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A CONSEQUENCE OF THE AXIOM OF CHOICE

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Let R, C be the additive groups of the real, complex numbers respectively. Using the Axiom of Choice (A.C.), these groups may be shown to be isomorphic. We show that this cannot be proved in Zermelo-Fraenkel set theory (see e.g. Fraenkel, Bar-Hillel and Levy (1973)) without the additional assumption of A.C. This is one of the most "concrete" used of the Axiom of Choice of which I know.

THEOREM 1 (assuming (A.C)). $C \cong R$.

PROOF. Regarding R as a vector space over the field of rationals, R has a basis, X. Since R is uncountable, X is infinite. A bijection thus exists between X and $(X \times \{0\}) \cup (\{0\} \times X)$ which is a basis for $R \oplus R$ regarded again as a vector space. This bijection then induces an isomorphism between R and $R \oplus R$ as vector spaces, and so as groups. But $C \cong R \oplus R$.

The proposition $C \cong R$ may be deduced, by this method, from restricted forms of the Axiom of Choice, such as the existence of a well-ordering of the reals, and so does not imply the full Axiom of Choice. (See Derrick and Drake (1967).) We show, however, that $C \cong R$ *does* imply a result known not to be provable without the Axiom of Choice.

We use the notation $\langle a, b \rangle$ for ordered pairs and (a, b) *etc.* for intervals in R.

THEOREM 2 (without A.C.). *If $C \cong R$ then there is a set of reals which is not Lebesgue measurable* (see e.g. Halmos (1950)).

PROOF. Assume $C \cong R$. Then, since $C \cong R \oplus R$, we have an isomorphism $f: R \oplus R \to R$. Now suppose that each subset of R is Lebesgue measurable. We shall derive a contradiction.

For $x \in R$, let S_x be the image under f of $R \oplus [x, x+1)$. The S_n, for integers n, partition R, and so the $S_n \cap (0,1)$ partition (0,1). We show that the $S_n \cap (0,1)$ all have the same measure, giving a contradiction to countable additivity as required.

First, since $R \oplus [x, x+1)$ may be translated to $R \oplus [0,1)$ in $R \oplus R$, then the same applies to their images. Thus $S_n \cap (0,1)$ translates to $S_0 \cap (y, y+1)$ for some $y \in R$. It is therefore sufficient to show that $S_0 \cap (y, y+1)$ and $S_0 \cap (0,1)$ have the same measure.

Let T be the set of images under f of pairs $\langle r, 0 \rangle$ for rational r. Then we have an enumeration of T. Also T is a subgroup of R isomorphic to the rationals and so can easily be shown to be dense in R. Finally, S_0 is closed under addition of elements of T.

Since T is dense in R, one can use the enumeration of T to express $(y, y+1)$ as the union of the chain of intervals (s_k, t_k) $k = 0, 1, \cdots$ where $s_k, t_k \in T$. $t_k - s_k \to 1$ as $k \to \infty$, so $(0,1)$ is the union of the chain $(0, t_k - s_k)$. But, since S_0 is closed under addition of elements of T, each $S_0 \cap (s_k, t_k)$ translates to $S_0 \cap (0, t_k - s_k)$. Thus, taking measures of unions of countable chains, $S_0 \cap (y, y+1)$ and $S_0 \cap (0,1)$ have the same measure as required.

Corollary. *Theorem* 1 *cannot be proved in ZF* (*Zermelo-Fraenkel set theory*) *with out the additional use of A.C.*

Proof. Solovay (1960) demonstrates the existence of a model for ZF in which every set of real numbers is Lebesgue measurable, thus showing that the conclusion of Theorem 2, and so its premise, cannot be proved in ZF.

For those interested in weaker forms of A.C., is should be observed that in the model of Solovay (1970) the Principle of Dependent Choices (D.C.) holds. Thus $C \cong R$ cannot even be proved in ZF with D.C. (while for many other results in analysis—for example the Baire category theorem—this form of A.C. suffices).

A more familiar version of A.C. is the *Countable* Axiom of Choice (the existence of a choice function for every countable family of non-empty sets) usually designated by AC_ω.

AC_ω is easily shown to be implied by D.C., so the same model shows the Countable Axiom of Choice (while sufficient for many results in measure theory and analysis) is insufficient to prove $C \cong R$.

Remarks. If Q is the additive group of rationals, then $Q \oplus R \cong R$ as in Theorem 1. $Q \oplus R \cong R$ implies, as in Theorem 2, the existence of a non-measurable set of reals, and so its proof again requires A.C. The assertions $Q \oplus R \cong R$ and $R \oplus R \cong R$ do not seem to be related in the absence of A.C.

Similarly, without A.C. one cannot show the existence of any subgroup of R with countably index. I do not know, however, whether the existence of an uncountable $S < R$ with R/S uncountable requires A.C.

References

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Added in proof

In deriving the corollary to theorem 2, I have unwittingly assumed the existence of an inaccessible cardinal, since the model of [4] is constructed under this assumption. The corollary, and the following remarks are nevertheless demonstrable without this assumption, by appeal to the model of theorem 4.2–4.26 of Sacks (1969), *Measure theoretic uniformity*, Trans. A.M.S. **142**, 381–419, and to the fact that Theorem 2 may be applied to any countably additive translation invariant extension of Lebesgue measure.