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Second Order Operators on a Compact Lie Group

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Abstract. We describe the structure of the space of second order elliptic differential operators on a homogenous bundle over a compact Lie group. Subject to a technical condition, these operators are homotopic to the Laplacian. The technical condition is further investigated, with examples given where it holds and others where it does not. Since many spectral invariants are also homotopy invariants, these results provide information about the invariants of these operators.

1 Introduction

In this paper we shall give a description of the structure of the space of invariant, elliptic, second order differential operators on a compact Lie group. Briefly the result is that, when a certain restrictive technical condition holds, any such operator is homotopic through such operators to the Laplacian.

Previously, in [5], first order operators were studied. (For clarity of exposition unless otherwise stated the term "operator" means "invariant elliptic differential operator".) These proved to be Dirac operators, and from this information followed results on spectral symmetry and the η-function. The situation for second order operators is less rigid and more complicated: invariant operators other than the Laplacian exist [2, 7]. (To find out in general what the operators are means handling quadratic maps from a vector space with values in a ring of endomorphisms.) Here we find conditions under which the Laplacian is essentially the only such operator: for certain special types of representations of G, which we call *simple*, the only invariant second order operator turns out (eventually) to be the Laplacian in the sense that there is no second order elliptic symbol perpendicular to that of the Laplacian. (That is, decomposing $\mathcal{S}^2(\mathfrak{g})$ into irreducible representations, no summand except the trivial one gives rise to an elliptic operator.)

The more interesting cases are certainly when there are other elliptic operators. It is for these that one might hope to find a non-vanishing η-invariant. (The question of the non-vanishing of the η-invariant of second order operators was raised by Gilkey [6]. An example of a second order operator with non-vanishing η-invariant was given in [4], but this operator was only pseudo-differential, not differential.) As the Laplacian has a zero η-invariant there are no invariant differential operators with non-vanishing η-invariant in the cases we study.

Let G be a compact, semi-simple connected Lie group. Given a closed subgroup H of G, a representation $\pi\colon G \to \operatorname{Aut} E$ is H-*simple* if $\pi \mid H$ decomposes into distinct

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irreducible representations of H. We shall also refer to such representations as having H-simple multiplicities. We shall often just use the term simple instead of H-simple. The main results are given in section five. Let $\pi\colon G \to \operatorname{Aut} E$ be a T-simple representation and $\mathbf{E}$ be the associated homogeneous bundle. If D is an invariant, elliptic second order operator on $\mathbf{E}$ we can summarize the result as follows.

Theorem 1.1 *If $\mathbf{E}$ is T-simple then any invariant second order elliptic operator is of the form $c\Delta + Q$, where Q is non-elliptic and σQ is perpendicular to $\sigma\Delta$, σ being the symbol and the metric is that on $S^2(\mathfrak{g})$ induced from the Killing form.*

Depending on whether c is positive or negative, $c = 0$ is impossible, there is a further homotopy to either Δ or $-\Delta$.

Theorem 1.2 *The space of such operators is an open cone consisting of two convex subsets: one containing Δ and the other $-\Delta$.*

Notice that Theorem 1.2 can only be stated in the case when π is a real representation. In Section 6 we give two infinite families of representations with simple multiplicities for the groups $SU(n), n \geq 2$. There are other simple representations, and in Section 7 we list some low dimensional examples. These examples show some of these other simple representations and at the same time suggest they are unusual special cases. The earlier sections contain preparatory results. Section 2 establishes the basic notation and shows we need only consider the symbols of the operators. Section 3 establishes the result for the torus, and in Section 4 we show how the result on the torus for symbols passes to that on the group.

