a connected component of v in $M \setminus u$. If $M \setminus (u \cup v_1)$ is connected, then v_1 is called a *nonseparating component of v with respect to u*. Otherwise, v_1 is called a *separating component of v with respect to u.*

Let $M_1 = M \setminus u$. Let u_1 and u_2 be the two copies of u in M_1. Let v_1 be a nonseparating component of v with respect to u. If v_1 relates u_1 and u_2, then v_1 is called *two-sided*. Otherwise, v_1 is called *one-sided*.

LEMMA 1. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$ such that $i(u, v) \geqslant 2$. Let v have a separating component v_1 with respect to u. Then there exists $w \in \mathrm{S}(M) \setminus \{u, v\}$ such that $N(u, v) \subset N(w)$. Moreover, w is disjoint to u.*

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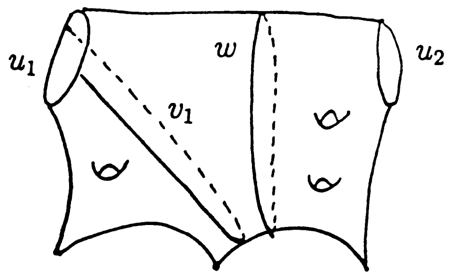

Figure 1. The separating component v_1 of v with respect to u in M_1

Proof (Compare Figure 1). Let $M_1 = M \setminus u$. Let u_1 and u_2 be the two copies of u in M_1. Then v_1 starts and ends in the same boundary geodesic of M_1, in u_1, say. Denote by V_1 and V_2 the two connected components of $M_1 \setminus v_1$ where the notation is chosen such that u_2 lies in V_2. In V_2, there is a unique simple closed geodesic w such that $V_2 \setminus w$ has a connected component W of genus zero which contains no cusps and which has v_1 in its boundary. Note that $w \neq u_2$ since v is simple. It follows that $w \in \mathrm{S}(M) \setminus \{u, v\}$. Put $X = W \cup V_1$.

Let $s \in N(u, v)$. Since $s \in N(u)$, it follows that $s \cap X$ has a connected component s_1 relating u_1 and w. Assume that $s \cap X$ has a second connected component s_2. Then s_2 starts and ends in w and therefore intersects v_1 at least twice (since v_1 separates X). But since $s \in N(v)$, s_2 cannot exist and therefore $s \in N(w)$. □

DEFINITION. A subset $\{u, v, w\} \subset \mathrm{S}(M)$ of three elements is called a *triple* if the three elements are mutually orthogonal and if M has a $(1, 1)$-subsurface which contains u, v, w.

LEMMA 2. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$, $u \perp v$. Then $\mathrm{S}(M)$ has exactly two different elements w such that $\{u, v, w\}$ is a triple. Moreover, $M \setminus (u \cup v \cup w)$ has three connected components; two of them are isometric hyperbolic triangles.*

Proof. Obvious. □

LEMMA 3. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$ such that $i(u, v) \geqslant 2$. Let v have a nonseparating component v_1 with respect to u. Then there exist $w, w' \in \mathrm{S}(M) \setminus \{u, v\}$ such that $N(u, v) \subset (N(w) \cup N(w'))$. Moreover, if v_1 is one-sided, then u, w, w' are mutually disjoint; if v_1 is two-sided, then $\{u, w, w'\}$ is a triple with*

$$i(u, v) = i(v, w) + i(v, w') \quad \textit{and} \quad \min\{i(v, w), i(v, w')\} > 0. \tag{1}$$

Proof (Compare Figure 2). Let $M_1 = M \setminus u$. Let u_1 and u_2 be the two copies of u in M_1.

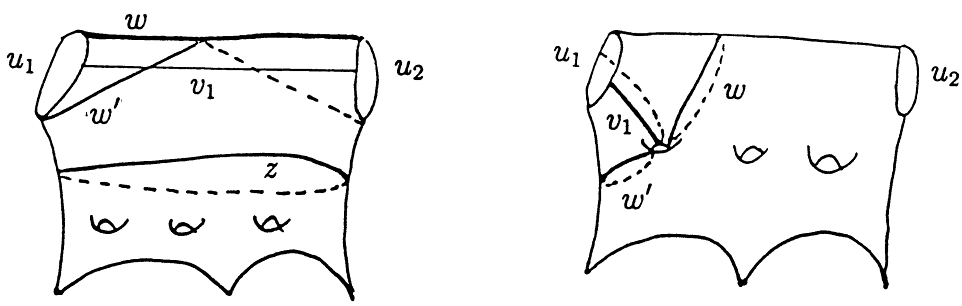

Figure 2. The nonseparating component v_1 of v with respect to u in M_1; v_1 is two-sided on the left-hand side and one-sided on the right-hand side, respectively.

(i) Assume first that v_1 is two-sided. Cut M_1 along v_1; then in the boundary of the resulting surface there is a simple closed curve which is freely homotopic to a unique simple closed geodesic z (in M_1) which is the boundary geodesic of a pair of pants Y (in M_1) which contains v_1; the two other boundary geodesics of Y are u_1 and u_2. In M, z separates a $(1, 1)$-subsurface Q from the rest (Q contains u).

Let $s \in N(u, v)$ and let s_1 be the connected component of $s \cap Q$ which intersects u. Assume that $s \cap Q$ has a second connected component s_2. Since $s \in N(u)$, s_2 does not intersect u. By construction, s_2 then intersects v_1. Since $s \in N(v)$, it follows that $s \cap Q$ has at most two connected components.

