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Selberg Integrals and Multiple Zeta Values

Dedicated to Tetsuji Shioda on his sixtieth birthday

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Abstract. For a suitable choice of f, the Selberg integral

$$\int f \prod_{3\leqslant i<j\leqslant n} (x_j - x_i)^{\alpha_{ij}} \prod_{i=3}^{n} x_i^{\alpha_{1i}} \prod_{i=3}^{n} (1-x_i)^{\alpha_{2i}} \mathrm{d}x_3 \cdots \mathrm{d}x_n ,$$

is a homolphic function on α_{ij}. In this paper, we show that the coefficients of the Taylor expansions of the Selberg integrals with respect to the variables α_{ij} can be expressed as linear combinations of multiple zeta values over $\mathbf{Q}$.

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1. Introduction

In this paper, we show that the coefficients of the Taylor expansions of Selberg integrals with respect to their exponent variables can be expressed as linear combinations of multiple zeta values over $\mathbf{Q}$. The Selberg integrals are the period integrals of an Abelian covering of the moduli space of n-points in $\mathbf{P}^1$. Let $2 \leqslant r \leqslant n$ be an integer and α_{ij} be positive real numbers. Let f be an element in $\mathbf{C}[\frac{1}{x_i - x_j}]$, and $x_1, \ldots, x_r$ elements of $\mathbf{R}$ such that $x_1 < x_r < x_{r-2} < \cdots < x_2$. The integral

$$\int_{D'} f \prod_{i<j} (x_j - x_i)^{\alpha_{ij}} \mathrm{d}x_{r+1} \cdots \mathrm{d}x_n , \tag{1.1}$$

where

$$D' = \{(x_{r+1}, \ldots, x_n) \mid x_1 < x_n < x_{n-1} < \cdots < x_r\}$$

is called a Selberg integral. It is considered as a period integral for an Abelian covering of the moduli space of $(n-r)$-distinct points in $\mathbf{C} - \{x_1, \ldots, x_r\}$. It is a function of $x_1, \ldots, x_r$ and the exponent parameters α_{ij}. If $r = 2$, the Selberg integrals are determined by their restriction to $x_1 = 0$ and $x_2 = 1$. In this case, the Selberg integrals are essentially functions of the exponent parameters α_{ij}. It is natural to ask about the arithmetic nature of these Selberg integrals.

We recall the definition of the multiple zeta values introduced by Euler. Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a sequence of integers such that $k_i \geqslant 1$ $(i = 1, \ldots, m-1)$ and $k_m \geqslant 2$. The multiple zeta value of index $\mathbf{k}$ is defined by

$$\zeta(\mathbf{k}) = \sum_{n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}.$$

The natural number $|\mathbf{k}| = \sum_{p=1}^{m} k_p$ is called the weight of the index $\mathbf{k}$ and the multiple zeta value $\zeta(\mathbf{k})$. By using the iterated integral expressions, multiple zeta values are regarded as the period integrals for the fundamental group $\pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$ of $\mathbf{P}^1 - \{0, 1, \infty\}$. (See Section 2.1 for the iterated integral expressions of multiple zeta values.) Notice that the motivic weight of $\zeta(\mathbf{k})$ is equal to $-2\,|\mathbf{k}|$.

For a sequence of integers $i_3, \ldots, i_n$ such that $1 \leqslant i_k \leqslant k-1$, we define a formal sum of graphs

$$\gamma = \sum_{\Gamma} a_\Gamma \Gamma = \emptyset(R) \wedge (3, i_3) \wedge \cdots \wedge (n, i_n)$$

(Section 3.1 (3.2)) and a Selberg integral $S_\gamma = \sum_\Gamma a_\Gamma S_\Gamma$. (For the definition of S_Γ, see Section 3.1 (3.1).) The integral S_γ can be expressed as the restriction to $x_1 = 0$ and $x_2 = 1$ of (1.1) with $r = 2$ and a suitable choice of $f \in \mathbf{Z}[1/(x_i - x_j), \alpha_{ij}]$. The main theorem of this paper is

THEOREM 1.1. *Let S_γ be the Selberg integral as above.*

(1) *The integral S_γ is a holomorphic function on the variable α_{ij} at the origin.*
(2) *The coefficients of the degree w monomials of α_{ij} in the Taylor expansion of the Selberg integral S_γ is a linear combinations of weight w multiple zeta values.*

Let us illustrate a primitive example for this statement. By the well-known equality

$$\log \Gamma(1-x) = \gamma x + \sum_{n \geqslant 2} \frac{\zeta(n) x^n}{n},$$

we have

$$\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} = \exp\left(\sum_{n \geqslant 2} \frac{\zeta(n)((-\alpha)^n + (-\beta)^n - (-\alpha-\beta)^n)}{n}\right).$$

In this example, we choose $r = 2$, $n = 3$ and $f = \alpha/(1-x)$. (See [2] and [6] for another expression of this quantity.) We can find a prototype of this theorem in [2]. This choice of f is equal to a β-nbc base after Falk and Terao [3].

Let us summerize the method of the proof of the main theorem. Let $\mathbf{C}\langle\langle X, Y \rangle\rangle$ be the formal noncommutative free algebra generated by X and Y, and $\mathbf{Q}[[\alpha_i]]$ and $\mathbf{C}[[\alpha_i]]$ be the formal power series rings with (commutative) variables α_i $(i \in I)$ over $\mathbf{Q}$ and $\mathbf{C}$, respectively. We construct a representation $\rho : \mathbf{C}\langle\langle X, Y \rangle\rangle \to M(r, \mathbf{C}[[\alpha_i]])$, where all the matrix elements of $\rho(X)$ and $\rho(Y)$ are homogeneous polynomial with

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rational coefficients of degree 1 in α_{ij}. To construct the representation ρ, we consider the higher direct image of a local system for the projection $X_n \to X_{n-1}$, where X_n is the moduli space of n-distinct points in $\mathbf{C}$. We compute the differential equation of the higher direct image in Section 3 and obtain an explicit description of ρ by a combinatorial method in Section 4.

For any solution s of the differential equation

$$\mathrm{d}s = \left(\frac{\rho(X)}{x} + \frac{\rho(Y)}{x-1}\right) s\,\mathrm{d}x,$$

we have

$$\lim_{x\to 1}((1-x)^{-\rho(Y)}s(x)) = \rho(\Phi(X,Y)) \lim_{x\to 0}(x^{-\rho(X)}s(x)),$$

where $\Phi(X, Y)$ is the associator in $\mathbf{C}\langle\langle X, Y\rangle\rangle$ defined by Drinfeld. (For the definition of $\Phi(X, Y)$, see Section 2.) It is known that the coefficients of $\Phi(X, Y)$ are expressed as $\mathbf{Q}$-linear combinations of the multiple zeta values by Le-Murakami [5], and as a consequence, the coefficients of the Taylor expansions of all the matrix elements of $\rho(\Phi(X, Y))$ are also $\mathbf{Q}$-linear combinations of multiple zeta values.

We construct a horizontal section s with the following properties.

(1) All the elements of $\lim_{x\to 0}(x^{-\rho(X)}s(x))$ can be expressed as $(n-3)$-dimensional Selberg integrals by taking a limit for some of $\alpha_i \to 0$.
(2) All the elements of $\lim_{x\to 1}((1-x)^{-\rho(Y)}s(x))$ can be expressed as $(n-2)$-dimensional Selberg integrals by taking the same limit $\alpha_i \to 0$.

The argument on the limit for $x \to 0$, $x \to 1$ and $\alpha_i \to 0$ in Section 5 gives the proof of the main theorem.

2. Preliminary

2.1. THE DRINFELD ASSOCIATOR

In this section, we recall known facts about the Drinfeld associator. Let $R = \mathbf{C}\langle\langle X, Y\rangle\rangle$ be the completion of the noncommutative polynomial ring in symbols X and Y with respect to its total degree on X and Y. Let $V = R$. Then X and Y act on V as left multiplication and under this action, X and Y are regarded as elements of $\mathrm{End}_{\mathbf{C}}(V)$. Now we consider the differential form ω on $\mathbf{C} - \{0, 1\}$ with coefficients in $\mathrm{End}_{\mathbf{C}}(V)$ defined by

$$\omega = \frac{X}{x}\,\mathrm{d}x + \frac{Y}{x-1}\,\mathrm{d}x,$$

where x is a coordinate on $\mathbf{C}$. Let $E(x) = \exp(\int_{x_0}^{x} \omega)$ be the solution of the differential equation for $\mathrm{End}_{\mathbf{C}}(V)$-valued function $\mathrm{d}E(x) = \omega E(x)$ with the initial condition

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$E(x_0) = 1$. Then by the standard argument for iterated integrals, $\exp(\int_{x_0}^{x} \omega)$ is expressed as

$$\exp\left(\int_{x_0}^{x} \omega\right) = 1 + \int_{x_0}^{x} \omega + \int_{x_0}^{x} \omega\omega + \cdots. \tag{2.1}$$

Here we used the convention for iterated integrals defined by the inductive relation

$$\int_p^q \omega_1 \cdots \omega_n = \int_p^q \left(\omega_1(q_1) \int_p^{q_1} \omega_2 \cdots \omega_n\right).$$

The expression (2.1) implies $\exp(\int_{x_0}^{x} \omega) \in \mathbf{C}\langle\langle X, Y\rangle\rangle^{\times}$, and the shuffle relation for iterated integrals implies that $E = \exp(\int_{x_0}^{x} \omega)$ is a group-like element, i.e.

$$\Delta(E) = E \otimes E \text{ in } \mathbf{C}\langle\langle X, Y\rangle\rangle \hat{\otimes} \mathbf{C}\langle\langle X, Y\rangle\rangle,$$

where the comultiplication

$$\Delta : \mathbf{C}\langle\langle X, Y\rangle\rangle \to \mathbf{C}\langle\langle X, Y\rangle\rangle \hat{\otimes} \mathbf{C}\langle\langle X, Y\rangle\rangle$$

is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad \text{and} \quad \Delta(Y) = Y \otimes 1 + 1 \otimes Y.$$