2 Second Order Operators on a Lie Group.

Let G be a compact Lie group, and $\pi\colon G \to \operatorname{Aut} E$ be a representation. The homogeneous bundle associated to π is $\mathbf{E} = G \times_\pi E$ and we wish to study invariant second order linear operators $D\colon C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$. Since the prolongation of G is trivial, this is equivalent to studying the symbol of

$$D_\sigma\colon \mathfrak{g} \to \mathcal{S}^2(\mathfrak{g}) \to \operatorname{End} E \tag{2.1}$$

Here the first map $\mathfrak{g} \to \mathcal{S}^2(\mathfrak{g})$ is given by $X \to X^2$, where $\mathcal{S}^2$ denotes the second symmetric power, and the second map, $\tilde{\sigma}\colon \mathcal{S}^2(\mathfrak{g}) \to \operatorname{End} E$, is linear. The symbol given in (2.1) is defined on a fibre of the usual expression involving bundles; the invariance of D yields passage between this fibre expression and that on bundles. Further, this invariance means that σ is equivariant:

$$\sigma(g \cdot X) = \pi(g)\sigma(X)\pi(g)^{-1}, \tag{2.2}$$

for $g \in G$ and $X \in \mathfrak{g}$. Similarly the invariance of D allows it to be given as a linear map on the fibre of a jet bundle:

$$D\colon J^2(E) \to E. \tag{2.3}$$

The triviality of the prolongation of G leads to an equivalent splitting of the jet bundle exact sequence:

$$0 \to \mathcal{S}^2(\mathfrak{g}) \to J^2(E) \to J^1(E) \to 0, \tag{2.4}$$

which, in turn, splits D as a sum $D = \tilde{\sigma} + D_1$. Here D_1 is an invariant first order operator. The condition that D is elliptic is that $\sigma(X)$ is an invertible element of $\operatorname{End}(E)$ if $X \neq 0$. However, while $\sigma(X) = \tilde{\sigma}(X, X)$, for $X \neq 0$, is invertible it does not of course follow that $\tilde{\sigma}(X, Y)$, for $X, Y \neq 0$, need be invertible if $X \neq Y$. Such operators are often natural [3].

3 Elliptic Operators on a Torus

Let $\pi \colon T \to \operatorname{Aut}(V)$ be an irreducible representation of the torus; $\mathbf{V}$, the associated homogeneous vector bundle. Since T is abelian left and right translations on T are essentially the same, $R_g = L_{g^{-1}}$. However, if π is nontrivial there is a difference on $\mathbf{V}$. If we trivialize $\mathbf{V}$ using left translation we have:

$$L_g(x, v) = (gx, v), R_g(x, v) = (xg^{-1}, \pi(g)v). \tag{3.1}$$

Let $\mathfrak{t}$ be the Lie algebra of T. As observed in Section two, we need only study the second order part of D which is given by an element of $S^2(\mathfrak{t}) \otimes \operatorname{End} V$. If $\{X_1, \dots, X_\ell\}$ is a basis of $\mathfrak{t}$ this is given by

$$D = \Sigma_{i,j} X_i X_j \otimes \phi_{ij}. \tag{3.2}$$

Using left translation to trivialize $\mathbf{V}$ a section $s \in C^\infty(\mathbf{V})$ is then given by $s = \Sigma_k \mathfrak{f}_k \otimes v_k$ where $\{v_k\}$ is a basis of V and $f_k \in C^\infty(T)$. Then

$$Ds = \Sigma_{i,j,k} X_i X_j f_k \otimes \phi_{ij}(vk). \tag{3.3}$$

This is automatically left invariant. To be right invariant the condition is that $\phi_{ij}\pi(g) = \pi(g)\phi_{ij}$, *i.e.*, the endomorphisms ϕ_{ij} commute with the action of T. The symbol of D is the map $\sigma \colon \mathfrak{t} \to \operatorname{End} V$ given by

$$\sigma(\Sigma_i \xi_i X_i) = \Sigma_{ij} \xi_i \xi_j \phi_{ij}. \tag{3.4}$$

Invariance of D means that this commutes with the action of T:

$$\sigma(X)\pi(g) = \pi(g)\sigma(X), \tag{3.5}$$

where the fact that T is abelian gives a trivial action on $\mathfrak{g}$.

