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COMPACTNESS OF SUBSETS OF TYCHONOFF SETS VIA EXPONENTIAL LAWS

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Abstract

Using the exponential map in multifunction context, the paper deduces a system of non-Hausdorff theorems which generalize all known Ascoli theorems for the space of continuous functions and the space of point-compact continuous multifunctions.

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1. Introduction

In [18] Noble studied the equivalence between the exponential laws and the Ascoli theorems for the space of continuous functions. This paper uses this approach in the space of point-compact continuous multifunctions. In order to eliminate the Hausdorff restriction, the paper uses the Levine generalization of the closed set [11]. To formulate theorems which may be interpreted either in terms of continuous functions or in terms of point-compact continuous multifunctions, the paper introduces the notion of a Tychonoff set.

All unexplained terminology is defined in [14]. In this paper, the following notations will be used:

(1) X, Y and Z denote topological spaces.

(2) Y^{mX}, $(Y^{mX})_0$, $\mathcal{C}_0(X, Y)$ denote, respectively, the set of all multifunctions on X to Y, the set of all point-compact members of Y^{mX} and the set of all continuous members of $(Y^{mX})_0$.

(3) $P\{Y_x: x \in X\}$ denote the m-product of the topological spaces Y_x, $x \in X$.

(4) τ_p, τ_c denote, respectively, the pointwise topology and the compact open topology.

(5) If $f \in Y^{mX}$ and $B \subseteq Y$, we write $f^{-}(B) = \{x\colon x \in X \text{ and } fx \cap B \neq \varnothing\}$, $f^{+}(B) = \{x\colon x \in X \text{ and } fx \subseteq B\}$.

2. Fundamental implications

An element $f \in Z^{m(X \times Y)}$ determines the function $\tilde{f}\colon x \to f(x, \cdot)$ on X to Z^{mY}. The function $\mu\colon f \to \tilde{f}$ called the *exponential map*, is a bijection of $Z^{m(X\times Y)}$ onto $(Z^{mY})^X$ such that $\mu(\mathcal{C}_0(X \times Y, Z)) \subseteq (\mathcal{C}_0(Y, Z))^X$. If τ is a topology on $\mathcal{C}_0(Y, Z)$ such that

$$\mu(\mathcal{C}_0(X \times Y, Z)) \subseteq C(X, (\mathcal{C}_0(Y, Z), \tau))$$

we say that (X, Y, Z, τ) satisfies the *partial exponential law*. If τ is a topology on $\mathcal{C}_0(Y, Z)$ such that

$$\mu^{-1}(C(X, (\mathcal{C}_0(Y, Z), \tau))) \subseteq \mathcal{C}_0(X \times Y, Z)$$

we say that (X, Y, Z, τ) satisfies the *inverse partial exponential law*.

Let $F \subseteq Z^{mY}$. We say that F is *pointwise bounded* if $\overline{F[y]}$ is compact for every $y \in Y$. A subset T of Z^{mY} will be called a *Tychonoff set* if, for every pointwise bounded subset F of T, $P\{\overline{F[y]}\colon y \in Y\} \cap T$ is τ_p-compact. For example, Z^Y is a Tychonoff set, by the classical Tychonoff theorem; also $(Z^{mY})_0$ is a Tychonoff set, by Corollary 7.6 of [14, page 17].

Following Levine [11, page 90], a subset A of a topological space will be called *g-closed* if $\bar{A} \subseteq U$ whenever U is an open set containing A.

If τ is a topology on $\mathcal{C}_0(Y, Z)$ we say that (Y, Z, τ) satisfies the *Ascoli theorem* if, for every Tychonoff subset T of $(Z^{mY})_0$, a subset F of $\mathcal{C}_0(Y, Z) \cap T$ is τ-compact, provided that

(i) F is g-closed,

(ii) F is pointwise bounded, and

(iii) τ_p is jointly continuous on the τ_p-closure of F in T.

If τ is a topology on $\mathcal{C}_0(Y, Z)$ we say that (Y, Z, τ) satisfies the *converse Ascoli theorem* if, for every Tychonoff subset T of $(Z^{mY})_0$, every τ-compact subset F of $\mathcal{C}_0(Y, Z) \cap T$ satisfies the conditions (i), (ii) and (iii).

2.1. THEOREM. *If (X, Y, Z, τ) satisfies the partial exponential law for every compact space X, then (Y, Z, τ) satisfies the Ascoli theorem.*

PROOF. Let T be a Tychonoff subset of $(Z^{mY})_0$ and let F be a subset of $\mathcal{C}_0(Y, Z) \cap T$ satisfying the conditions (i), (ii) and (iii). Let $\bar{F}$ be the τ_p-closure of F in T. By (iii), $\omega\colon (\bar{F}, \tau_p) \times Y \to Z$ is continuous, and, in particular, $\bar{F} \subseteq \mathcal{C}_0(Y, Z)$.