The set

$$\hat{G} = \{\Delta(g) = g \otimes g \mid g \in \mathbf{C}\langle\langle X, Y\rangle\rangle^{\times}\}$$

is called the set of group-like elements and closed under the multiplication. By the theory of differential equation only with regular singularities, the limit

$$\lim_{x\to 1} \exp\left(\int_x^0 \frac{Y}{x-1}\,\mathrm{d}x\right) \exp\left(\int_{x_0}^{x} \omega\right)$$

exists. In the same way, the limit

$$\Phi(X, Y) = \lim_{x\to 1, y\to 0} \exp\left(\int_x^0 \frac{Y}{x-1}\,\mathrm{d}x\right) \exp\left(\int_y^x \omega\right) \exp\left(\int_1^y \frac{X}{x}\,\mathrm{d}x\right)$$

exists and belongs to $\mathbf{C}\langle\langle X, Y\rangle\rangle^{\times}$. The element $\Phi(X, Y)$ in $\mathbf{C}\langle\langle X, Y\rangle\rangle$ is called the Drinfeld associator. Since

$$\exp\left(\int_x^0 \frac{Y}{x-1}\mathrm{d}x\right) \quad \text{and} \quad \exp\left(\int_1^y \frac{X}{x}\,\mathrm{d}x\right)$$

are elements in $\hat{G}$, and $\hat{G}$ is a closed subset of $\mathbf{C}\langle\langle X, Y\rangle\rangle^{\times}$, the limit $\Phi(X, Y)$ is an element in $\hat{G}$.

We recall relations between the multiple zeta values and the coefficients of the Drinfeld associator. Let $k_1, \ldots, k_n$ be integers such that $k_i \geqslant 1$ for $i = 1, \ldots, n$ and $k_n \geqslant 2$. Set $\mathbf{k} = (k_1, \ldots, k_n)$. The series

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 < m_2 < \cdots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

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is called the multiple zeta value of index $\mathbf{k} = (k_1, \dots, k_n)$. The number $|\mathbf{k}| = \sum_{i=1}^{n} k_i$ is called the weight of the index $\mathbf{k}$. Let L_w be the finite-dimensional $\mathbf{Q}$ vector subspace of $\mathbf{C}$ generated by $\zeta(\mathbf{k})$, with $|\mathbf{k}| = w$. The following iterated integral expression of the multiple zeta value is fundamental.

$$\zeta(k_1, \dots, k_n) = \int_0^1 \underbrace{\frac{\mathrm{d}x}{x} \cdots \frac{\mathrm{d}x}{x}}_{k_n - 1} \frac{\mathrm{d}x}{1-x} \underbrace{\frac{\mathrm{d}x}{x} \cdots \frac{\mathrm{d}x}{x}}_{k_{n-1} - 1} \frac{\mathrm{d}x}{1-x} \cdots \underbrace{\frac{\mathrm{d}x}{x} \cdots \frac{\mathrm{d}x}{x}}_{k_1 - 1} \frac{\mathrm{d}x}{1-x}.$$

By using this expression and the shuffle relation, we have $L_{w_1} \cdot L_{w_2} \subset L_{w_1 + w_2}$. Using this fact, we define the homogeneous multiple zeta value ring (homogeneous MZV ring for short) H in $\mathbf{C}\langle\langle X, Y \rangle\rangle$ by

$$H = \hat{\bigoplus}_{w \geqslant 0} \bigoplus_{W:\text{word of length } w \text{ on } X, Y} L_w \cdot W.$$

The following proposition is due to Le and Murakami [5].

PROPOSITION 2.1. $\Phi(X, Y) \in H$.

It is very useful to specialize this universal result to a special class of representations of $\mathbf{C}\langle\langle X, Y \rangle\rangle$. Let R be a homogeneous complete ring generated by degree 1 elements $\alpha_1, \dots, \alpha_m$ over $\mathbf{Q}$, i.e. R is topologically generated by degree 1 homogeneous elements $\alpha_1, \dots, \alpha_m$ with homogeneous relations and is complete under the topology defined by the total degree. The formal decomposition of R with respect to its degree is denoted by $R = \hat{\oplus}_{d \geqslant 0} R_d$. Let $R_{\mathbf{C}}$ be the completion of $R \otimes \mathbf{C}$ with respect to the topology defined by the total degree. A ring homomorphism $\rho: \mathbf{C}\langle\langle X, Y \rangle\rangle \to M(r, R_{\mathbf{C}})$ is called a homogeneous rational representation of degree 1 if and only if all the matrix elements of $\rho(X)$ and $\rho(Y)$ are homogeneous elements of degree 1 in R. The homogeneous MZV ring H_R for R is defined by $H_R = \hat{\oplus}_{d \geqslant 0}(R_d \otimes L_d)$. The following corollary is a direct consequence of Proposition 2.1.

COROLLARY 2.2. *Let* $\rho: \mathbf{C}\langle\langle X, Y \rangle\rangle \to M(r, R_{\mathbf{C}})$ *be a homogeneous rational representation of degree* 1. *Then all the matrix elements of* $\rho(\Phi(X, Y))$ *are elements of* H_R.

3. Selberg Integrals

3.1. COMBINATORIAL ASPECTS

Let $[n] = \{1, \dots, n\}$. A graph Γ consists of sets of vertices V_Γ and edges E_Γ. We assume that every edge has two distinct terminals. Moreover, we assume that for any two vertices p and q, there exists at most one edge whose terminals are p and q. An edge is written as (p, q), where p and q are its terminals. For a graph Γ, we can associate a one-dimensional simplicial complex by the usual manner and we use the standard terminologies, for example, connected component, tree, and so on. Moreover, if the order of E_Γ is specified, it is called an ordered graph. A specified

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vertex in a connected component is called the root of the component. If roots are specified for all the components, the graph is called a rooted graph. The set of the roots of a rooted graph Γ is denoted by $R = R_\Gamma$. For two sets V, R such that $R \subset V$, we define $\Omega^i(V \bmod R)$ by $\wedge^i(\Omega^1_{X_V}/p^*\Omega^1_{X_R})$, where $X_V = \{(x_i)_{i\in V} \mid x_i \neq x_j \text{ for } i \neq j\}$, $X_R = \{(x_i)_{i\in R} \mid x_i \neq x_j \text{ for } i \neq j\}$, and p is the natural projection $X_V \to X_R$. Then it is easy to see that $\Omega^{\#V-\#R}(V \bmod R)$ is an $\mathcal{O}_{X_n}$ module of rank 1 generated by $\wedge_{i\in V-R}\mathrm{d}x_i$. For an edge $e = (p, q), p, q \in V_\Gamma$, we define

$$\omega_e = \mathrm{d}\log(x_p - x_q) \in \Omega^1(V \bmod R).$$

For an ordered tree, we define ω_Γ as

$$\omega_\Gamma = \omega_{e_r} \wedge \cdots \wedge \omega_{e_1} \text{ in } \Omega(V \bmod R),$$

where $E_\Gamma = \{e_1, \ldots, e_r\}$ and $e_1 < \cdots < e_r$. It is easy to see the following lemma.

LEMMA 3.1. *Assume $\#E = \#V - \#R$. Then Γ is a tree if and only if $\omega_\Gamma \neq 0$*

Let R be a subset of $[n]$ such that $\{1, 2\} \subset R$. We define an order $\ll$ on $[n]$ by $1 \ll n \ll \cdots \ll 3 \ll 2$. We define $D(R)$ by $\{(x_1, \ldots, x_i)_{i\in R} \mid x_i < x_j \text{ for } i \ll j\}$. For two subsets V and R of $[n]$ such that $R \subset V$, the fiber of the map $D(V) \to D(R)$ at $(x_i)_{i\in R} \in D(R)$ is denoted by $D(V/R, x_i)_{i\in R}$. Let $\alpha_{i,j}$ $(i, j \in V)$ be positive real numbers. We choose a branch of $\Phi(V) = \prod_{i\ll j}(x_j - x_i)^{\alpha_{i,j}}$ on $D(V)$ with $\Phi \in \mathbf{R}_+$. For an ordered rooted graph Γ whose root set is R, we define a function $S_\Gamma = S_\Gamma(V/R, x_i)_{i\in R}$ on $D(R)$ by

$$S_\Gamma(V/R, x_i)_{i\in R} = \int_{D(V/R,x_i)_{i\in R}} \Phi(V) \prod_{(i,j)\in E_\Gamma} \alpha_{i,j}\omega_\Gamma. \tag{3.1}$$

If R is fixed, it is denoted by S_Γ. Then S_Γ is a function of $(x_i)_{i\in R}$ and $\alpha_{i,j}$. The free abelian group generated by the ordered rooted graphs whose root set and the vertex set are R and V, is denoted by $\Gamma(V, R)$. For an element $\gamma = \sum a_\Gamma \Gamma$ in $\Gamma(V, R)$, we define S_γ by $S_\gamma = \sum a_\Gamma S_\Gamma$. The function S_γ is called the Selberg integral for γ.

Before the statement of the main theorem, we introduce several combinatorial notions. For two natural numbers n, r such that $2 \leqslant r \leqslant n$, we set $R = [r]$ and $V = [n]$. For an ordered rooted graph Γ, whose vertex set and root set are $[n]$ and $[r]$, we define an element $\Gamma \wedge (n+1, i)$ in $\Gamma([n+1], [r])$ for $i \in [n]$ by the following recipe.

(1) Choose a subset A of edges whose elements re adjacent to i. (A may be an empty set.)
(2) Replace the number i by $n+1$ for all the edges contained in A chosen in (1).
(3) Make a graph Γ_A by adding the edge $(n+1, i)$ to the graph made in (2) and extend the original ordering to the ordering of Γ_A such that $(n+1, i)$ is the biggest edge.
(4) Consider the sum $\sum_A \Gamma_A$ of Γ_A, where A runs through all the subsets of edges adjacent to i. This summation is denoted by $\Gamma \wedge (n+1, i)$.

