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OSCILLATION THEOREMS FOR DELAY DIFFERENTIAL EQUATIONS VIA LAPLACE TRANSFORMS

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ABSTRACT. Sufficient conditions for the oscillation of all solutions of the delay differention equation (1) below are obtained.

1. **Introduction**. Consider the delay differential equation

$$\dot{x}(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = f(t), \ t \geqq 0 \tag{1}$$

where $f \in C[0, \infty)$ and $p_i \in (-\infty, \infty)$, $\tau_i \in [0, \infty)$ for $i = 1, 2, \ldots, n$.

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of (1). The arguments rely on a known result (Lemma 1) about the abscissa of convergence of the Laplace transform of a non-negative function. Our results apply when, for example, the coefficients p_i are positive and the function $f(t)$ is a finite linear combination of sines and cosines. (See Corollary 1.) By using Laplace transforms we also obtain a remarkably short proof of the following well-known theorem:

THEOREM 0. *Every solution of*

$$\dot{x}(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0 \tag{2}$$

is oscillatory if and only if the characteristic equation

$$P(\lambda) = \lambda + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0 \tag{3}$$

has no real roots.

For other proofs of Theorem 0 see [1], [2], [4], and [5].

As usual, a solution $x(t)$ of (1) is called oscillatory if it has arbitrarily large zeros.

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2. **Forced Oscillations**. Without loss of generality we will assume the coefficients p_i of (1) are all nonzero and that $\tau_1 = \max\{\tau_1, \ldots, \tau_n\}$. Then a solution $x(t)$ of (1) is defined for $t \geqq -\tau_1$ and $x \in C[-\tau_1, \infty) \cap C^1[0, \infty)$.

We first recall some facts about Laplace transforms. If $X(s)$ is the Laplace transform of $x(t)$,

$$X(s) = \int_0^\infty e^{-st} x(t) dt,$$

then the abscissa of convergence of $X(s)$ is defined by

$$b = \inf\{\sigma \in R : X(\sigma) \text{ exists}\}.$$

Then $X(s)$ is analytic for $\text{Re}s > b$.

Let $x_c(t)$ denote $x(t+c)$. Then, for any $c \in R$, the Laplace transform $X_c(s)$ of $x_c(t)$ exists and has the same abscissa of convergence as $X(s)$ as we can see by noting that the defining integrals of $X(s)$ and $X_c(s)$ converge or diverge for the same values of s. Moreover, for $\text{Re}s > b$ we can write

$$X_c(s) = e^{sc}\left[X(s) - \int_0^c e^{-st} x(t) dt\right] \tag{4}$$

The last integral defines an entire function of the complex variable s so we see that $X(s)$ and $X_c(s)$ have their singularities at the same points. We will use the following known result from Widder [6].

Lemma 1. *If $X(s)$ is the Laplace transform of a non-negative function $x(t)$ and has abscissa of convergence $b > -\infty$, then $X(s)$ has a singularity at the point $s = b$.*

We call a function $x(t)$ eventually positive if there is a $c \geqq 0$ such that $x_c(t) > 0$ for all $t > 0$. Our discussion of the abscissa of convergence of $X_c(s)$ implies that Lemma 1 holds when $X(s)$ is the Laplace transform of an eventually positive function.

We assume that, for some $\alpha > 0$, $f(t) = o(e^{\alpha t})$. Then the Laplace transform $F(s)$ of $f(t)$ exists. Let $x(t)$ be a solution of (1). Then (see e.g. [3]) there is a $\beta > 0$ such that $x(t) = o(e^{\beta t})$. This shows that the Laplace transform $X(s)$ of $x(t)$ exists with an abscissa of convergemce b less than infinity.

We now state our first result.

Theorem 1. *Let $a \in R$ and assume that the following conditions are satisfied: (H_1) Equation (3) has no (real) roots in $[a, \infty)$; (H_2) a is the abscissa of convergence of $F(s)$, $F(s)$ has a singularity on* $\text{Re}s = a$, *but $F(s)$ is analytic at $s = a$.*

Then every solution of (1) is oscillatory.