Since T is a Tychonoff set and F is pointwise bounded, by an obvious modification of the proof of Lemma 7.8 of [14, page 18], we can conclude that $\bar{F}$ is τ_p-compact. Then, by the hypothesis, $\tilde{\omega}$: $\bar{F} \to (\mathcal{C}_0(Y, Z), \tau)$ is continuous. Since $\tilde{\omega}$ is the inclusion map, $\bar{F} = \tilde{\omega}(\bar{F})$ is τ-compact. But $F \subseteq \bar{F} \subseteq \mathcal{C}_0(Y, Z) \cap T$. Then (i) implies, by Theorem 2.9 of [11], that F is g-closed in $(\bar{F}, \tau)$. Since $(\bar{F}, \tau)$ is compact, Theorem 3.1 of [11] implies that F is τ-compact.

2.2. THEOREM. *Let Z be a regular space and let τ be a regular topology on $\mathcal{C}_0(Y, Z)$. If (X, Y, Z, τ) satisfies the inverse partial exponential law for every compact space X, then (Y, Z, τ) satisfies the converse Ascoli theorem.*

PROOF. Let F be a τ-compact subset of $\mathcal{C}_0(Y, Z) \cap T$, where T is a Tychonoff subset of $(Z^{mY})_0$. Then F is pointwise bounded because, if $y \in Y$, $F[y] = \mathrm{pr}_y(F)$ is compact [3, page 116]. Since $(\mathcal{C}_0(Y, Z) \cap T, \tau)$ is regular, Theorem 3.5 of [11] implies that F is g-closed in $\mathcal{C}_0(Y, Z) \cap T$.

In order to prove the condition (iii), it suffices, by Theorem 10.1 of [14, page 21], to show that F satisfies the condition (G). Let F_0 be a τ_c-closed subset of F and let V be an open subset of Z. It must be shown that $\bigcap_{f \in F_0} f^-(V)$ and $\bigcap_{f \in F_0} f^+(V)$ are open in Y. Since the inclusion map i: $F \to \mathcal{C}_0(Y, Z)$ is continuous, by the hypothesis, $\mu^{-1}(i) = \omega$: $F \times Y \to Z$ is continuous. This implies, in particular, that τ is finer than τ_c [21, page 49]. Therefore F_0 is τ-closed in F, hence τ-compact.

Let $y_0 \in \bigcap_{f \in F_0} f^-(V)$ and let ω: $F_0 \times Y \to Z$. Then $F_0 \times \{y_0\} \subseteq \omega^-(V)$. Since F_0 is compact and $\omega^-(V)$ is open, by the theorem of Wallace, there exists a neighbourhood U of y_0 such that $F_0 \times U \subseteq \omega^-(V)$. Let $y \in U$. For $f \in F_0$, $fy \cap V \neq \varnothing$, that is, $y \in \bigcap_{f \in F_0} f^-(V)$. We have, therefore, $U \subseteq \bigcap_{f \in F_0} f^-(V)$, showing that $\bigcap_{f \in F_0} f^-(V)$ is open. Similarly, we show that $\bigcap_{f \in F_0} f^+(V)$ is open.

3. Ascoli theorems

Let $F \subseteq Z^{mY}$. Following [15], we say that F is *evenly continuous* if, whenever $y \in Y$, K is a compact subset of Z and V is a neighbourhood of K, there exist neighbourhoods U, W of y, K, respectively, such that, for all $f \in F$, $fx \cap W \neq \varnothing$ implies $U \subseteq f^-(V)$ and $fx \subseteq W$ implies $U \subseteq f^+(V)$.

Let $Z = (Z, \mathcal{U})$ be a uniform space and let $F \subseteq Z^{mY}$. Following [22], we say that F is *equicontinuous* if, for $y \in Y$ and $U \in \mathcal{U}$, there exists a neighbourhood V of y such that, for all $f \in F$, $f(V) \subseteq U[fy]$ and $fx \cap U[z] \neq \varnothing$ whenever $(x, z) \in V \times fy$.

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Since (X, Y, Z, τ_c) satisfies the partial exponential law [14, page 18], we deduce from Theorem 2.1 the following consequence:

3.1. THEOREM. *(Y, Z, τ_c) satisfies the Ascoli theorem.*

3.2. COROLLARY. *Let Z be a regular space and let T be a Tychonoff subset of $(Z^{mY})_0$. A subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact if*
(a) *F is g-closed,*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous.*

PROOF. By Lemma 4.1 of [15], (c) implies the condition (iii).