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We extend the operation $\wedge(n+1, i_{n+1})$ from $\Gamma([n],[r])$ to $\Gamma([n+1],[r])$ by linearity. For an element $\gamma \in \Gamma([l],[r])$ and $(l+1, i_{l+1}), \ldots, (n, i_n)$, where $i_{l+1} \in [l], \ldots, i_n \in [n-1]$, we define $\gamma \wedge (l+1, i_{l+1}) \wedge \cdots \wedge (n, i_n)$ inductively by

$$\begin{aligned}\gamma \wedge (l+1, i_{l+1}) \wedge \cdots \wedge (n, i_n) \\ = (\gamma \wedge (l+1, i_{l+1}) \wedge \cdots \wedge (n-1, i_{n-1})) \wedge (n, i_n).\end{aligned} \tag{3.2}$$

The graph Γ with $V_\Gamma = R$ and $E_\Gamma = \emptyset$ is denoted by $\emptyset(R)$. A graph is denoted by $e_1 e_2 \cdots e_b$, where the set of edges is $\{e_1 < e_2 < \cdots < e_b\}$.

EXAMPLE 3.2. If $R = \{1, 2\}, i_3 = 2, i_4 = 2$, then

$$\emptyset(R) \wedge (3, 2) \wedge (4, 2) = (2, 3) \wedge (4, 2) = (2, 3)(4, 2) + (4, 3)(4, 2).$$

We state the main theorem. Let H_α be the homogeneous MZV ring for $\mathbf{Q}\langle\langle \alpha_{i,j}, \alpha_{1,k}, \alpha_{2,k}\rangle\rangle_{3 \leqslant i,j,k \leqslant n, i \neq j}$.

THEOREM 3.3 [Main Theorem]. *Let* $R = \{1, 2\}$. *For any* $i_3 \in [2], \ldots, i_n \in [n-1]$, *put* $\gamma = \emptyset(R) \wedge (3, i_3) \wedge \cdots \wedge (n, i_n)$. *Then* $S_\gamma([n]/[2], 0, 1)$ *is a holomorphic function of* $\alpha_{i,j}$, *and is an element of* H_α.

3.2. THE DIFFERENTIAL EQUATION SATISFIED BY THE SELBERG INTEGRAL

First we compute the higher direct image of local system for the morphism π: $X_n \to X_{n-1}$ defined by $(x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1})$, where $X_n = \{(x_1, \ldots, x_n) \mid x_i \neq x_j \text{ for } i \neq j\}$ is the moduli space for n-distinct points in $\mathbf{C}$. Let $A_{i,j} \in M(d, \mathbf{C})$ be matrices for $1 \leqslant i \neq j \leqslant n$ satisfying the following relations:

(1) $A_{i,j} = A_{j,i}$.
(2) $[A_{i,j}, A_{k,l}] = 0$ for all distinct i, j, k, l.
(3) $[A_{i,j} + A_{j,k}, A_{i,k}] = 0$ for all distinct i, j, k.

These relations are called the infinitesimal pure braid relations. The matrix valued 1-form

$$\omega = \sum_{1 \leqslant i<j \leqslant n} A_{i,j} d\log(x_i - x_j)$$

defines an integrable connection ∇ on $\mathcal{O}_{X_n}^d = \{v = {}^t(v_1, \ldots, v_d)\}$ by $\nabla v = \mathrm{d}v - \omega v$. Let v be a horizontal section of the connection ∇ on $D([n])$, i.e. $\mathrm{d}v = \omega v$. For $i \in [n-1]$, and $(x_1, \ldots, x_{n-1}) \in D([n-1])$, we define w_i as

$$w_i = \int_{D([n]/[n-1], x_1, \ldots, x_{n-1})} \frac{A_{n,i}}{x_n - x_i} v \,\mathrm{d}x_n.$$

Then w_i is a function on $(x_1, \ldots, x_{n-1}) \in D([n-1])$. We have the following proposition. (See [1].)

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PROPOSITION 3.4. (1) $w_1 + \cdots + w_{n-1} = 0$.

(2) *Let* $W = {}^t(w_1, \ldots, w_{n-1})$. *Then* W *satisfies the differential equation*

$$\mathrm{d}W = \sum_{1 \leqslant i<j \leqslant n-1} \frac{A'_{i,j}(\mathrm{d}x_i - \mathrm{d}x_j)}{x_i - x_j} W,$$

where

$$A'_{i,j} = \begin{pmatrix} A_{ij} & \overset{i}{\cdots} & \overset{j}{\cdots} & 0 \\ \vdots & A_{ij} + A_{nj} & -A_{ni} & \vdots \\ \vdots & -A_{nj} & A_{ij} + A_{ni} & \vdots \\ 0 & \cdots & \cdots & A_{ij} \end{pmatrix} \begin{matrix} \\ i \\ j \\ \\ \end{matrix}. \tag{3.3}$$

Proof. (1) By the equality

$$\frac{\partial v}{\partial x_n} = \sum_{j=1}^{n-1} \frac{A_{nj}}{x_n - x_j} v,$$

and Stokes' theorem, we have the required equality.

(2) By using the differential equation for v, we have

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left(\frac{A_{nj}}{x_n - x_j} v\right) \\ &\quad = \frac{A_{nj}}{x_n - x_j}\left(\sum_{k\neq i,j,n} \frac{A_{ik}v}{x_i - x_k} + \frac{A_{ij}v}{x_i - x_j} + \frac{A_{ni}v}{x_i - x_n}\right) \\ &\quad = \sum_{k\neq i,j,n} \frac{A_{ik}A_{nj}v}{(x_i - x_k)(x_n - x_j)} + \frac{A_{nj}v}{x_i - x_j}\left\{-\frac{A_{ni}v}{x_n - x_i} + \frac{(A_{ij} + A_{ni})v}{x_n - x_j}\right\}. \end{aligned}$$

By the commutativity condition of A_{ij}, we have

$$\frac{\partial}{\partial x_i} w_j = \sum_{k\neq i,j,1 \leqslant k \leqslant n-1} \frac{A_{ik}}{x_i - x_k} w_k + \frac{1}{x_i - x_j}\{(A_{ij} + A_{ni})w_j - A_{nj}w_i\}$$

for $i \neq j$. Using the relation in (1), we have

$$\frac{\partial}{\partial x_i} w_i = -\sum_{j\neq i,1 \leqslant j \leqslant n-1} \frac{\partial}{\partial x_i} w_j.$$

Therefore we obtain the statement of (2). □

Remark 3.5. If the set of matrices $\{A_{ij}\}$ satisfies the infinitesimal pure braid relations, then the set $\{A'_{ij}\}$ defined by (3.3) also satisfies the infinitesimal pure braid relations for $n - 1$. Therefore the connection ∇' on $(\mathcal{O}^{\oplus d})^{\oplus(n-1)}$ given by

$$\nabla' W = dW - \sum_{1 \leqslant i<j \leqslant n-1} \frac{(\mathrm{d}x_i - \mathrm{d}x_j)A'_{ij}}{x_i - x_j} W$$

is integrable.

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Let V_n be the local system of horizontal sections of the connection ∇ and $V_n \mid_{\pi^{-1}(x_1^0,\ldots,x_{n-1}^0)}$ its restriction to the fiber $\pi^{-1}(x_1^0,\ldots,x_{n-1}^0)$ of $\pi : X_n \to X_{n-1}$. Then the Euler–Poincaré characteristic of $V_n \mid_{\pi^{-1}(x_1^0,\ldots,x_{n-1}^0)}$ is $-(\mathrm{rank} V)\cdot(n-2)$. Therefore under certain nonresonance condition, $\dim H^1(\pi^{-1}(x_1^0,\ldots,x_{n-1}^0), V_n)$ is equal to $\mathrm{rank} V\cdot(n-2)$. By a direct computation, the submodule

$$(\mathcal{M})^{\mathrm{red}} = \left\{ W = {}^t(w_1,\ldots,w_{n-1}) \mid w_i \in \mathcal{O}^{\oplus d}, \sum_{i=1}^{n-1} w_i = 0 \right\}$$

turns out to be a sub connection of $\mathcal{M} = ((\mathcal{O}^{\oplus d})^{\oplus(n-1)}, \nabla')$. As a consequence, the space of horizontal sections of $(\mathcal{M})^{\mathrm{red}}$ is equal to the higher direct image of V under the projection π. This construction is compatible with the sub local system in V.

We apply this inductive formula to compute the differential equations satisfied by Selberg integrals. Note that the similar computation is executed in [1] with a different choice of base of de Rham cohomology. In Section 5.2, we show that for our base, the Selberg integrals are holomorphic with respect to α_{ij}. This base is nothing but the β-nbc base introduced in [3].

Let R be a ring. For a set of elements $\mathbf{a} = \{a_{pq}\}_{1 \leqslant p<q \leqslant k}$ satisfying the infinitesimal pure braid relation, we define a set of elements $\mathrm{Ind}(\mathbf{a}) = \{\mathrm{Ind}(\mathbf{a})_{ij}\}_{1 \leqslant i<j \leqslant k-1}$ in $M(k-1, R)$ by

$$\mathrm{Ind}(\mathbf{a})_{ij} = \begin{pmatrix} a_{ij} & \overset{i}{\cdots} & \overset{j}{\cdots} & 0 \\ \vdots & a_{ij}+a_{kj} & -a_{ki} & \vdots \\ \vdots & -a_{kj} & a_{ij}+a_{ki} & \vdots \\ 0 & \cdots & \cdots & a_{ij} \end{pmatrix} \begin{matrix} \\ i \\ j \\ \\ \end{matrix}.$$

Let $2 \leqslant r \leqslant n$ be integers and $V_{r,n}$ be a $\mathbf{C}$ vector space of dimension $r(r+1)\cdots(n-1)$ whose coordinates are given by $v_{i_{r+1},\ldots,i_n}$ for $1 \leqslant i_{r+1} \leqslant r, \ldots, 1 \leqslant i_n \leqslant n-1$. We define $\mathbf{A}^{(p)} = \{A_{ij}^{(p)}\}_{1 \leqslant i<j \leqslant p}$ for $p = r, \ldots, n-1$ by $\mathbf{A}^{(p)} = \mathrm{Ind}(\mathbf{A}^{(p+1)})$ and $A_{ij}^{(n)} = \alpha_{ij}$.