Proof. Suppose (1) had an eventually positive solution $x(t)$ with Laplace transform $X(s)$ having abscissa of convergence b. Then $X(s)$ is analytic in the half-plane $\text{Re}s > b$ and, by Lemma 1, cannot be analytically continued at $s = b$. That is, there is no complex neighborhood of b on which we can find an analytic function which agrees with $X(s)$ for Re$s > b$. By taking the Laplace transform of both sides of (1) we find that

$$P(s)X(s) = x(0) - \phi(s) + F(s) \tag{5}$$

where P is defined by (3) and $\phi(s) = \phi_1(s) + \cdots + \phi_n(s)$ with

$$\phi_i(s) = p_i e^{-s\tau_i} \int_{-\tau_i}^{0} e^{-s\xi} x(\xi) d\xi.$$

for $i = 1, \ldots, n$.

By analyticity, (5) holds for Re$s > \max\{a, b\}$. Note that ϕ is an entire function. Now $a > b$ is impossible because (5) and (H_2) would imply a singularity of $X(s)$ in Re$s > b$.

On the other hand, $a \leqq b$ is impossible because we could then use (H_1), (H_2), and (5) to analytically continue $X(s)$ at $s = b$. Thus (1) cannot have an eventually positive solution. □

COROLLARY 1. *Assume that p_i, $\tau_i \in [0, \infty)$ for $i = 1, \ldots, n$ and that $f(t)$ is a finite linear combination of sines and cosines. Then every solution of (1) is oscillatory.*

THEOREM 2. *Suppose that: (H_3) Equation (3) has no real roots; (H_4) The abscissa of convergence of $F(s)$ is $-\infty$ and, for some $\epsilon > 0$, $|F(s)| = 0(e^{-s(\tau_i - \epsilon)})$ as $s \longrightarrow -\infty$. Then every solution of (1) is oscillatory.*

PROOF. Otherwise (1) has a solution $x(t)$ such that for some $c \geqq 0$, $x_c(t) > 0$ for $t \geqq 0$. Let $(1')$ denote equation (1) with f replaced by f_c. Then $x_c(t)$ is a positive solution of $(1')$. It is easily checked using (4) that $F_c(s)$ also satisfies (H_4). Since we are seeking a contradiction, we may as well assume that $x(t) > 0$ for $t \geqq -\tau_1$. Then in view of (5), and by Lemma 1, if tollows that the abscissa of convergence of $X(s)$ is $-\infty$. Clearly, for all real s we have $X(s) > 0$, and by (H_3), $P(s) > 0$. Now consider $\lim_{s \longrightarrow -\infty} X(s)$. (H_3) implies that p_1, the coefficient corresponding to the largest delay, is positive. Take $\epsilon > 0$ small enough so that $\tau_1 - \epsilon > \tau_i$ for $i = 2, \ldots, n$. By continuity and the assumed positivity of $x(t)$ in $[-\tau_1, 0]$, we can conclude that, eventually as $s \longrightarrow -\infty$,

$$\phi_1(s) > e^{-s(\tau_1 - \epsilon)} \longrightarrow \infty.$$

On the other hand, as $s \longrightarrow -\infty$,

$$|\phi_i(s)| = o(e^{-s\tau_i}) = o(\phi_1(s))$$

for $i = 2, \ldots, n$. This, together with (H_4) and (5), implies that $\lim_{s \longrightarrow -\infty} X(s) = -\infty$. This contradiction concludes the proof. □

The *if* statement of Theorem 0 follows, of course, from Theorem 2 with $f(t) = 0$. Our argument reduces to the following short, direct proof of this: Assume, for the sake of contradiction, that (2) has a positive solution $x(t)$ with Laplace transform $X(s)$. Then, as in (5),

$$P(s)X(s) = x(0) - \phi(s). \tag{6}$$

for $s \in (-\infty, \infty)$. But both $P(s)$ and $X(s)$ are positive while $\phi(s) \longrightarrow \infty$ as $s \longrightarrow -\infty$. Hence (6) leads to a contradiction. The converse part of Theorem 0 is obvious.

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