Let now w, w' be simple closed geodesics in Q such that $\{u, w, w'\}$ is a triple and such that $w \cap Y$ and $w' \cap Y$ are homotopic to v_1 (the homotopy is such that the endpoints may vary on u_i, $i = 1, 2$). It follows that v_1 intersects each of w and w' at most once. Let s_1 be disjoint to v_1. Then, by Lemma 2, s_1 is disjoint to one of w, w' and intersects once the other one. If s_2 does not exist, we are done. If s_2 exists, then s_2 intersects once each of w, w', and we are done again. So assume that s_1 intersects v_1 (and that s_2 does therefore not exist). It follows by Lemma 2 that s_1 intersects once one of w, w'. We thus have proved that $s \in N(w) \cup N(w')$.

By Lemma 2, the triangle inequality and the fact that v cannot intersect transversally v_1, it follows that $i(u, v) = i(v, w) + i(v, w')$. Let $v' \subset v$ be the connected component of v in Q which contains v_1. Then v' intersects u at least twice and therefore, v' cannot be connected in $Q \setminus w$ nor in $Q \setminus w'$. This proves that v intersects both w and w' and therefore (1) holds.

(ii) Assume now that v_1 starts and ends in u_1. Then v_1 separates u_1 into two parts u_{1a} and u_{1b}. Let w be the simple closed geodesic in M_1 which is freely homotopic to $u_{1a} \cup v_1$; let w' be the simple closed geodesic in M_1 which is freely homotopic to $u_{1b} \cup v_1$. Then u_1, w, w' are the boundary geodesics of a (unique) pair of pants Y, embedded in M_1. Note that $v_1 \subset Y$. Since v_1 is a nonseparating component, w and w' are in $\mathrm{S}(M) \setminus \{u, v\}$. Let $s \in N(u, v)$. It follows that $s \cap Y$ has a connected component s_1 starting in u_1 and ending in w or in w'. Let s_2 be another connected component of $s \cap Y$. Then s_2 must relate w and w' and therefore intersects v_1. It follows that $s \cap Y$ has at most two connected components and therefore, $s \in N(w) \cup N(w')$. □

Remark. Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$, let $k = i(u, v)$. If $k = 0$, then $N(u, v)$ is not empty. If $k \geqslant 2$, then $i(v, w) < k$ where w is defined as in Lemma 1 or in Lemma 3. It follows by induction with respect to k that v and w are in the same connected component of $\mathrm{G}(M)$ and hence also u and v. This proves that $\mathrm{G}(M)$ is connected.

LEMMA 4. *Let M be a (g, n)-surface. Let F be a subset of* $\mathrm{S}(M)$ *such that there exists $v \in \mathrm{S}(M)$ with $i(u, v) = 0$ for all $u \in F$. If $N(F)$ has an element w which intersects v, then $N(F)$ is an infinite set.*

Proof. Let $w \in N(F)$ such that w intersects v. Execute a full twist deformation along v. The result is a surface M_1 isometric to M. As marked geodesics, the elements of F are not changed by this deformation, but w has become a different element $w_1 \in \mathrm{S}(M)$. Of course, $w_1 \in N(F)$. The same argument holds for a twist deformation along v of k full twists (for any integer k). This proves the lemma. □

LEMMA 5. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$ be two disjoint elements. Then there do not exist elements $w, w' \in \mathrm{S}(M) \setminus \{u, v\}$ such that $N(u, v) \subset (N(w) \cup N(w'))$.*

Proof. Assume that there exist $w, w' \in \mathrm{S}(M) \setminus \{u, v\}$ such that $N(u, v) \subset (N(w) \cup N(w'))$. Let $z \in \mathrm{S}(M)$ such that $i(w, z) > 0$ and $i(u, z) = i(v, z) = 0$. Let $z' \in \mathrm{S}(M)$ such that $i(w', z') > 0$ and $i(u, z') = i(v, z') = 0$. Let $T \subset N(u, v)$ be the subset of elements which intersect both z and z'; it is clear that T is not empty.

Let $t \in T$. By Lemma 4 we can 'twist' t along z in order to obtain $t' \in T$ such that the number of intersections of t' with w becomes big. By the same argument we then can twist t' along z' so that also the number of intersections with w' becomes big. It is therefore impossible that $T \subset (N(w) \cup N(w'))$. □

THEOREM 6. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $u, v \in \mathrm{S}(M)$, $u \neq v$. Then u and v are disjoint if and only if $\phi(u)$ and $\phi(v)$ are disjoint for every $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. In other words,* $\mathrm{G}(M)$ *recognizes whether the elements of* $\mathrm{S}(M)$ *are disjoint or not disjoint.*

Proof. This follows by Lemma 1, Lemma 3, and Lemma 5. □

DEFINITION. Let M be a (g, n)-surface, $g \geqslant 1$. A *partition* $P \subset \mathrm{S}(M)$ is a set of $3g - 3 + n$ mutually disjoint elements.

CONVENTION. Let M be a (g, n)-surface. Let $\gamma \in \Gamma(M)$, taken as a self-homeomorphism of M. Let u be a simple closed geodesic of M. Then $\gamma(u)$ is a simple closed curve in M, and in the homotopy class of $\gamma(u)$, there is a unique simple closed geodesic. Therefore, γ induces a map, also denoted by γ, of the simple closed geodesics of M to the simple closed geodesics of M (of course, this map does not change if γ is replaced by a γ' isotopic to γ). We will use this interpretation of the elements of $\Gamma(M)$.

COROLLARY 7. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $P \subset \mathrm{S}(M)$ be a partition. Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Then $\phi(P)$ is a partition. Moreover, there exists $\gamma \in \Gamma(M)$ such that $\gamma(u) = \phi(u)$ for all $u \in P$.*

Proof. It is clear by Theorem 6 that $\phi(P)$ is a partition. It is therefore sufficient to prove that the boundary components of a pair of pants (induced by P) are mapped, by ϕ, to boundary components of a pair of pants (induced by $\phi(P)$); this would imply the existence of γ as claimed.

In the sequel let Y be a pair of pants induced by P with boundary components u, v, w.