We define the $V_{k,n}$-valued functions $S^{(k)}(x_1,\ldots,x_k)$ on $D([k])$ inductively by

$$S^{(k)} = \begin{pmatrix} \int_{D([k+1]/[k],x_i)_{i\in[k]}} \frac{A_{k+1,1}^{(k+1)}}{x_{k+1}-x_1} S^{(k+1)}(x_1,\ldots,x_{k+1})\mathrm{d}x_{k+1} \\ \vdots \\ \int_{D([k+1]/[k],x_i)_{i\in[k]}} \frac{A_{k+1,k}^{(k+1)}}{x_{k+1}-x_k} S^{(k+1)}(x_1,\ldots,x_{k+1})\mathrm{d}x_{k+1} \end{pmatrix}$$

for $k = r, \ldots, n-1$ and

$$S^{(n)} = \prod_{1 \leqslant i \ll j \leqslant n} (x_j - x_i)^{\alpha_{ij}}.$$

Note that the components of $S^{(r)}$ are indexed by $\{(i_{r+1},\ldots,i_n) \mid i_k \in [k-1]\}$. We have the following corollary of Proposition 3.4.

COROLLARY 3.6. *The $V_{k,n}$-valued function $S^{(k)}$ satisfies the following differential equation* $\mathrm{d}S^{(k)} = \Omega_k S^{(k)}$, *where*

$$\Omega_k = \sum_{1 \leqslant i<j \leqslant k} \frac{A_{ij}^{(k)} d(x_i - x_j)}{x_i - x_j}.$$

The next proposition is used in the proof of the Main Theorem 3.3.

PROPOSITION 3.7. *Let $S_{i_{r+1},\dots,i_n}^{(r)}$ be the $(i_{r+1},\dots,i_n)$-component of $S^{(r)}$. Then we have*

$$\sum_{i'_p=1}^{p-1} S_{i_{r+1},\dots,i_{p-1},i'_p,i_{p+1},\dots,i_n} = 0. \tag{3.4}$$

Proof. If $p = r+1$, then it is nothing but the first statement of Proposition 3.4. Suppose $p > r+1$. Then the $(i_{r+1},\dots,i_{p-1})$-part of $S^{(r)}$ is a linear combination of

$$\int \prod_{i=r+1}^{p-1} A_{p_i q_i}^{(p)} \prod_{j=r+1}^{p-1} \frac{1}{x_j - x_{i_j}} S^{(p)} \mathrm{d}x_{r+1} \cdots \mathrm{d}x_{p-1}. \tag{3.5}$$

Since the set $\{(a_{i_p,\dots,i_n}) \mid \sum_{i'_p=1}^{p-1} a_{i'_p,\dots,i_n} = 0\}$ is stable under the action of $A_{ab}^{(p)}$, (3.5) satisfies the relation $\sum_{i'_p=1}^{p-1} a_{i'_p,\dots,i_n} = 0$. □

DEFINITION 3.8. We define

$$V_{k,n}^{\mathrm{red}} = \left\{ v(i_{k+1},\dots,v_n) \middle| \sum_{i_p=1}^{p-1} v(i_{k+1},\dots,i_p,\dots,i_n) = 0 \right\}.$$

4. Combinatorial Propositions

4.1. STATEMENT OF THE COMBINATORIAL THEOREM

In this section, we present combinatorial facts which are used for the computation of Selberg integrals. Let P_n be the quotient of the free noncommutative ring $\mathbf{C}\langle a_{ij}\rangle$ by the two-sided ideal generated by the infinitesimal pure braid relations. We inductively define the set of matrices $\mathbf{A}^{(k)} = \{A_{ij}^{(k)}\}_{1 \leqslant i,j \leqslant k}$ in $M(k(k+1)\cdots(n-1), P_n)$ by the relations: $\mathbf{A}^{(k)} = \mathrm{Ind}(\mathbf{A}^{(k+1)})$ for $k = r,\dots,n-1$ and $A_{ij}^{(n)} = a_{ij}$. We introduce the degree of P_n by $\deg a_{ij} = 1$. Then all the matrix elements of $A_{ij}^{(k)}$ are of degree 1 for $k = r,\dots,n$ and $A_{ij}^{(k)}$ satisfies the pure braid relations. In other words, a ring homomorphism $P_r \to M(r(r+1)\cdots(n-1), P_n)$ is defined by attaching $A_{ij}^{(r)}$ to $a_{ij} \in P_r$. We inductively define the vector

$$w_k \in P_n^{k(k+1)\cdots(n-1)} \otimes \mathbf{C}\left[\frac{1}{x_i - x_j}\right]$$

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by the relation

$$w_k = \begin{pmatrix} \frac{A^{(k+1)}_{k+1,1}}{x_{k+1}-x_1} w_{k+1} \\ \vdots \\ \frac{A^{(k+1)}_{k+1,k}}{x_{k+1}-x_k} w_{k+1} \end{pmatrix} \tag{4.1}$$

for $k = r, \dots, n-2$ and

$$w_{n-1} = \begin{pmatrix} \frac{A^{(n)}_{n,1}}{x_n - x_1} \\ \vdots \\ \frac{A^{(n)}_{n,n-1}}{x_n - x_{n-1}} \end{pmatrix}.$$

In this section, we express each coordinate of w_r in terms of the combinatorics introduced in Section 3.1. For an ordered rooted tree Γ with the vertex set $[k]$ and root set $R = [r]$, we define $A^{(k)}_\Gamma$ by

$$A^{(k)}_\Gamma = \prod_{i=l}^{1} A^{(k)}_{p_i, q_i} \in M(k(k+1)\cdots(n-1), P_n),$$

where

$$E_\Gamma = \{e_1 < \cdots < e_l\} \quad \text{and} \quad e_i = (p_i, q_i).$$

Here we used the notation $\prod_{i=l}^{1} a_i = a_l a_{l-1} \cdots a_1$ in a noncommutative ring. We define the matrix-valued differential form $\eta_\Gamma \in \Omega([k] \bmod [r]) \otimes M(k(k+1)\cdots (n-1), P_n)$ by $\eta_\Gamma = A^{(k)}_\Gamma \omega_\Gamma$, where ω_Γ is defined in Section 3.1. For an element $\gamma \in \Gamma([k], [r])$, we define η_γ by $\eta_\gamma = \sum_\Gamma a_\Gamma \eta_\Gamma$, where $\gamma = \sum_\Gamma a_\Gamma \Gamma$.

THEOREM 4.1. *Let us denote the* $(i_{r+1}, \dots, i_n)$*-component of* w_r *by* $w_r(i_{r+1}, \dots, i_n)$. *Then*

$$w_r(i_{r+1}, \dots, i_n) \mathrm{d}x_n \wedge \cdots \wedge \mathrm{d}x_{r+1} = \eta_\gamma,$$

where $\gamma = \emptyset([r]) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n, i_n)$.

For the rest of this section, we prove Theorem 4.1.

4.2. SEVERAL LEMMATA

Let Γ be an ordered rooted tree with the root set $[r]$ and the vertex set $[n-1]$. The edge set is denoted by $E = \{e_1 < \cdots < e_l\}$, where $e_i = (p_i, q_i)$. Suppose that p and q are contained in the same connected component. Then there exists a unique path P connecting p and q in Γ. We write $P = \{e_{t_1}, \dots, e_{t_m}\}$. The subgraph P looks like Figure 1.

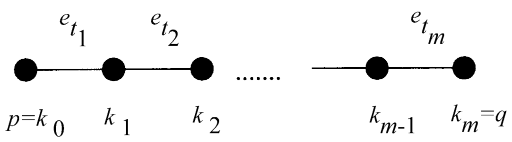

Figure 1.

LEMMA 4.2. *Let* $A_\Gamma^{(n-1)} \in M(n-1, P_n)$ *be defined as in Section* 4.1

(1) *If the* qth *component of*

$$A_\Gamma^{(n-1)} \begin{pmatrix} 0 \\ \vdots \\ a_{np} \\ \vdots \\ 0 \end{pmatrix} \in P_n^{(n-1)} \tag{4.2}$$

is not zero, then p *and* q *are contained in the same connected component, and* $t_1 < t_2 < \cdots < t_m$.

(2) *Suppose that* $t_1 < t_2 < \cdots < t_m$. *We write the vertices of the path* P *as* $p = k_0, k_1, \ldots, k_m = q$, (see Figure 1) *and define* B_i $(i = 1, \ldots, l)$ *by*

$$B_i = \begin{cases} -a_{k_j,n} & (\textit{if } i = t_j), \\ a_{p_iq_i} + a_{nq_i} & (\textit{if } t_j < i < t_{j+1} \textit{ and} \\ & e_i = (p_i, q_i) \textit{ ajacents to } k_j \textit{ and put } p_i = k_j), \\ a_{p_iq_i} & (\textit{if } t_j < i < t_{j+1} \textit{ and } e_i \textit{ does not ajacent to } k_j). \end{cases}$$

(For the second case see Figure 2.) *Then the* qth *component of* (1) *is equal to* $\prod_{i=l}^{1} B_i$.

Proof. For a vector $v = {}^t(v_1, \ldots, v_{n-1}) \in P_n^{n-1}$, we set $\mathrm{Supp}(v) = \{i \mid v_i \neq 0\}$.

(1) If $\{i, j\} \cap \mathrm{Supp}(v) = \emptyset$, then $\mathrm{Supp}(A_{ij}^{(n-1)} v) = \mathrm{Supp}(v)$ and the kth component of $A_{ij}^{(n-1)} v$ is equal to $a_{ij} v_k$ for $k \in \mathrm{Supp}(v)$.

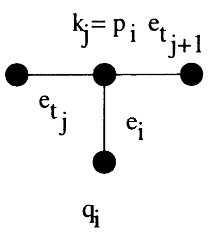

Figure 2.