3.3. COROLLARY. *Let Z be a regular space and let T be a Tychonoff subset of $(Z^{mY})_0$. A subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact if*
(a) *F is g-closed,*
(b) *F is pointwise bounded, and*
(c) *F satisfies the condition* (G).

PROOF. By Theorem 10.1 of [14, page 21], (c) implies the condition (iii).

3.4. COROLLARY. *Let Z be a uniform space and let T be a Tychonoff subset of $(Z^{mY})_0$. A subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact if*
(a) *F is a g-closed,*
(b) *F is pointwise bounded, and*
(c) *F is equicontinuous.*

PROOF. By Lemmas 6 and 7 of Smithson [21, pages 257–258], (c) implies (iii).

Let $Z = (Z, \mathcal{U})$ be a uniform space and let cZ denote the set of all non-empty compact subsets of Z. Following Michael [13, page 153], we define on cZ the uniformity $c\mathcal{U}$ having as base the sets of the form $\{(K_1, K_2): K_2 \subseteq \bigcup_{x \in K_1} U[x]$ and $K_2 \cap U[y] \neq \varnothing$ for all $y \in K_1\}$, where $U \in \mathcal{U}$. Since, in the proof of Theorem 3.3 of [13, page 160], the restriction that the members of cZ be closed in Z is not used, we may conclude that $c\mathcal{U}$ induces the Vietoris or finite topology on cZ. Accordingly we can identify $f \in \mathcal{C}_0(Y, Z)$ with the function $f^*: y \to fy \in cZ$ to obtain the equation $\mathcal{C}_0(Y, Z) = C(Y, cZ)$. This identification being understood, we write f for f^*. It turns out that the topology of uniform convergence τ_u on $\mathcal{C}_0(Y, Z)$, as defined by Smithson [22, page 253], is the same as the topology of uniform convergence, in the classical sense, on $C(Y, cZ)$, using the uniformity

$c\mathcal{U}$. Consequently, a base for the neighbourhood filter of an element $f \in (\mathcal{C}_0(Y, Z), \tau_u)$ consists of all sets of the form $N_{d,\varepsilon}(f) = \{g: d(f(y), g(y)) < \varepsilon$ for all $y \in Y\}$, where $\varepsilon > 0$ and d is a continuous pseudometric on $(cZ, c\mathcal{U})$.

A subset of a topological space X is called a *zero-set* if it is the inverse image of 0 under some continuous real-valued function on X [8, page 53]. We note that if f is a real-valued continuous function on X, then $\{x \in X: f(x) \geqslant 0\}$ is a zero-set. Let X, Y be topological spaces and let $\Pi_X: X \times Y \to X$ be the first projection. Then Π_X is said to be *z-closed* [7] if it maps every zero-set of $X \times Y$ onto a closed subset of X.

3.5. THEOREM. *If Z is a uniform space and $\Pi_X: X \times Y \to X$ is z-closed, then (X, Y, Z, τ_u) satisfies the partial exponential law.*

PROOF. Let $f \in \mathcal{C}_0(X \times Y, Z)$, let $x_0 \in X$ and let $N_{d,\varepsilon}(\tilde{f}(x_0)) = \{g \in \mathcal{C}_0(Y, Z): d(\tilde{f}(x_0)(y), g(y)) < \varepsilon$ for all $y \in Y\}$, where $\varepsilon > 0$ and d is a continuous pseudometric on $(cZ, c\mathcal{U})$. It must be shown that $\tilde{f}^{-1}(N_{d,\varepsilon}(\tilde{f}(x_0)))$ is a neighbourhood of x_0. Let $\Gamma_f: X \times Y \to Y \times cZ$ be defined by the formula $\Gamma_f(x, y) = (y, f(x, y))$. Since $\Pi_{cZ} \circ \Gamma_f = f$ and $\Pi_Y \circ \Gamma_f$ is the second projection on $X \times Y$, Γ_f is continuous. Let $S = \{(y, K) \in Y \times Z: d(f(x_0, y), K) \geqslant \varepsilon\}$. Then $\Gamma_f^{-1}(S) = \{(x, y) \in X \times Y: d(f(x_0, y), f(x, y)) \geqslant \varepsilon\}$, so $\Pi_X(\Gamma_f^{-1}(S)) = \{x \in X: d(f(x_0, y), f(x, y)) \geqslant \varepsilon$ for some $y \in Y\}$ and therefore $X - \Pi_X(\Gamma_f^{-1}(S)) = \{x \in X: d(f(x_0, y), f(x, y)) < \varepsilon$ for all $y \in Y\} = \tilde{f}^{-1}(N_{d,\varepsilon}(\tilde{f}(x_0)))$. Since $\Gamma_f^{-1}(S) = \{(x, y) \in X \times Y: (d \circ (\tilde{f}(x_0) \times 1_{cZ}) \circ \Gamma_f)(x, y) \geqslant \varepsilon\}$, $\Gamma_f^{-1}(S)$ is a zero-set. Then $\Pi_X(\Gamma_f^{-1}(S))$ is a closed set not containing x_0, so $\tilde{f}^{-1}(N_{d,\varepsilon}(\tilde{f}(x_0)))$ is an open set containing x_0.