(i) Assume first that $u, v, w \in P$ and then assume that $N(u, v, w)$ is empty, so also is $N(\phi(u), \phi(v), \phi(w))$. This implies that $M \setminus (\phi(u) \cup \phi(v) \cup \phi(w))$ is not connected. On the other hand, there exists $u' \in N(u)$ disjoint to v. It follows from Theorem 6 that $\phi(u')$ is disjoint in $\phi(v)$. This implies that $M \setminus (\phi(u) \cup \phi(v))$ is connected. The same argument shows that $M \setminus (\phi(u) \cup \phi(w))$ is connected and that $M \setminus (\phi(v) \cup \phi(w))$ is connected. Therefore, $\phi(u), \phi(v), \phi(v)$ are the boundary geodesics of a pair of pants.

(ii) Assume now that w is a cusp and that only u and v are in P. Then there does not exist $u' \in N(u)$ disjoint to v. By Theorem 6 this property is respected by ϕ which implies that $M \setminus (\phi(u) \cup \phi(v))$ is not connected.

Further, there exists $z \in \mathrm{S}(M)$ disjoint to $u \cup v$ such that z intersects all elements of $P \setminus \{u, v\}$. By Theorem 6 this property is respected by ϕ. Therefore, $\phi(u)$ and $\phi(v)$ are the boundary components of a pair of pants induced by $\phi(P)$. □

COROLLARY 8. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $\{u, v, w\} \subset \mathrm{S}(M)$ be a triple. Then $\{\phi(u), \phi(v), \phi(w)\}$ is also a triple for every $\phi \in \mathrm{Aut}(\mathrm{G}(M))$.*

Proof. By definition of a triple, there exists a $(1, 1)$-subsurface Q of M with boundary component z such that u, v, w are in Q. If $(g, n) = (1, 1)$, the corollary holds, so we can exclude this case in the sequel and assume that z is a simple closed geodesic.

(i) Let $g \geqslant 2$. Then there exists a $(1, 2)$-subsurface $R \subset M$ with boundary components $x, y \in \mathrm{S}(M)$ which contains Q. Moreover, there exists $t \in N(x, y)$, t disjoint to z. By Corollary 7, $\phi(u), \phi(v), \phi(w)$ lie in a $(1, 2)$-subsurface $R' \subset M$ with boundary components $\phi(x), \phi(y)$. Since $\phi(t)$ is disjoint to $\phi(u), \phi(v), \phi(w)$ and intersects the boundary of R', it follows that $\phi(u), \phi(v), \phi(w)$ lie in a $(1, 1)$-subsurface.

(ii) Assume now that $g = 1$. Then M has a partition

$$P = \{u, x_1, \ldots, x_{n-1}\} \subset \mathrm{S}(M)$$

such that $x_i \in N(v, w)$, $\forall i = 1, \ldots, n-1$. To x_i there exists a unique simple closed geodesic z_i in M such that z_i is disjoint to v and to all elements of $P \setminus \{x_i\}$, $i = 1, \ldots, n-1$. Since $\{u, v, w\}$ is a triple, w is disjoint to z_i, $i = 1, \ldots, n-1$. By Lemma 4 it follows that if the elements of $P \setminus \{x_i\}$ are fixed, then there are infinitely many different possibilities to choose x_i with the required properties. By Corollary

7, $P' = \phi(P)$ is a partition. Since we had infinitely many different possibilities to choose x_i, it follows that to every $x' \in P' \setminus \{\phi(u)\}$, there must exist a simple closed geodesic z' which is disjoint to $\phi(v)$ and $\phi(w)$ and to all elements of $P' \setminus \{x'\}$. This implies that $\{\phi(u), \phi(v), \phi(w)\}$ is a triple. □

THEOREM 9. *Let M be a (g, n)-surface, $g \geqslant 1$. Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Let $u, v \in \mathrm{S}(M)$. Then $i(u, v) = i(\phi(u), \phi(v))$.*

Proof. If $i(u, v) = 0$, then the theorem follows by Theorem 6. Assume that the theorem holds for all $u, v \in \mathrm{S}(M)$ with $i(u, v) \leqslant k - 1$ for a $k \geqslant 2$.

Let $u, v \in \mathrm{S}(M)$ such that $i(u, v) = k$. In order to prove the theorem, it is sufficient to show that $i(u, v) = i(\phi(u), \phi(v))$. In the sequel, a component of v is always a component with respect to u.

(i) Assume that there exists a two-sided component v_1 of v. Let w, w' be defined as in Lemma 3. Then $\{u, w, w'\}$ is a triple and (1) in Lemma 3 holds. Since v intersects both w and w', it follows by hypothesis on k that $i(v, w) = i(\phi(v), \phi(w))$ and $i(v, w') = i(\phi(v), \phi(w'))$. By Corollary 8, $\{\phi(u), \phi(w), \phi(w')\}$ is a triple, therefore, by the triangle inequality and Lemma 2,

$$i(\phi(v), \phi(w)) + i(\phi(v), \phi(w')) \geqslant i(\phi(u), \phi(v))$$

which implies $i(u, v) \geqslant i(\phi(u), \phi(v))$. It follows by hypothesis on k (applied to ϕ^{-1}) that $i(u, v) = i(\phi(u), \phi(v))$.