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(2) If $\{i, j\} \cap \mathrm{Supp}(v) = \{i\}$, then $\mathrm{Supp}(A_{ij}^{(n-1)}v) \subset \mathrm{Supp}(v) \cup \{j\}$ and the jth component and the ith component of $A_{ij}^{(n-1)}v$ are equal to $-a_{nj}v_i$ and $(a_{ij} + a_{nj})v_i$, respectively.

Therefore if we define S_i inductively by

$$S_{i+1} = \begin{cases} S_i & (\text{if } e_{i+1} \cap S_i = \emptyset), \\ S_i \cup \{j\} & (\text{if } e_{i+1} \cap S_i = \{k\} \text{ and } e_{i+1} = \{j, k\}), \end{cases}$$

and $S_0 = \{p\}$. Then $\mathrm{Supp}(\prod_{i=l}^{1} A_{p_i,q_i}^{(n-1)})v \subset S_l$. If $q \in S_l$, then $t_1 < \cdots < t_m$. This proves (1). For the claim (2), we can prove

$$\left[\left(\prod_{i=s}^{1} A_{p_i,q_i}^{(n-1)} \begin{pmatrix} 0 \\ \vdots \\ a_{nk} \\ \vdots \\ 0 \end{pmatrix}\right)\right]_{k_j} = a_{n,k_j} \cdot \prod_{j=s}^{1} B_j.$$

if $t_j \leqslant s < t_{j+1}$ by induction on s using the infinitesimal pure braid relation (1) and (2). (In case $s \geqslant t_m$ and $s < t_1$, $a_{n,k_j} = a_{n,k_m}$ and $a_{n,k_j} = a_{n,k_0}$ respectively.) This completes the proof of (2). □

Next we rewrite $\emptyset([r]) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n, i_n)$ by using the notion of principal tree.

DEFINITION 4.3. For an index set $I = (i_{k+1}, \ldots, i_n)$, $(1 \leqslant i_p \leqslant p-1)$, we define the ordered rooted tree P_I as follows.

(1) The set of vertices is $[n]$,
(2) the set of roots is $[k]$, and
(3) the set of ordered edges is $\{(k+1, i_{k+1}) < \cdots < (n, i_n)\}$.

The tree P_I is called the principal tree of the index set I.

Let p, q be two vertices contained in the same connected component of P_I. The unique shortest path connecting p, q in P_I is denoted by $\gamma(p, q)$ and the minimal edge of $\gamma(p, q)$ is denoted by $\min(p, q)$. Then by the construction of the principal tree, we have the following lemma.

LEMMA 4.4. *Let us write a path $\gamma(p, q)$ connecting p, q in P_I as in Figure 1: Suppose that e_{t_s} is the minimal edge of $\gamma(p, q)$. Then $t_1 > \cdots > t_s < \cdots < t_m$*

A graph Γ is called a support of $\gamma = \sum_\Gamma a_\Gamma \Gamma$ if $a_\Gamma \neq 0$. The set of supports of γ is denoted by $\mathrm{Supp}(\gamma)$. Let $p, q \in [n]$ be vertices contained in the same connected component in P_I. We set $\gamma = \emptyset(\{1, \ldots, k\}) \wedge (k+1, i_{k+1}) \wedge \cdots \wedge (n, i_n)$. By the construction of γ, if $\Gamma \in \mathrm{Supp}(\gamma)$ and (p, q) is an edge in Γ, then (p, q) is the mth edge of Γ, where $e_m = \min(p, q)$. (To define $\min(p, q)$, we used the principal tree P_I.) Conversely, for any pairs $(p_{k+1}, q_{k+1}), \dots, (p_n, q_n)$ such that

(1) p_i and q_i are contained in the same connected component of P_I, and
(2) $\min(p_j, q_j)$ is the jth edge (j, i_j) of P_I,

we have $a_\Gamma = 1$ for $\Gamma = (p_{k+1}, q_{k+1}) \cdots (p_n, q_n)$. We use the distributive notation

$$\{(p_{k+1}, q_{k+1}) + (p'_{k+1}, q'_{k+1})\}(p_{k+2}, q_{k+2}) \cdots (p_n, q_n)$$
$$= (p_{k+1}, q_{k+1})(p_{k+2}, q_{k+2}) \cdots (p_n, q_n) + (p'_{k+1}, q'_{k+1})(p_{k+2}, q_{k+2}) \cdots (p_n, q_n).$$

Here the right hand side has a meaning as a formal linear combination of ordered graphs. The following proposition is nothing but a restatement of the definition of $\wedge$.

PROPOSITION 4.5. *Let* $S_i = \sum_{1 \leqslant p<q \leqslant n, \min(p,q)=e_i \text{ in } P_I}(p, q)$. *Then*
$$\emptyset(\{1, \dots, r\}) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n, i_n) = S_{r+1} \cdot S_{r+2} \cdots S_n.$$

We finish this subsection by computing $\mathrm{Res}_{x_n \to x_k}(\omega_\Gamma)$ for an ordered rooted tree $\Gamma = \{e_{r+1}, \dots, e_n\} \in \mathrm{Supp}(\gamma)$, where $\gamma = \emptyset(\{1, \dots, k\}) \wedge (k+1, i_{k+1}) \wedge \cdots \wedge (n, i_n)$. Until the end of this subsection we assume $\Gamma \in \mathrm{Supp}(\gamma)$ and (n, k) is an edge of Γ. Put $R_- = \{k' \mid (n, k') \in \Gamma, \min(n, k') < \min(n, k)\}$ and $R_+ = \{k' \mid (n, k') \in \Gamma, \min(n, k') \geqslant \min(n, k)\}$. We introduce a labeling on $R_+ = \{k_1 = i_n, k_2, \dots, k_s = k\}$ such that $\min(n, k_1) > \cdots > \min(n, k_s)$. Set $e_{t_i} = \min(n, k_i)$. For the figure of principal tree see Figure 3. If $s \geqslant 2$, we put $P = P(\Gamma, k)$ as the power set of $R_+ - \{k_1, k_s\}$. For an element $p \in P$, we define the graph $\Gamma(p) \in \Gamma([n-1], [r])$ as follows. For $i = 2, \dots, s$, put $m(p, i) = \max\{j \mid k_j \in p \cup \{k_1\}, j < i\}$. The t_ith edge of $\Gamma(p)$ is equal to (k_i, k_m), where $m = m(p, i)$. The jth edge is the same as Γ if $j \neq t_i, n$ $(i = 2, \dots, s)$. The set of ordered graphs $\{\Gamma(p) \mid p \in P(\Gamma, k)\}$ is denoted by $R(\Gamma, k)$, and is called the set of residue graphs of Γ with respect to k.

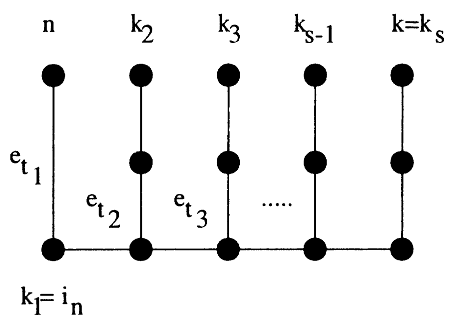

Figure 3.

PROPOSITION 4.6. *If* $\# \mid R_+ \mid \geqslant 2$, *then*

$$\mathrm{Res}_{x_n \to x_k}(\omega_\Gamma) = \sum_{p \in P(\Gamma,k)} (-1)^{\#p+1} \omega_{\Gamma(p)}.$$

Here $\mathrm{Res}_{x_n \to x_k}\omega = \eta \mid_{x_n = x_k}$, *where* $\omega = d\log(x_n - x_k) \wedge \eta$.

Proof. We write $\mathrm{d}\log(x_p - x_q) = \langle p, q\rangle$ for short. First we prove the equality

$$\begin{aligned}&\langle k, k_1\rangle \cdots \langle k, k_{s-1}\rangle \\ &\quad = \sum_{p \subset \{k_2,\ldots,k_{s-1}\}} (-1)^{s+\#p} \langle m(p,2), k_2\rangle \cdots \langle m(p,s-1), k_{s-1}\rangle \langle m(p,s), k_s\rangle\end{aligned}$$

by induction on the cardinality of $\{k_1, \ldots, k_{s-1}\}$. By the inductive assumption for $\{k_1, \ldots k_{s-2}\}$, we have

$$\begin{aligned}&\langle k, k_1\rangle \cdots \langle k, k_{s-2}\rangle \langle k, k_{s-1}\rangle \\ &\quad = \sum_{q \subset \{k_2,\ldots,k_{s-1}\}} (-1)^{s+\#q} \langle m(q,2), k_2\rangle \cdots \langle m(q,s-1), k\rangle \langle k, k_{s-1}\rangle \\ &\quad = \sum_{q \subset \{k_2,\ldots,k_{s-1}\}} (-1)^{s+\#q} \langle m(q,2), k_2\rangle \cdots \langle m(q,s-2), k_{s-2}\rangle \times \\ &\qquad \times (\langle m(q,s-1), k_{s-1}\rangle \langle k, k_{s-1}\rangle - \langle k_{s-1}, m(q,s-1)\rangle \langle k, m(q,s-1)\rangle)\end{aligned}$$

and the last expression gives the expression of $\{k_1, \ldots, k_{s-1}\}$. Therefore we have

$$\begin{aligned}&\mathrm{Res}_{x_n \to x_k} \langle n, k_1\rangle \cdots \langle n, k_s\rangle \\ &\quad = (-1)^{s-1} \langle n, k_1\rangle \cdots \langle n, k_{s-1}\rangle \\ &\quad = \sum_{p \subset \{k_2,\ldots,k_{s-1}\}} (-1)^{\#p+1} \langle m(p,2), k_2\rangle \cdots \langle m(p,s-1), k_{s-1}\rangle \langle m(p,s), k_s\rangle.\end{aligned}$$