The irreducible real representation of T can be divided into two cases: $\dim V = 1$ and $\dim V = 2$. If $\dim V = 1$ then σ is real quadratic form on $\mathfrak{t}$. If $\dim V = 2$ then we can decompose $\operatorname{End} V = (\operatorname{End} V)^T \oplus (\operatorname{End} V)^\perp$. Here

$$(\operatorname{End} V)^T = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \tag{3.6}$$

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and

$$(\mathrm{End}\,V)^{\perp} = \left\{ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}. \tag{3.7}$$

Of these, $(\mathrm{End}\,V)^T$ decomposes further into two trivial representations, while $(\mathrm{End}\,V)^{\perp}$ is irreducible. If $A \in (\mathrm{End}\,V)^T$, then $A\pi(g) = \pi(g)A$ for all g. However, if $g \neq 1$ and $B \in (\mathrm{End}\,V)^{\perp}$, then $B\pi(g) \neq \pi(g)B$, for $B \neq 0$. Note further that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3.8}$$

with $\theta = \arctan(b/a)$ so $(\mathrm{End}\,V)^T$ is made up of dilations and rotations.

The condition that D be an elliptic operator is that $\sigma(X)$ is invertible if $X \neq 0$. If $\dim V = 1$ then $\sigma(X)$ is real quadratic form. Hence if $\sigma(X)$ is invertible for all X and $\dim T \geq 2$ then $\sigma(X)$ is either positive definite for all X or negative definite for all X. In either case there is a constant c such that $D = c\Delta$, where Δ is the Laplacian with respect to a suitable metric. If $\dim V = 2$ the situation is more complicated. To ensure that $\mathfrak{t} - \{0\}$ is connected, we shall only consider the case of $\dim T \geq 2$ and use the following notation for a rotation matrix:

$$\mathrm{rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.9}$$

Now $\sigma \colon \mathfrak{t} \to (\mathrm{End}\,V)^T$, by invariance, and so using the decomposition (3.8) there are two real valued functions r, θ of $\mathfrak{t}$ such that

$$\sigma(X) = r(X)\,\mathrm{rot}\,\big(\theta(X)\big). \tag{3.10}$$

Rescale the metric so $r(x) = 1$ for $X \in S^{n-1}$; that is so that $\sigma(X)$ is a pure rotation for unit length X. Using the fact that σ is a quadratic map yields the following.

Lemma 3.1 *Let X_1 and X_2 be two orthogonal unit vectors in $\mathfrak{t}$, and set $X_t = \sin tX_1 + \cos tX_2$ then $\theta(X_t)$ is one of the three possibilities: $\theta(X_t) = 0, \theta(X_t) = 2t$ or $\theta(X_t) = -2t$.*

Proof This is an explicit calculation. First normalize σ so that $\sigma(X_1) = I$. Now define r, θ_1 and θ_2 by

$$\sigma(X_1, X_2) = r\,\mathrm{rot}(\theta_1), \sigma(X_2) = \mathrm{rot}(\theta_2). \tag{3.11}$$

Let $f(t) = \sigma(X_t)$; then since σ is quadratic we have

$$f(t) = \sin^2 tI + 2r \sin t \cos t\,\mathrm{rot}(\theta_1) + \cos^2 t\,\mathrm{rot}(\theta_2). \tag{3.12}$$

Now set $g(t) = \det f(t)$. Since our normalization gives $f(t) = \mathrm{rot}(\theta(t))$ for some $\theta(t)$, we have that $g(t) = 1$ for all t thus $g'(t) = 0$ for all t. A computation of $g'(t)$ for $t = 0, \pi/2$ and $\pi/3$ yields the following system of three equations:

$$\begin{gathered} 4r\cos\sigma_1\cos\sigma_2 + 4r\sin\sigma_1\sin\sigma_2 = 0 \\ -4r\cos\sigma_1 = 0 \\ \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{3}\cos\sigma_2 - 2r\cos\sigma_2 - r^2\sqrt{3} - 2r\sin\sigma_1\sin\sigma_2 = 0. \end{gathered} \tag{3.13}$$

There are 3 solutions to these:

$$\begin{array}{rrll} \text{(i)} & r = 0, & \cos\sigma_2 = 1, & \\ \text{(ii)} & r* = 0, & \cos\sigma_1 = 0, & \sin\sigma_1 = 1, \\ \text{(iii)} & r = 1, & \cos\sigma_1 = 0, & \sin\sigma_1 = -1 \end{array} \tag{3.14}$$

which correspond to the three solutions $\theta(X_t) = 0, 2t, -2t$. ■

Corollary 3.2 *If* $\dim T \geq 3$ *then* $\sigma(X) =\| X \|^2$, *for some norm on* $\mathfrak{t}$.