(ii) Let u_1, u_2 be the two copies of u in $M' = M \setminus u$. By (i) we can assume that all components of v are separating ore one-sided. Let M_i be the smallest embedded subsurface of M' (the boundary components of M_i being simple closed geodesics or cusps) such that M_i contains all components of v with endpoints on u_i, $i = 1, 2$. Then M_1 and M_2 have disjoint interior. Since u is nonseparating, M_1 and M_2 have a common boundary component $x = x_1 = x_2 \in \mathrm{S}(M)$ or $M' \setminus (M_1 \cup M_2)$ has a connected component M_3 which has a common boundary component $x_i \in \mathrm{S}(M)$ with M_i, $i = 1, 2$. Then there exists a simple curve $\tau_i \subset M_i$ which relates u_i and x_i and is disjoint to v, $i = 1, 2$. Let $\tau \subset M'$ be a simple curve which relates u_1 and u_2 such that $\tau \cap M_i = \tau_i$, $i = 1, 2$. Let t be a geodesic segment homotopic to τ (the homotopy is such that the endpoints may vary on u_i, $i = 1, 2$). Treat t as a component of a simple closed geodesic with respect to u. Then define w, w' as in Lemma 3. It follows as in Lemma 3 that $i(u, v) = i(v, w)+ i(v, w')$. If v intersects both w and w', it follows by the same argument as in (i) that $i(u, v) = i(\phi(u), \phi(v))$.

If $g = 1$, then v must intersect both w and w' since otherwise, v is separating. This proves the theorem for $g = 1$.

Assume that $g \geqslant 2$. Note that we can interchange the role of u and v and, by the same argument as above, construct a triple $\{v, \bar{w}, \bar{w}'\}$ such that $i(u, v) = i(u, \bar{w})+ i(u, \bar{w}')$ (where $\bar{w}, \bar{w}'$ are orthogonal to x_1). We therefore can assume that v does not intersect w and that u does not intersect $\bar{w}$. This implies that in $\mathcal{M} = M \setminus x_1$, both u, v are nonseparating. $\mathcal{M}$ is homeomorphic to a

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$(g-1, n+2)$-surface, also denoted by $\mathcal{M}$. Of course, ϕ induces canonically an element in $\mathrm{Aut}(\mathrm{G}(\mathcal{M}))$. It then follows by induction with respect to g that $i(u, v) = i(\phi(u), \phi(v))$. □

LEMMA 10. $\mathrm{G}(1, 1)$ *and* $\mathrm{G}(0, 4)$ *are isomorphic.*

Proof. Let $\Gamma(1)$ be the modular group, let $\Gamma(3)$ be the principal congruence subgroup of $\Gamma(1)$ of level three and let Γ' be the commutator subgroup of $\Gamma(1)$. Then $M' = \mathsf{H}/\Gamma'$ is a $(1, 1)$-surface (the so-called modular torus) and $M = \mathsf{H}/\Gamma(3)$ is a $(0, 4)$-surface (H is the upper halfplane). It is well known (see [1, 3]) that there exists a natural bijection between the simple closed geodesics of M and the simple closed geodesics of M'. This bijection induces an isomorphism between $\mathrm{G}(1, 1)$ and $\mathrm{G}(0, 4)$. □

LEMMA 11. *Let* M *be a* $(0, 4)$-*surface. Let* u *and* v *be simple closed geodesics of* M *with* $i(u, v) = 2$.

(i) *Let* w *be a simple closed geodesic of* M *such that* $i(u, w) = 2$. *Then there exists* $\gamma \in \Gamma(M)$ *such that* $\gamma(u) = u$ *and* $\gamma(v) = w$.
(ii) *There are exactly two simple closed geodesics* w_i *of* M *such that* $i(u, w_i) = i(v, w_i) = 2$, $i = 1, 2$. *Moreover, there exists* $\gamma \in \Gamma(M)$ *such that* $\gamma(u) = u$, $\gamma(v) = v$, *and* $\gamma(w_1) = w_2$.

Proof. (i) A twist deformation along u will do the job.

(ii) The first statement is a reformulation of Lemma 2, applying the bijection defined in the proof of Lemma 10. The existence of an (orientation reversing) involution $\gamma \in \Gamma(M)$ with the properties required is obvious. □

THEOREM 12. *Let* M *be a* (g, n)-*surface,* $g \geqslant 1$.

(a) If $(g, n) \notin \{(1, 1), (1, 2), (2, 0)\}$, then $\mathrm{Aut}(\mathrm{G}(M))$ is isomorphic to $\Gamma(M)$.
(b) If $(g, n) \in \{(1, 1), (1, 2), (2, 0)\}$, then $\mathrm{Aut}(\mathrm{G}(M))$ is isomorphic to $\Gamma(M)/H$ where H is the subgroup generated by the hyperelliptic involution.

Proof. (i) Let $\gamma \in \Gamma(M)$. It follows by our convention that γ is an automorphism of $\mathrm{Aut}(\mathrm{G}(M))$. Therefore, we have a group homomorphism, denoted by $\Psi(g, n)$,

$$\Psi(g, n) : \Gamma(M) \longrightarrow \mathrm{Aut}(\mathrm{G}(M)).$$

The kernel of $\Psi(g, n)$ is trivial, except in the cases $(g, n) \in \{(1, 1), (1, 2), (2, 0)\}$. In these three cases, the kernel of $\Psi(g, n)$ contains the isotopy class of the identity and the isotopy class of the (unique) hyperelliptic involution. This has been proved in [4] for closed surfaces (that is $n = 0$), the general case easily follows.

(ii) We have to prove that $\Psi(g, n)$ is surjective. Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Let $m = 3g - 3 + n$. Let $P = \{u_1, \ldots, u_m\} \subset \mathrm{S}(M)$ be a partition of M. By Corollary

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7 we can assume that $\phi(u_i) = u_i$, $i = 1, \ldots, m$. To every u_i, $i = 1, \ldots, m$, there exists $v_i \in \mathrm{S}(M)$ with $i(u_i, v_i) = 2$ and $i(u_j, v_i) = 0$ for all $j = 1, \ldots, m$, $j \neq i$. By Lemma 11 (i) we can assume that $\phi(v_i) = v_i$, $i = 1, \ldots, m$.

For $i = 1, \ldots, m$, there exists $w_i \in \mathrm{S}(M)$ such that $i(u_j, w_i) = 0$ for all $j = 1, \ldots, m$, $j \neq i$, and such that $i(u_i, w_i) = 2$ and $i(v_i, w_i) = 2$. By Lemma 11 (ii) we can assume that $\phi(w_i) = w_i$, $i = 1, \ldots, m$.