This implies the proposition. □

4.3. PROOF OF THEOREM 4.1

We prove Theorem 4.1 by induction. By Remark 3.5, the morphism

$$\rho : A_{ij}^{(n-1)} \mapsto \mathrm{Ind}(\mathbf{a})_{ij}$$

defines a ring homomorphism from P_{n-1} to $M(n-1, P_n)$. We assume Theorem 4.1 for $n-1$. We define $W_k \in P_{n-1}^{r(r+1)\cdots(n-1)}$ for $k = 1, \ldots, n-3$ inductively by the relation similar to (4.1) and

$$W_{n-2} = \begin{pmatrix} \frac{A_{n-1,1}^{(n-1)}}{x_{n-1} - x_1} \\ \vdots \\ \frac{A_{n-1,n-2}^{(n-1)}}{x_{n-1} - x_{n-2}} \end{pmatrix}.$$

Then by the inductive assumption,

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$$\eta_\gamma = W_r(i_{r+1}, \dots, i_{n-1})\mathrm{d}x_{n-1} \wedge \cdots \wedge \mathrm{d}x_{r+1}$$

for $\gamma = \emptyset(\{1, \dots, r\}) \wedge (r+1, i_{r+1}) \wedge \cdots \wedge (n-1, i_{n-1})$ in $P_{n-1} \otimes \Omega([n-1] \bmod [r])$. Here $W_r(i_{r+1}, \dots, i_{n-1})$ is the $(i_{r+1}, \dots, i_n)$th component of W_r. By applying the above ring homomorphism ρ, we have

$$\rho(\eta_\gamma) = \rho(W_r(i_{r+1}, \dots, i_{n-1}))\mathrm{d}x_{n-1} \wedge \cdots \wedge \mathrm{d}x_{r+1}$$

in $M(n-1, P_n) \otimes \Omega([n-1] \bmod [r])$. By the definition of w_r, $w_r(i_{r+1}, \dots, i_n)$ is equal to the i_nth component of the vector

$$\rho(W_r(i_{r+1}, \dots, i_{n-1}))\begin{pmatrix} \frac{a_{n1}}{x_n - x_1} \\ \vdots \\ \frac{a_{nn-1}}{x_n - x_{n-1}} \end{pmatrix}.$$

Therefore by taking the residue $\mathrm{Res}_{x_n \to x_k}$, it is enough to prove that

$$\rho(W_r(i_{r+1}, \dots, i_{n-1}))\begin{pmatrix} 0 \\ \vdots \\ a_{nk} \\ \vdots \\ 0 \end{pmatrix}_{i_n} = \mathrm{Res}_{x_n \to x_k}(\eta_{\bar{\gamma}}) \tag{4.3}$$

for all $k = 1, \dots, n-1$, where $\bar{\gamma} = \emptyset(\{1, \dots, k\}) \wedge (k+1, i_{k+1}) \wedge \cdots \wedge (n, i_n)$. We compute the left-hand side and right-hand side by using Lemma 4.2 and Proposition 4.6. The left-hand side of (4.3) is expressed as a linear combination of η_Γ, where Γ is in the support of γ. On the other hand, the expression given in Proposition 4.6 gives an expression of the right-hand side by a linear combination of η_Γ, where Γ is a support of γ. By comparing the coefficient of ω_Γ, it is enough to prove the following proposition.

PROPOSITION 4.7. *Assume that* $\Gamma \in \mathrm{Supp}(\gamma)$, *and* k *and* i_n *are contained in the same connected component of* Γ.

(1) $\mathrm{length}(k, i_n) = \#p + 1$ *if* $\bar{\Gamma}(p) = \Gamma$. *Here* $\mathrm{length}(k, i_n)$ *is the length of the path connecting* k *and* i_n *in* Γ.

(2) $$(-1)^{\mathrm{length}(k, i_n)} \prod_{i=r+1}^{n} B_i = \sum_{\{\bar{\Gamma} \in \mathrm{Supp}(\bar{\gamma}) \mid \Gamma \in R(\bar{\Gamma}, k)\}} A_{\bar{\Gamma}}^{(n)},$$

were B_i *is defined in Lemma* 4.2 *and* $R(\bar{\Gamma}, k)$ *is defined in Proposition* 4.6.

Proof. Let $\Gamma \in \mathrm{Supp}\,\gamma$ and suppose $R(\bar{\Gamma}, k) \ni \Gamma$. As in Lemma 4.2, we make the labeling of the path in Γ from k to i_n as $e_{t_1} = (n, k_1), e_{t_2} = (k_1, k_2), \dots, e_{t_m} = (k_{s-1}, k)$ and $k_s = k, k_1 = i_n$. First we claim the set $L = L(\bar{\Gamma}) = \{l \mid (n, l) \in \bar{\Gamma}\}$ contains

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$k_1, \ldots, k_s$. Since $\mathrm{Res}_{x_n \to x_k}(\bar{\Gamma})$ is not zero, $\bar{\Gamma}$ contains (n, k), i.e. $k = k_s \in L$. If the path connecting k and i_n in the corresponding graph $\bar{\Gamma}(p)$ is $k_1, \ldots, k_s$, then $p = \{k_2, \ldots, k_{s-1}\}$. Therefore $L \supset \{k_1, \ldots, k_s\}$. If $l \in L - \{k_1, \ldots, k_s\}$ and q is the minimal element satisfying $\min(i_n, l) < \min(i_n, k_q)$, then $\bar{\Gamma}(p)$ contains an edge (l, k_q) by the definition of $\bar{\Gamma}(p)$. That is $(l, k_q) \in G(\Gamma, k, i_n)$, where

$$G(\Gamma, k, i_n) = \{e : \text{edge} \mid \text{There exists } i \text{ such that } e \ni k_i, \\ \min(k_{i-1}, i_n) \leqslant e < \min(k_i, i_n)\}.$$

Therefore $L - \{k_1, \ldots, k_s\} \subset G(\Gamma, k, i_n)$.

Conversely, for any subset L of $\{1, \ldots, n\}$ satisfying

(1) L is contained in the same connected component of i_n,
(2) $L \supset \{k_1, \ldots, k_s\}$, and
(3) $L - \{k_1, \ldots, k_s\} \subset G(\Gamma, k, i_n)$,

there exists a unique $\bar{\Gamma}(L)$ satisfying

(1) $L(\bar{\Gamma}(L)) = L$,
(2) $\bar{\Gamma} \in \mathrm{Supp}(\bar{\gamma})$, and
(3) $\mathrm{Supp}(\mathrm{Res}_{x_n \to x_k}(\bar{\Gamma})) \ni \Gamma$.

Therefore

$$\sum_{\{\bar{\Gamma} \mid \Gamma \in R(\bar{\Gamma}, k), \bar{\Gamma} \in \mathrm{Supp}(\bar{\gamma})\}} A^{(n)}_{\bar{\Gamma}} = \sum_{\substack{L \supset \{k_1, \ldots, k_s\}, L - \{k_1, \ldots, k_s\} \subset G(\Gamma, k, i_n), \\ L \text{ is contained in the same connected component of } i_n}} A^{(n)}_{\bar{\Gamma}(L)} \\ = (-1)^{\mathrm{length}(k, i_n)} \prod_{i=k+1}^{n} B_i. \qquad \square$$

5. Proof of The Main Theorem

5.1. SOME LEMMATA FOR THE ASYMPTOTIC BEHAVIORS

In this subsection, we investigate the asymptotic behavior of solutions of a linear differential equation with regular singularities. Let $A \in (1/x)M(d, \mathcal{O}_x)$, where $\mathcal{O}_x$ is the germs of holomorphic functions at $x = 0$. We are interested in the differential equation for $r \times r$-matrix-valued function V: $\mathrm{d}V/\mathrm{d}x = AV$. We write $A = Rx^{-1} + \sum_{i=0}^{\infty} A_i x^i$, where $R, A_i \in M(r, \mathbf{C})$. If all the eigenvalues of R are small enough, then the solution V can be written as $V = Fx^R C_0$, where F is an $r \times r$-valued holomorphic function of $I + xM(r, \mathcal{O}_x)$, and $C_0 \in GL(r, \mathbf{C})$. In the rest of this section, we assume that all the eigen values of R are sufficiently small positive real numbers and R is semi-simple. The eigenvalues of R are denoted by $0 < \lambda_1 < \cdots < \lambda_s$.

LEMMA 5.1. *Let* $\mathbf{C}^r = \oplus_{i=1}^s W_i$ *be the eigenspace decomposition of* $\mathbf{C}^r$ *with respect to* R.

(1) *If* $w_i \in W_i$, *then each component* a_k *of the vector* $Fx^R w_i$ *satisfies the estimation* $|a_k| \leqslant |x|^{\lambda_i} c$, *for some constant* c *for* $k = 1, \ldots, r$. *Moreover we have* $\lim_{x\to 0}(x^{-\lambda_i} Fx^R w_i) = w_i$.
(2) *Let* $\lambda > \lambda_i$. *If* $w_i \in W_i$ *and all the components* a_k *of* $Fx^R w_i$ *satisfy* $|a_i| \leqslant x^{\lambda} c$ *for some constant* c, *then* $w = 0$.
(3) *Let* $p\colon W_i \to \mathbf{C}^l$ *be a linear map and denote by* $\tilde{p}$ *the composite* $\mathbf{C}^r \to W_i \to \mathbf{C}^l$. *Then we have*

$$\tilde{p}\left(\lim_{x\to 0} x^{-\lambda_i} Fx^R w\right) = \tilde{p}\left(\lim_{x\to 0} x^{-R} Fx^R w\right)$$

for any $w \in W$.