By definition, a topological space X is *pseudocompact* if every real-valued continuous function on X is bounded [8, page 67]. It is known that the product of a pseudocompact space by a compact space is pseudocompact [16, page 20].

3.6. THEOREM. *If Z is a uniform space and Y is a pseudocompact space, then (Y, Z, τ_u) satisfies the Ascoli theorem.*

PROOF. This follows from Theorems 2.1, 3.5 and Theorem 2.5 of [18, page 397].

3.7. COROLLARY. *Let Y be a pseudocompact space, let Z be a uniform space and let T be a Tychonoff subset of $(Z^{mY})_0$. A subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_u)$ is compact if*

(a) *F is g-closed,*
(b) *F is pointwise bounded, and*
(c) *F is equicontinuous.*

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PROOF. Same as proof of Corollary 3.4.

4. Converse Ascoli theorems

A topological space X is a k_3-*space* if $C_k(X, Y) = C(X, Y)$ for every regular space Y [19, page 195]. Thus a k-space is a k_3-space but not conversely. In fact, the product of uncountably many copies of the real line, which is not a k-space, is a k_3-space [19, Theorem 5.6 (i)].

If X is compact, Y is a k_3-space and Z is regular, then (X, Y, Z, τ_c) satisfies the inverse partial exponential law [15, Theorem 3.4]. From this fact, Theorem 2.2 and Theorem 2 of Smithson [21, page 48], we deduce the following consequence:

4.1. THEOREM. *If Y is a k_3-space and Z is regular, then (Y, Z, τ_c) satisfies the converse Ascoli theorem.*

4.2. COROLLARY. *Let Y be a k-space, let Z be a regular space and let T be a Tychonoff subset of $(Z^{mY})_0$. If a subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact, then*

(a) *F is g-closed,*

(b) *F is pointwise bounded, and*

(c) *F is evenly continuous.*

PROOF. By (iii), ω: $(F, \tau_p) \times Y \to Z$ is continuous, therefore $F = \{\omega(f, \cdot)$: $f \in F\}$ is evenly continuous [15, Lemma 4.2].

4.3. COROLLARY. *Let Y be a k_3-space, let Z be a regular space and let T be a Tychonoff subset of $(Z^{mY})_0$. If a subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact, then*

(a) *F is g-closed,*

(b) *F is pointwise bounded, and*

(c) *F satisfies the condition* (G).

PROOF. By (iii), τ_p on F is jointly continuous. Then, by Corollary 10.6 of [14, page 23], F satisfies (G).

4.4. COROLLARY. *Let Y be a k_3-space, let Z be a uniform space, and let T be a Tychonoff subset of $(Z^{mY})_0$. If a subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_c)$ is compact, then*

(a) *F is g-closed,*

(b) *F is pointwise bounded, and*

(c) *F is equicontinuous.*

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PROOF. By (iii), τ_p on F is jointly continuous. Then, by Lemma 8 of Smithson [22, page 258], F is equicontinuous.

4.5. THEOREM. *If Z is a uniform space, then (X, Y, Z, τ_u) satisfies the inverse partial exponential law.*

PROOF. Let $f \in (Z^{m(X\times Y)})_0$ be such that the function $\tilde{f}\colon X \to (\mathcal{C}_0(Y, Z), \tau_u)$ is continuous. Let $(x_0, y_0) \in X \times Y$ and let $\{U_i\}_{1\leq i\leq n}$ be a finite sequence of open subsets of Z such that $f(x_0, y_0) \subseteq \bigcup_{i=1}^{n} U_i$ and $f(x_0, y_0) \cap U_i \neq \varnothing$ for all $i = 1,\ldots,n$. It must be shown that $f^{-1}(\langle U_1,\ldots,U_n\rangle)$ is open in $X \times Y$.