Let $P' = \{u_i, v_i, w_i : i = 1, \ldots, m\}$. It now follows that $x \in \mathrm{S}(M)$ is uniquely determined by the $3m$ intersection numbers $i(x, y)$, $y \in P'$. This was first proved by Dehn [2] and rediscovered by Thurston, see [13] for a proof. Therefore, by Theorem 9, $\phi(x) = x$ for all $x \in \mathrm{S}(M)$ so that ϕ is the identity and $\Psi(g, n)$ clearly is surjective. □

Let M be a (g, n)-surface, $g \geqslant 1$. Recall that we have already seen (after Lemma 3) that $\mathrm{G}(M)$ is connected.

3. Proof of the Main Theorem if $g = 0$

LEMMA 13. *Let M be a $(0, 4)$-surface. Let $\mathrm{S}(M)$ be the set of simple closed geodesics of M. Let $\mathrm{G}(M)$ be the following graph. $\mathrm{S}(M)$ is the set of vertices of $\mathrm{G}(M)$ and*

$$\{(u, v) \in \mathrm{S}(M) \times \mathrm{S}(M) : i(u, v) = 2\}$$

is the set of (nonoriented) edges. Then the automorphism group $\mathrm{Aut}(\mathrm{G}(M))$ of $\mathrm{G}(M)$ is isomorphic to $\Gamma(M)/H$ where H is the subgroup of order four generated by the three hyperelliptic involutions of M. Moreover, $\mathrm{G}(M)$ is connected.

Proof. This follows by the the corresponding result for $(g, n) = (1, 1)$ by virtue of Theorem 12 and of Lemma 10. □

Remark. For the rest of this section we can therefore exclude the case $(g, n) = (0, 4)$.

DEFINITION. Let M be a $(0, n)$-surface, $n \geqslant 5$.

(i) Let $\mathrm{S}(M)$ be the set of simple closed geodesics of M which separate a pair of pants (with two cusps) from the rest of the surface. These two cusps are called the *cusps of u*.

Let $u, v \in \mathrm{S}(M)$. Then I say that u and v are *orthogonal* and write $u \perp v$ if $i(u, v) = 2$.

(ii) Let $\mathrm{G}(M)$ be the following *graph*. $\mathrm{S}(M)$ is the set of vertices of $\mathrm{G}(M)$ and

$$\{(u, v) \in \mathrm{S}(M) \times \mathrm{S}(M) : u \perp v\}$$

is the set of (nonoriented) edges.

(iii) Denote by $\mathrm{O}(M)$ the set of simple geodesics in M which relate two different cusps.

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Let $u \in \mathrm{O}(M)$ such that u relates the cusps A and B. Then A and B are called *the cusps of* u.

Let $u, v \in \mathrm{O}(M)$. Then I say that u and v are *orthogonal* and write $u \perp v$ if u and v have one common cusp and do not intersect in the interior of M.

Let $\mathrm{G}'(M)$ be the following *graph*. $\mathrm{O}(M)$ is the set of vertices of $\mathrm{G}'(M)$ and

$$\{(u, v) \in \mathrm{O}(M) \times \mathrm{O}(M) : u \perp v\}$$

is the set of (nonoriented) edges.

(iv) Let $F = \{u_1, \ldots, u_k\} \subset \mathrm{O}(M)$. Then define

$$N(F) := N(u_1, \ldots, u_k) := \{u \in \mathrm{O}(M) : u \perp u_i, i = 1, \ldots, k\}.$$

The analogous definition is used if $F \subset \mathrm{S}(M)$.

LEMMA 14. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Then $\mathrm{G}(M)$ and $\mathrm{G}'(M)$ are canonically isomorphic.*

Proof. Let $u \in \mathrm{S}(M)$. Then u separates M into a pair of pants $Y(u)$ and a second surface M' which is not a pair of pants (since $n \geqslant 5$). In $Y(u)$ there is a unique element of $\mathrm{O}(M)$. It is clear that this defines an isomorphism between the two sets of vertices $\mathrm{S}(M)$ and $\mathrm{O}(M)$.

Let $u, v \in \mathrm{O}(M)$, $u \perp v$. Then the corresponding elements in $\mathrm{S}(M)$ intersect twice so that they are orthogonal as well.

Let $u \in \mathrm{S}(M)$ and let $Y(u)$ be defined as above. Let $v \in \mathrm{S}(M)$, $v \perp u$. Then v intersects twice the boundary geodesic u of $Y(u)$ and it follows that $Y(v)$ and $Y(u)$ have a common cusp so that the corresponding elements of u, v in $\mathrm{O}(M)$ are orthogonal. □

LEMMA 15. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $u, v \in \mathrm{O}(M)$ such that u and v are different, but neither orthogonal nor disjoint. Let v have a connected component $v_1 \subset v$ in $M \setminus u$ which starts and ends on u (u includes the cusps of u). Then there exists $w \in \mathrm{O}(M) \setminus \{u, v\}$ such that $N(u, v) \subset N(w)$.*

Proof. By hypothesis there exists $w \in \mathrm{O}(M) \setminus \{u, v\}$ which has the same cusps as u and is homotopic to v_1 (the homotopy is such that the endpoints may vary on u). Let $s \in N(u, v)$. Then s has a common cusp with u and hence with w. On the other hand, s cannot intersect the interior of w since then s would also intersect the interior of u or of v_1. This proves $s \in N(w)$. □

LEMMA 16. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $u, v \in \mathrm{O}(M)$ such that u and v are different, but neither orthogonal nor disjoint. Let v have a connected component $v_1 \subset v$ in $M \setminus u$ which starts on u and ends in a cusp A which is not a cusp of u. Then there exist $w, w' \in \mathrm{O}(M) \setminus \{u, v\}$ such that $N(u, v) \subset (N(w) \cup N(w'))$.*