Proof. Since $F = I + xm, m \in M(r, \mathcal{O}_x)$, using the identity $\lim_{x\to 0} x^{-\lambda} xmx^R = \lim_{x\to 0} x^{-R} xmx^R = 0$, we get the statements. □

Let n, k be integers such that $2 \leqslant k \leqslant n$ and define $A_{ij}^{(k)}$ $(1 \leqslant i < j \leqslant k)$ and the reduced part $V^{\mathrm{red}} = V_{k,n}^{\mathrm{red}}$ as in Section 3.2. The restriction of $A_{ij}^{(k)}$ to V^{red} is denoted by $A_{ij,\mathrm{red}}^{(k)}$. For a subset S of $[i, k]$, we define $A_S^{(k)}$ and $A_{S,\mathrm{red}}^{(k)}$ by

$$A_S^{(k)} = \sum_{i<j, i,j\in S} A_{ij}^{(k)}, \; A_{S,\mathrm{red}}^{(k)} = \sum_{i<j, i,j\in S} A_{ij,\mathrm{red}}^{(k)}.$$

From now on, α_{ij} are sufficiently general small positive real numbers. For a semisimple matrix A, the formal sum of eigenvalues of A counting their multiplicities is denoted by $\sigma(A)$: $\sigma(A) = \sum(\text{eigenvalues of } A)$. The set of eigenvalues is denoted by $\mathrm{Supp}(\sigma(A))$.

PROPOSITION 5.2. *Under the above notations and assumptions,* $A_S^{(k)}$ *and* $A_{S,\mathrm{red}}^{(k)}$ *are semi-simple and*

$$\sigma(A_S^{(k)}) = \sum_{T\subset[k+1,n]} (k - l; \mid T^c \mid)(l; \mid T \mid) a_{S\cup T},$$

$$\sigma(A_{S,\mathrm{red}}^{(k)}) = \sum_{T\subset[k+1,n]} (k - l - 1; \mid T^c \mid)(l; \mid T \mid) a_{S\cup T},$$

where $a_U = \sum_{i<j, i,j\in U} \alpha_{ij}$ *for a subset* $U \subset [1, n]$. *For a subset* $T \subset [k+1, n]$, $T^c = [k+1, n] - T$ *and* $l = \# \mid S \mid -1$ *and* $(a; b) = a(a+1)\cdots(a+b-1)$.

To prove the above proposition, we use the following two elementary lemmata.

LEMMA 5.3. *Let* X *be a* $kN \times kN$*-matrix. We assume that there exist semi-simple matrices* $B, D \in M(N, \mathbf{C})$ *and matrices* $C_1, \ldots, C_k \in M(N, \mathbf{C})$ *such that*

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$$A\begin{pmatrix}0\\ \vdots\\ 1\\ -1\\ \vdots\\ 0\end{pmatrix}\begin{matrix}\\ \\ i\\ i+1\\ \\ \\ \end{matrix} = \begin{pmatrix}0\\ \vdots\\ B\\ -B\\ \vdots\\ 0\end{pmatrix}$$

$$A\begin{pmatrix}C_1\\ \vdots\\ C_k\end{pmatrix} = \begin{pmatrix}C_1 D\\ \vdots\\ C_k D\end{pmatrix},$$

with $\mathrm{Supp}(\sigma(B)) \cap \mathrm{Supp}(\sigma(D)) = \emptyset$. *Then*

(1) $\sigma(A) = (k-1)\sigma(B) + \sigma(D)$.

(2) *The* $(k-1)N$*-dimensional subspace* $V^{\mathrm{red}} = \{(v_1, \dots, v_k) \mid v_i \in \mathbf{C}^N, \sum v_i = 0\}$ *is stable under the action of* A. *Let* A^{red} *be the restriction of* A *to* V^{red}. *Then* $\sigma(A^{\mathrm{red}}) = (k-1)\sigma(B)$.

LEMMA 5.4. *Let* $a_{ij} \in P_k$ *and set* $A_{ij} = \mathrm{Ind}(\mathbf{a})_{ij}$ *for* $1 \leqslant i < j \leqslant k-1$,

$$A_{[1,k-1]} = \sum_{1 \leqslant i<j \leqslant k-1} A_{ij}, \qquad a_{[1,k-1]} = \sum_{1 \leqslant i<j \leqslant k-1} a_{ij} \quad \text{and} \quad a_{[1,k]} = \sum_{1 \leqslant i<j \leqslant k} a_{ij}.$$

Then we have

$$A_{[1,k-1]}\begin{pmatrix}0\\ \vdots\\ 1\\ -1\\ \vdots\\ 0\end{pmatrix}\begin{matrix}\\ \\ i\\ i+1\\ \\ \\ \end{matrix} = \begin{pmatrix}0\\ \vdots\\ a_{[1,k]}\\ -a_{[1,k]}\\ \vdots\\ 0\end{pmatrix},$$

$$A_{[1,k-1]}\begin{pmatrix}a_{k1}\\ \vdots\\ a_{k-1}\end{pmatrix} = \begin{pmatrix}a_{k1}a_{[1,k-1]}\\ \vdots\\ a_{kk-1}a_{[1,k-1]}\end{pmatrix}.$$

Proof. The first equality follows from the expression

$$A_{[1,k-1]} = \begin{pmatrix} a_{[1,k-1]} + \sum_{j\neq 1} a_{k,j} & -a_{k1} & \cdots \\ -a_{k2} & a_{[1,k-1]} + \sum_{j\neq 2} a_{kj} & \cdots \\ -a_{k3} & -a_{k3} & \cdots \\ \vdots & & \vdots \end{pmatrix}.$$

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The second equality is obtained directly by the equality

$$A_{ij}\begin{pmatrix} a_{k1} \\ \vdots \\ a_{kk-1} \end{pmatrix} = \begin{pmatrix} a_{k1}a_{ij} \\ \vdots \\ a_{kk-1}a_{ij} \end{pmatrix}. \qquad \square$$

Proof of Proposition 5.2. We prove the proposition by induction. By the two lemmata, we have

$$\sigma(A_S^{(k)}) = (k-l)\sigma(A_S^{(k+1)}) + l\sigma(A_{S\cup\{k+1\}}^{(k+1)}),$$
$$\sigma(A_{S,\mathrm{red}}^{(k)}) = (k-l-1)\sigma(A_{S,\mathrm{red}}^{(k+1)}) + l\sigma(A_{S\cup\{k+1\},\mathrm{red}}^{(k+1)}),$$

using the homomorphism $P^{(k+1)} \to M((k+1)(k+2)\cdots(n-1), \mathbf{C})$ and the assumption of the independence of α_{ij}. $\square$

5.2. RELATION BETWEEN SELBERG INTEGRALS AND THE DRINFELD ASSOCIATOR

In this section, we will compare vectors whose elements are given by Selberg integrals with the Drinfeld associator. Let $n \geqslant 3$ be an integer and we define $A_{ij}^{(k)}$ as in Section 3.2. We set $V = V_{3,n} = \mathbf{C}^{3\cdot 4\cdots(n-1)}$. Let $S = S([n]/[3], x_1, x_2, x_3, \ \alpha_{ij})$ be a V-valued function on x_1, x_2, x_3 whose $(i_4, \ldots, i_n)$-component is given by

$$S_{\emptyset(\{1,2,3\})\wedge(4,i_4)\wedge\cdots\wedge(n,i_n)}([n]/[3], x_1, x_2, x_3, \alpha_{ij}).$$

Then by Theorem 4.1 and Corollary 3.6, S satisfies the differential equation

$$\mathrm{d}S = \{A_{13}^{(3)}\mathrm{d}\log(x_1 - x_3) + A_{23}^{(3)}\mathrm{d}\log(x_2 - x_3)\}S.$$

We set $\bar{S}(x_3) = S([n]/[3], 0, 1, x_3)$. Then $\bar{S}$ satisfies the equation

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}x_3} = \left(A_{13}^{(3)}\frac{\mathrm{d}x_3}{x_3} + A_{23}^{(3)}\frac{\mathrm{d}x_3}{x_3 - 1}\right)\bar{S}.$$

Since all the elements of $A_{13}^{(3)}, A_{23}^{(3)}$ are homogeneous polynomials of degree 1 in α_{ij}, the representation

$$\rho\colon \mathbf{C}\langle\langle X, Y\rangle\rangle \to M(3\cdot 4\cdots(n-1), \mathbf{Q}[[\alpha_{ij}]])$$

given by $\rho(X) = A_{13}^{(3)}, \rho(Y) = A_{23}^{(3)}$ is a rational representation of degree 1. By the definition of the Drinfeld associator, we have

$$\lim_{x\to 1}(1 - x_3)^{-A_{23}^{(3)}}\bar{S}(x_3) = \rho(\Phi(X, Y))\lim_{x_3\to 0} x_3^{-A_{13}^{(3)}}\bar{S}(x_3).$$

LEMMA 5.5. (1) *For $i_4, \ldots, i_n$ such that $i_k \in [k-1]$, we put $\gamma = \emptyset(\{1, 2, 3\})\wedge (4, i_4) \wedge \cdots \wedge (n, i_n)$. Then for sufficiently small x_3, we have an estimation*

$$|S_\gamma([n]/[3], 0, 1, x_3)| < cx_3^{\alpha_{\max}} \tag{5.1}$$

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for some constant c. Here $\alpha_{\max}$ *is the maximal eigenvalue* $\sum_{1\leqslant i<j\leqslant n, i,j\neq 2}\alpha_{ij}$ *of* $A_{13}^{(3)}$.

(2) *For* $\Gamma\in\Gamma([n],[3])$,

$$\lim_{x_3\to 0} x_3^{-\alpha_{\max}} S_\Gamma([n]/[3],0,1,x_3) = \begin{cases} S_{\Gamma'}([n]-\{2\}/\{1,3\},0,1) & (\textit{if there is no edges containing } 2),\\ 0 & (\textit{otherwise}).\end{cases}$$

Here $\Gamma'\in\Gamma([n]-\{2\},\{1,3\})$ *is the ordered graph obtained by deleting* 2 *from the graph* Γ.