Since $f(x_0, y_0) \in \langle U_1,\ldots,U_n\rangle$, there exists $\varepsilon > 0$ and a continuous pseudometric d on $(cZ, c\mathcal{U})$ such that $\mathcal{B}_{d,\varepsilon}(f(x_0, y_0)) = \{K \in cZ\colon d(K, f(x_0, y_0)) < \varepsilon\} \subseteq \langle U_1,\ldots,U_n\rangle$. Since $\tilde{f}(x_0)$ is continuous, $W = \tilde{f}(x_0)^{-1}(\mathcal{B}_{d,\varepsilon/2}(f(x_0, y_0)))$ is a neighbourhood of y_0. Moreover, since $\tilde{f}$ is continuous, $V = \tilde{f}^{-1}(N_{d,\varepsilon/2}(\tilde{f}(x_0)))$ is a neighbourhood of x_0. Then $V \times W$ is a neighbourhood of (x_0, y_0) contained in $f^{-1}(\langle U_1,\ldots,U_n\rangle)$. In fact, let $(x, y) \in V \times W$. Then $\tilde{f}(x) \in N_{d,\varepsilon/2}(\tilde{f}(x_0))$ and $f(x_0)(y) \in \mathcal{B}_{d,\varepsilon/2}(f(x_0, y_0))$, that is, $d(f(x, t), f(x_0, t)) < \varepsilon/2$ for all $t \in Y$ and $d(f(x_0, y), f(x_0, y_0)) < \varepsilon/2$. So, in particular, $d(f(x, y), f(x_0, y_0)) < \varepsilon$ and therefore $f(x, y) \in \mathcal{B}_{d,\varepsilon}(f(x_0, y_0)) \subseteq \langle U_1,\ldots,U_n\rangle$.

4.6. THEOREM. *If Z is a uniform space, then (Y, Z, τ_u) satisfies the converse Ascoli theorem.*

PROOF. This follows from Theorems 2.2 and 4.5.

4.7. COROLLARY. *Let Z be a uniform space and let T be a Tychonoff subset of $(Z^{mY})_0$. If a subset F of $(\mathcal{C}_0(Y, Z) \cap T, \tau_u)$ is compact, then*

(a) *F is g-closed,*

(b) *F is pointwise bounded, and*

(c) *F is equicontinuous.*

PROOF. Same as proof of Corollary 4.4.

4.8. REMARK. Referring to the equivalence relation on a regular space introduced in [14, page 11], we note that if F^* is closed then $F \subseteq \overline{F} \subseteq F^*$ and therefore F is g-closed; moreover, if F is compact then F^* is closed [14, Theorem 4.1].

4.9. REMARK. Corollary 3.2 together with Corollary 4.2 is the Theorem 5.1 of [15], which, in the case $T = Z^Y$, contains the Ascoli theorem 4.1 of [4, page 635]. This latter generalizes the Ascoli theorems of Kelley-Morse [10, page 236], Bagley-Yang [2, page 704], Noble [18, Corollary 4.4] and Kaul [9, Theorem B].

4.10. REMARK. If we take $T = (Z^{mY})_0$ in Corollaries 3.3 and 4.3, we obtain a k_3-space generalization of Theorem 10.10 of [14, pages 23–24], which in turn contains the function Ascoli theorem of Gale [5, page 304] and the multifunction Ascoli theorem of Mancuso [12, page 470].

4.11. REMARK. If we take $T = (Z^{mY})_0$ in Corollaries 3.4 and 4.4 we obtain, because of Theorem 12.2 of [14, page 28], a k_3-space generalization of Theorem 12.8 of [14, page 31], which in turn contains the function Ascoli theorems of Arens [1, page 491], Myers [17, pages 497–498] and Bagley-Yang [2, page 705], also the multifunction Ascoli theorem of Smithson [22, page 259].

4.12. REMARK. If we take $T = Z^Y$, Corollary 3.7 together with Corollary 4.7 generalizes the Ascoli theorem of Noble [18, Corollary 4.3 (i)], which in turn generalizes the Ascoli theorem of Glicksberg [6, page 257].

4.13. REMARK. There has recently appeared another definition of even continuity in the space $(Z^{mY})_0$ [20, page 14]. Using a suitable modification of the arguments used in the proofs, it can be shown that this "even continuity" has the properties stated in Lemma 4.1, and, with the additional point-compact condition, has the property stated in Lemma 4.2 of [15]. These properties established, we can deduce, with greater generality, the Ascoli theorem 3.1 of [20, page 150] from Theorems 3.1 and 4.1.

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