Proof. Let A_1 and A_2 be the cusps of u. There exist w and w' in $\mathrm{O}(M)$, $w \neq w'$, which both have the cusp A and both are homotopic to v_1 (the homotopy is such that the endpoint on u may vary); the second cusp of w is A_1, the second cusp of w' is

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A_2. Note that $v_1 = v$ is impossible (otherwise $u \perp v$), therefore $w, w' \in \mathrm{O}(M) \setminus \{u, v\}$. Let $s \in N(u, v)$ and assume that s has the cusp A_1. Then s cannot intersect the interior of w or w' since s would then also intersect the interior of u or of v_1. It follows that if the second cusp of s is A, then $s \in N(w')$ and if the second cusp of s is not A, then $s \in N(w)$. An analogous argument holds if A_2 is a cusp of s. □

LEMMA 17. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let u, v be two disjoint elements of* $\mathrm{S}(M)$. *Then there do not exist elements w, w' in* $\mathrm{S}(M) \setminus \{u, v\}$ *such that* $N(u, v) \subset (N(w) \cup N(w'))$.

Proof. Assume that there exist w, w' in $\mathrm{S}(M) \setminus \{u, v\}$ such that $N(u, v) \subset (N(w) \cup N(w'))$. Let $t \in N(u, v)$ (of course, $N(u, v)$ is not empty). Note first that if $t \in N(w)$, then w intersects u or v. The same is true for w'. Therefore, at least one of w, w' must intersect $u \cup v$. If both w and w' intersect $u \cup v$, then by twisting t along u and v (as in the proof of Lemma 4) we can produce $t' \in N(u, v)$ such that both $i(t', w)$ and $i(t', w')$ become arbitrarily big, hence $t' \notin N(w) \cup N(w')$, a contradiction. If only w intersects $u \cup v$, then $N(u, v) \subset N(w)$. But by twisting t along u and v, we again can produce $t' \in N(u, v)$ such that $i(t', w)$ becomes arbitrarily big. □

THEOREM 18. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Let $u, v \in \mathrm{S}(M)$, $u \neq v$. Then u and v are disjoint if and only if $\phi(u)$ and $\phi(v)$ are disjoint for every $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. In other words, $\mathrm{G}(M)$ recognizes whether the elements of $\mathrm{S}(M)$ are disjoint or not disjoint.*

Proof. Note first that (from Lemma 14) we could also formulate Lemmas 15 and 16 for $\mathrm{S}(M)$. Therefore, the theorem follows from Lemmas 15, 16, and 17. □

DEFINITION. Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $\{x, y, z\} \subset \mathrm{O}(M)$. Then $\{x, y, z\}$ is called a *0-triple* if x, y, z are mutually orthogonal and if $M \setminus (x \cup y \cup z)$ has a connected component which is an embedded triangle.

LEMMA 19. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $\{x, y, z\} \subset \mathrm{O}(M)$ be a 0-triple.*

(i) *Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Then $\{\phi(x), \phi(y), \phi(z)\}$ is a 0-triple.*
(ii) *Let x', y', z' be the corresponding (to x, y, z) elements in $\mathrm{S}(M)$. Let $u \in \mathrm{S}(M)$. Then $i(u, x') \leqslant i(u, y') + i(u, z')$.*

Proof. (i) Since $\{x, y, z\}$ is a 0-triple, there exists a unique $(0, 4)$-subsurface $Q \subset M$ which contains x, y, z. Let $M' = M \setminus Q$, let A be a cusp of M'. Then there are $n - 4$ mutually orthogonal elements $v_i \in \mathrm{O}(M)$, $i = 1, \ldots, n - 4$, which all have the cusp A and which lie in M'. By Theorem 18, $\phi(x), \phi(y), \phi(z)$ are all disjoint to $\phi(v_i)$, $i = 1, \ldots, n - 4$. It follows that $\phi(x), \phi(y), \phi(z)$ lie in a $(0, 4)$-subsurface of M which proves (i).

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Assertion (ii) follows by the triangle inequality (recall that $M \setminus (x \cup y \cup z)$ has a connected component which is a triangle). □

THEOREM 20. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Then $i(u, v) = i(\phi(u), \phi(v))$ for all $u, v \in \mathrm{S}(M)$.*

Proof. If $i(u, v) = 0$, then the theorem follows by Theorem 18. Assume that the theorem holds for all $u, v \in \mathrm{S}(M)$ with $i(u, v) \leqslant k$ for a $k \geqslant 2$.

Let $u, v \in \mathrm{S}(M)$ with $i(u, v) = k + 2$ (note that $i(u, v)$ is always even). In order to prove the theorem, it is sufficient to prove that $i(\phi(u), \phi(v)) = i(u, v)$. Let u', v' be the elements in $\mathrm{O}(M)$ corresponding to u, v. Let A_i, $i = 1, 2$, be the cusps of u'.

(i) Assume that v' has a cusp $A \notin \{A_1, A_2\}$. Let $v_1 \subset v'$ be the connected component of v' in $M \setminus u'$ which starts in A and ends on u'. Let $w, w' \in \mathrm{O}(M)$ be defined as in Lemma 16. Then $\{u', w, w'\}$ is a 0-triple. Let $w_1 \in \mathrm{S}(M)$ correspond to w and $w_2 \in \mathrm{S}(M)$ correspond to w'. It follows by Lemma 19(ii) that

$$i(v, w_1) + i(v, w_2) = i(u, v) \tag{2}$$

(since v cannot intersect v_1 transversally). Since v', w, w' have the common cusp A, it follows that $i(v, w_i) > 0$, $i = 1, 2$. By hypothesis on k this implies

$$i(\phi(v), \phi(w_i)) = i(v, w_i), \quad i = 1, 2. \tag{3}$$

By Lemma 19(i), $\{\phi(u'), \phi(w), \phi(w')\}$ is a 0-triple and it follows by (2), (3) and by Lemma 19(ii) that $i(\phi(u), \phi(v)) \leqslant i(u, v)$. If $i(\phi(u), \phi(v)) < i(u, v)$, then a contradiction follows by hypothesis on k (applied to ϕ^{-1}). This proves that $i(\phi(u), \phi(v)) = i(u, v)$.