Proof. By Proposition 5.2, we have $\alpha_{\max}=\sum_{1\leqslant i<j\leqslant n,i,j\neq 2}\alpha_{ij}$. To prove the statement, it is enough to prove that

$$\int_D \prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^{\alpha_{ij}}\omega_\Gamma\ |_{x_1=0,x_2=1}$$

satisfies the estimation of (5.1) for an ordered rooted tree Γ with the root set [3]. We change the variables by $x_p=\xi_p x_3$ for $p=4,\dots,n$. Then

$$\omega_\gamma=\pm\prod_{(p_i,q_i)\in E_\Gamma,\text{not adjacent to }2}\frac{d\xi_{p_i}-d\xi_{q_i}}{\xi_{p_i}-\xi_{q_i}}\prod_{(p_i,2)\in E_\Gamma}\frac{x_3 d\xi_{p_i}}{-1}\cdot(1+o(1)) \tag{5.2}$$

and

$$\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^{\alpha_{ij}}=\prod_{1\leqslant i<j\leqslant n,i,j\neq 2}(\xi_i-\xi_j)^{\alpha_{ij}}\cdot x_3^{\alpha_{\max}}\dot{(}1+o(1)).$$

Here we put $\xi_3=1$, $\xi_1=0$. In particular, $\lim_{x_3\to 0}x_3^{-\alpha_{\max}}\int_D\Phi\omega_\Gamma=0$ if Γ contains an edge adjacent to 2. The signature in (5.2) arises from the substitution for separating edges of Γ adjacent to 2 and those which are not adjacent to 2. If Γ contains no edges adjacent to 2, we get the second statement. □

From Lemma 5.1, we have the following corollary.

COROLLARY 5.6. *The* $(i_4,\cdots,i_n)$th *component of* $\lim_{x_3\to 0}x_3^{-A_{13}^{(3)}}\bar{S}(x_3)$ *is equal to* $S_\gamma([n]-\{2\}/\{1,3\},0,1)$ *if* $i_p\neq 2$ *for* $p=4,\dots,n$, *where* $\gamma=\emptyset(\{1,3\})\wedge(4,i_4)\wedge\cdots\wedge(n,i_n)$, *and* 0 *otherwise.*

Proof. By the definition of $\gamma=\emptyset([3])\wedge(4,i_4)\wedge\cdots\wedge(n,i_n)$, if $i_p=2$ for some p, then all $\Gamma\in\mathrm{Supp}(\gamma)$ have an edge adjacent to 2. If $i_p\neq 2$ for all p, then all $\Gamma\in\mathrm{Supp}(\gamma)$ contains no edges adjacent to 2. Therefore the statement follows from Proposition 5.5. □

Next we consider the asymptotic behavior for $x_3\to 1$. Let $\mathbf{I}$ be the set $\{I=(i_4,\dots,i_n)\mid i_p\neq 2,3\}$. By the definition of $A_{23}^{(3)}$, the projection $p\colon V\to\mathbf{C}^{\mathbf{I}}$ to the $\mathbf{I}$-th components factors through α_{23} eigen projection. Therefore we have

$$p\left(\lim_{x_3\to 1}(1-x_3)^{-A_{23}^{(3)}}\bar{S}(x_3)\right)=p\left(\lim_{x_3\to 1}(1-x_3)^{-\alpha_{23}}\bar{S}(x_3)\right).$$

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by Lemma 5.1. On the other hand, it is easy to see the following lemma.

LEMMA 5.7. *If* Γ *contains no edges containing* 2 *and* 3*, then*

$$\lim_{x_3\to 1}(1-x_3)^{-\alpha_{23}}S_\Gamma([n]/[3],0,1,x_3)=S_{\Gamma'}([n]-\{3\}/[2],0,1,\alpha'_{ij}),$$

where Γ' *is the ordered graph obtained by deleting* 3 *from the graph* Γ*, and* $\alpha_{ij}=\alpha'_{ij}$ *if* $i,j\neq 2$ *and* $\alpha'_{2j}=\alpha_{2j}+\alpha_{3j}$.

DEFINITION 5.8. The vectors

$$\lim_{x_3\to 0}x_3^{-A_{13}^{(3)}}\bar{S}(x_3)\quad\text{and}\quad\lim_{x_3\to 1}(1-x_3)^{-A_{23}^{(3)}}\bar{S}(x_3)$$

are denoted by $V^{(1)}$ and $V^{(2)}$, respectively. Then we have

$$p(V^{(2)})=p(\rho(\Phi(X,Y))V^{(1)})\tag{5.3}$$

By Lemma 5.7, the $(i_4,\dots,i_n)$th component of $V^{(2)}$ with $i_p\neq 2,3$ is equal to $S_\gamma(0,1,\alpha'_{ij})$, where $\alpha'_{ij}=\alpha_{ij}$ if $i,j\neq 2$ and $\alpha'_{2,j}=\alpha_{2j}+\alpha_{3j}$, where $\gamma=\emptyset(\{1,2\})\wedge(4,i_4)\wedge\cdots\wedge(n,i_n)$. We compute the limit of all the components of $V^{(1)}$ for the limit $\alpha_{3i}\to 0$. For this purpose, we compute in the next subsection $\lim_{\alpha_{3i}\to 0}S_\gamma(\alpha_{ij})$ for $\gamma=\emptyset(\{1,3\})\wedge(4,i_4)\wedge\cdots\wedge(n,i_n)$ with $i_p\neq 2$.

5.3. LIMIT FOR $\alpha_{3i}\to 0$

In this subsection, we change numbering from that of the last subsection. Let Γ be an ordered graph with the root set [2] and vertex set $[n]$. We set $\Phi=\prod_{i\ll j}(x_i-x_j)^{\alpha_{ij}}$, and

$$S(\alpha_{ij})=\int_{D([n]/[2],0,1)}\eta_\Gamma\Phi.$$

Before proving Proposition 5.10, we remark the following lemma.

LEMMA 5.9. *Let* $F(x)$ *be a continuous function defined on* $(p,1]$*. Suppose that* $F(x)$ *is integrable on* $(p,p+\epsilon]$*. Then we have*

$$\lim_{\alpha\to 0}\int_p^1\alpha(1-x)^{\alpha-1}F(x)\mathrm{d}x=F(1).$$

Proof. This is a fundamental property of δ-function $\lim_{\alpha\to 0}\alpha(1-x)^{\alpha-1}$. □

PROPOSITION 5.10. (1) *If* $\lim_{\alpha_{2i}\to 0}S(\alpha_{ij})\neq 0$*, then* (1) Γ *contains no edges adjacent to* 2*, or* (2) (2, 3) *is the unique edge adjacent to* 2*.*

(2) *If* (2, 3) *is the unique edge in* Γ *adjacent to* 2*, then* $\lim_{\alpha_{2i}\to 0}S(\alpha_{ij})$ *is equal to* $S_{\Gamma'}(\alpha'_{ij})$*, where* Γ' *is the ordered graph obtained by deleting the edge* (2, 3) *and by replacing the numbering* 3 *of the original edge by the new numbering* 2 *and* $\alpha'_{ij}=\alpha_{ij}$ *if* $i,j\neq 2$ *and* $\alpha'_{2,k}=\alpha_{3,k}$.

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Proof. Suppose that Γ contains an edge adjacent to 2. Let $p \leqslant 3$ be the minimal number such that $(2, p)$ is an edge of Γ. Set

$$F(x_p, \dots, x_n) = \prod_{(pq)\in E_\Gamma, \neq(2,p)} a_{pq} \prod_{1 \leqslant i \leqslant n, p \leqslant j \leqslant n, i<j, (i,j)\neq(2,p)} (x_i - x_j)^{\alpha_{ij}+\epsilon_{ij}} \times \\ \times \int_{\{x_p<\cdots<x_3<1\}} \prod_{1 \leqslant i<j \leqslant p-1} (x_i - x_j)^{\alpha_{ij}+\epsilon_{ij}} dx_{p-1} \cdots dx_3,$$

where $\epsilon_{ij} = -1$ if (i, j) is an edge of Γ, and 0 otherwise. Then

$$\lim_{\alpha_{2p}\to 0} S_\Gamma = \int_{\{0<x_n<\cdots<x_p<1\}} F(x_p, \dots, x_n)\alpha_{2p}(1 - x_p)^{\alpha_{2p}-1}.$$

Therefore $S_\Gamma = 0$ if $p \neq 3$, or there exist at least two p's such that $(2, p)$ is an edge of Γ. If $p = 3$ and there is no edge adjacent to 2 other than $(2, 3)$, then $\lim_{\alpha_{23}\to 0} S_\Gamma = S_{\Gamma'}(\alpha'_{ij})$. □

We define $S_\gamma(\alpha_{ij})$ by $\sum a_\Gamma S_\Gamma(\alpha_{ij})$, where $\gamma = \sum a_\Gamma \Gamma \in \Gamma([2], [n])$.

COROLLARY 5.11. *Let* $\gamma = \emptyset(\{1, 2\}) \wedge (3, i_3) \wedge \cdots \wedge (n, i_n)$.

(1) *If there exists* $k \neq 3$ *such that* $i_k = 2$, *then* $\lim_{\alpha_{2i}\to 0} S_\gamma(\alpha_{ij}) = 0$
(2) *If* $i_3 = 2$ *and* $i_k \neq 2$ *for* $k \neq 3$, *then*

$$\lim_{\alpha_{2i}\to 0} S_\gamma(\alpha_{ij}) = S_{\gamma'}(\alpha'_{ij}),$$

where γ' *is* $\emptyset(\{1, 3\}) \wedge (4, i_4) \wedge \cdots \wedge (n, i_n)$.

Proof of the Main Theorem 3.3. We can proceed by the induction on n. We consider the limit of (5.3) for $\alpha_{3i} \to 0$. Then all the entries of $\lim_{\alpha_{3i}\to 0}(\rho(\Phi(X, Y)))$ are contained in H_α by Corollary 2.2. By Corollary 5.11, all the entries of $\lim_{\alpha_{3i}\to 0} V^{(1)}$ are contained in H_α. Therefore all the entries of $\lim_{\alpha_{3i}\to 0} p(V^{(2)})$ are also contained in H_α. Therefore $S_\gamma(0, 1, \alpha_{ij})$ is an element of H_α for $\gamma = \emptyset(\{1, 2\}) \wedge (3, i_3) \wedge \cdots \wedge (n, i_n)$ under the restriction (R): $i_k \neq 2$ for all k. On the other hand, by the relation (3.4), the restriction (R) is not necessary. This completes the proof. □

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