(ii) Assume now that u' and v' have the same cusps. Let $v_1 \subset v'$ be the component of v' in $M \setminus u'$ which starts in A_1. Then $M \setminus (u' \cup v_1)$ has a connected component V such that the interior of V is disjoint to v' (and such that A_1 is on the boundary of V). Let A be a cusp of M in V, $A \notin \{A_1, A_2\}$. Then there exists $t \in N(u', v')$ with cusps A, A_1. Let $v_0 \in \mathrm{O}(M)$ have cusps A_1, A_2 such that v_0 is homotopic to v_1 (the homotopy fixes A_1 while the second point on u' may vary). Then there exists $t' \in N(v_0)$ such that $\{u, t, t'\}$ is a 0-triple. Let $s \in \mathrm{S}(M)$ correspond to t and $s' \in \mathrm{S}(M)$ correspond to t'. By the choice of V it follows by Lemma 19(ii) that $i(s, v) + i(s', v) = i(u, v)$. We then conclude by the same argument as in (i) that $i(\phi(u), \phi(v)) = i(u, v)$. □

THEOREM 21. *Let M be a $(0, n)$-surface, $n \geqslant 5$. Then the automorphism group $\mathrm{Aut}(\mathrm{G}(M))$ of $\mathrm{G}(M)$ is isomorphic to $\Gamma(M)$.*

Proof. (i) We use the same convention for the elements of $\Gamma(M)$ as in the case $g \geqslant 1$ in Section 2. As in the proof of Theorem 12 we then have a natural group homomorphism

$$\Psi(0, n) : \Gamma(M) \longrightarrow \mathrm{Aut}(\mathrm{G}(M)).$$

The kernel of $\Psi(0, n)$ is trivial (compare the proof of Theorem 12) so it remains to prove that $\Psi(0, n)$ is surjective.

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(ii) Let $\phi \in \mathrm{Aut}(\mathrm{G}(M))$. Let $F \subset \mathrm{O}(M)$ be a maximal set such that every two elements of F are disjoint or orthogonal. Then the elements of F induce a triangulation of M, each triangle corresponds to a 0-triple. By Theorem 18 and by Lemma 19 this structure is respected by ϕ. It is then clear that there exists $\gamma \in \Gamma(M)$ such that $\gamma(u) = \phi(u)$ for every $u \in F$. We therefore may assume that $\phi(u) = u$ for every $u \in F$. Let $F' \subset \mathrm{S}(M)$ be the to F corresponding set in $\mathrm{S}(M)$. Then $\phi(u') = u'$ for every $u' \in F'$.

(iii) Let u, t, v be three elements of F such that u and v are disjoint and such that $t \in N(u, v)$. Let u', t', v' be the corresponding elements in F'. t is partitioned by $u' \cup v'$ into three parts, denote by t_0 that part which contains none of the cusps of t. Let Y be the unique pair of pants embedded in M which has u and v among its boundary geodesics and which contains t_0. Let z be the third boundary component of Y. Let $w \in \mathrm{S}(M)$. It then follows by Theorem 20 that $i(u', w) = i(u', \phi(w))$, $i(v', w) = i(v', \phi(w))$, and $i(t', w) = i(t', \phi(w))$. It follows by the proof of Dehn's theorem (see [13], compare the proof of Theorem 12), that $i(w, z)$ is determined by $i(u', w)$, $i(v', w)$, and $i(t', w)$. Therefore, $i(z, w) = i(z, \phi(w))$ (note that we cannot apply Theorem 20 directly since z is in general not in $\mathrm{S}(M)$).

(iv) Let $F_0 \subset F'$ be a maximal subset of disjoint elements. Let $F_1 \supset F_0$ be a set of $n - 3$ mutually disjoint simple closed geodesics of M. Let $w \in \mathrm{S}(M)$. Repeating the argument in (iii), it follows that $i(z, w) = i(z, \phi(w))$ for all $z \in F_1$. It then follows by Dehn's theorem that $\phi(w) = w$. Therefore, ϕ is the identity and $\Psi(0, n)$ is surjective. □

Remark. Let M be a $(0, n)$-surface, $n \geqslant 5$. It follows by Lemma 15 and Lemma 16 that $\mathrm{G}(M)$ is connected; compare the remark in Section 2 after Lemma 3.

Remark. We have proved in Lemma 10 that $\mathrm{Aut}(\mathrm{G}(1, 1))$ is isomorphic to $\mathrm{Aut}(\mathrm{G}(0, 4))$. One can also prove that $\mathrm{Aut}(\mathrm{G}(0, 6))$ is isomorphic to $\mathrm{Aut}(\mathrm{G}(2, 0))$ by the following argument.

Let M be a $(2, 0)$-surface. Let H be the subgroup of the automorphism group of M generated by the hyperelliptic involution ψ. Then $M' = M/H$ corresponds to a $(0, 6)$-surface. Let $u \in \mathrm{S}(M)$. Then u passes through two fixed points A and B of ψ (for a formal proof of this fact see, for example, [4]). Therefore, in M', u corresponds to an element $u' \in \mathrm{O}(M)$. This correspondence induces an isomorphism between $\mathrm{Aut}(\mathrm{G}(2, 0))$ and $\mathrm{Aut}(\mathrm{G}(0, 6))$ by virtue of Lemma 14.

It easily follows from the main theorem and its proof that there are no further isomorphisms between groups $\mathrm{Aut}(\mathrm{G}(g, n))$ and $\mathrm{Aut}(\mathrm{G}(g', n'))$, $(g, n) \neq (g', n')$ (compare for example partitions $P \subset \mathrm{S}(M)$ and $P' \subset \mathrm{S}(M')$ where M is a (g, n)-surface and M' is a (g', n')-surface).

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4. Some Further Graphs

In the Introduction, I have defined the systolic complex of curves SC(M) of a (g, n)-surface M. Taking this complex as a graph, SC(M) has a number of interesting subgraphs, some of them are shortly presented here, without complete proofs. One subgraph is G(M) which we have already discussed in Sections 2 and 3. The complex of curves C(M) induces another natural subgraph which is the intersection of C(M) and SC(M).

DEFINITION. Let M be a (g, n)-surface. Let S(M) be defined as in Section 2 if $g \geqslant 1$ and as in Section 3 if $g = 0$. Let $C_S(M)$ be the following graph. S(M) is the set of vertices of $C_S(M)$ and

$$\{(u, v) \in S(M) \times S(M) : u \text{ is disjoint to } v\}$$

is the set of (nonoriented) edges.

THEOREM 22. *Let M be a (g, n)-surface, $(g, n) \notin \{(0, 4), (1, 1), (1, 2)\}$. Then the automorphism group* $\mathrm{Aut}(C_S(M))$ *of* $C_S(M)$ *is isomorphic to* $\mathrm{Aut}(G(M))$ *if and only if* $g \neq 1$.

Proof. By the main theorem for Aut(G(M)) it is sufficient (for $g \neq 1$) to prove that $\gamma \in \mathrm{Aut}(C_S(M))$ recognizes orthogonal elements. For $g \geqslant 2$ we can use the argument of Ivanov (proof of Lemma 1 in [8]), slightly adapted since in our present situation we cannot work with simple closed geodesics which are not in S(M). For $g = 0$ we can use a similar argument. The interesting case is however $g = 1$ so suppose $g = 1$ in the sequel.

Let $\{a, b, c\} \subset S(M)$ be a triple. Define $S(a) = \{x \in S(M) : i(a, x) \equiv 0 (\mathrm{mod}\ 2)\}$. Let $u \in S(M)$ be disjoint to a and let $\psi \in \Gamma(M)$ map a to u. Since $i(a, x) \equiv i(u, x) (\mathrm{mod}\ 2)$ for all $x \in S(M)$, it follows that $\psi(S(a)) = S(a)$. Now define the following bijection ψ' of S(M). On $S(a)$, put $\psi' = \psi$ while on $S(M) \setminus S(a)$, let ψ' be the identity. Let $v, w \in S(M)$ be disjoint. Then $i(a, v) \equiv i(a, w) (\mathrm{mod}\ 2)$. It follows by the definition of ψ' that $\psi'(v)$ and $\psi'(w)$ are disjoint. Since we can apply the same argument to $(\psi')^{-1}$, it follows that $v, w \in S(M)$ are disjoint if and only if $\psi'(v)$ and $\psi'(w)$ are disjoint. This proves that $\psi' \in \mathrm{Aut}(C_S(M))$. But ψ' maps the triple $\{a, b, c\}$ to $\{u, b, c\}$ which is not a triple. Therefore, there is no element of $\Gamma(M)$ which can induce ψ'. This proves the theorem for $g = 1$. □

Remark. It follows by the argument given in the proof of Theorem 22 that the graph $C_S(M)$ is not connected if $g = 1$.

Here is the definition of two other interesting subgraphs of SC(M).

DEFINITION. Let M be a (g, n)-surface, $g \geqslant 1$. Let $\tilde{S}(M)$ be the set of simple closed geodesics which are either nonseparating or separate a pair of pants from the rest of

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the surface. Let $\tilde{G}_i(M)$ be the following graph, $i = 0, 1$. The set of vertices of $\tilde{G}_i(M)$ is $\tilde{S}(M)$, $i = 0, 1$ (the same set of vertices than the graph $SC(M)$ has). Two vertices u and v are related by an edge if u and v are disjoint (this is $\tilde{G}_0(M)$) or if $u \perp v$ (this is $\tilde{G}_1(M)$), respectively.

Here, $u \perp v$ means the following. If u and v are nonseparating, then $i(u, v) = 1$. If at least one of u, v is separating, then $i(u, v) = 2$.

THEOREM 23. *Let M be a (g, n)-surface, $g \geqslant 1$.*

(i) $\mathrm{Aut}(\tilde{G}_1(M))$ *and* $\mathrm{Aut}(G(M))$ *are isomorphic groups.*
(ii) *If $(g, n) \notin \{(1, 1), (1, 2)\}$ then* $\mathrm{Aut}(\tilde{G}_0(M))$ *and* $\mathrm{Aut}(G(M))$ *are isomorphic groups.*

Proof. (i) Let $u, v \in \tilde{S}(M)$, $u \perp v$. One then proves that there exists $w \in \tilde{S}(M)$ with $N(u, v, w) = \emptyset$ if and only if u, v are both nonseparating or both separating. This implies that $\phi \in \mathrm{Aut}(\tilde{G}_1(M))$ either maps all nonseparating elements of $\tilde{S}(M)$ to separating elements or to nonseparating elements. One verifies that the latter must be the case. Finally, it remains to show that if ϕ is the identity if restricted to $S(M)$, then ϕ is also the identity in $\mathrm{Aut}(\tilde{G}_1(M))$.

(ii) By (i) it is sufficient to show that $\phi \in \mathrm{Aut}(\tilde{G}_0(M))$ maps orthogonal elements $u, v \in \tilde{S}(M)$ to orthogonal elements. This is done by analysing the action of ϕ on some particular partitions of M which are related to u, v. □

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