Proof The map $\sigma\colon S^{n-1} \to S^1$ gives rise to a map of homotopy groups

$$\sigma_*\colon \pi_1(S^{n-1}) \to \pi_1(S^1).$$

In the case $\theta(X_t) = 0$ this is the trivial map. If $\theta(X_t) = \pm 2t$ then 2 (resp., -2) is in the image of σ_*. However, for $n \geq 3$ the fundamental group $\pi_1(S^{n-1}) = 0$ and this is impossible. Thus $\sigma(X) = \sigma(X_1) \| X \|^2$ when the normalization used in Theorem 3.1 is reversed. Since σ is invariant, $\sigma(X_1)$ commutes with the action of T. Thus $\sigma(X_1) = cI$ for a constant c, which is then incorporated into the norm. Note that the norm may be either positive definite, if $c > 0$, or negative definite, $c < 0$. ■

Corollary 3.3 *If* $\dim T = 2$ *and* T *has an action by a Weyl group of a rank* 2 *compact semisimple Lie group induced by identifying T as the maximal torus then, for* σ *invariant under this additional action,* $\sigma(X) =\| X \|^2$ *for some norm on* $\mathfrak{t}$.

Proof Since $\dim T = 2$ we can choose $\{X_1, X_2\}$ to be an orthonormal basis of $\mathfrak{t}$. We shall show $\sigma(X_t) = \mathrm{rot}(2t)$ is not invariant under the action of a Weyl groups. The case $\sigma(X_t) = \mathrm{rot}(-2t)$ is similar. ■

Let W be the Weyl group, then there is a matrix valued function $M(W)$ so that $\sigma(W \cdot X_t) = M(W)^{-1}\sigma(X_t)M(W)$. Thus $\sigma(W \cdot X_t)$ and $\sigma(X_t)$ have the same eigenvalues. The eigenvalues of $\sigma(X_t)$ are:

$$\cos(2t) + i\sin(2t), \cos(2t) - i\sin(2t). \tag{3.15}$$

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There are three compact semisimple rank 2 lie groups: $SU(3)$, Spin(5) and G_2. A case by case check shows that the eigenvalues of (3.15) are not always constant.

(1) $SU(3)$. The Weyl group has element sending X_0 to $X_{\pi/3}$.

(2) Spin(5). The Weyl group has an element sending X_t to $X_{\pi/2-t}$.

(3) G_2. The Weyl group has an element sending X_0 to $X_{2\pi/3}$.

4 Passage Between the Group and the Torus

Let G be a compact semisimple Lie group and T its maximal torus. Let $\pi\colon G \to \operatorname{Aut}(E)$ be a representation and $\mathbf{E}$ the associated homogenous bundle with fibre E. If D is an invariant second order elliptic differentiable operator then its symbol $\sigma\colon g \to \operatorname{End} E$ satisfies

$$\sigma(\operatorname{Ad} g \cdot X) = \pi(g)\sigma(X)\pi(g^{-1}) \tag{4.1}$$

and $\sigma(X)$ is invertible for $X \neq 0$. Now under the adjoint action of $\mathfrak{g}$ each element of $\mathfrak{g}$ is congugate to an element of $\mathfrak{t}$. A consequence of this is:

Lemma 4.1 *The symbol σ is determined by its restriction to* $\mathfrak{t}$.

Proof Let $Y \in \mathfrak{g}$ then there is $g \in G$ and $X \in t$ such that $Y = \operatorname{Ad} g X$. Thus $\sigma(Y) = \pi(g)\sigma(X)\pi(g^{-1})$. ■

The converse of Lemma 4.1 is not quite true. If $\sigma\colon t \to \operatorname{End} E$ is a Weyl group invariant map then σ extends to an invariant map on $\mathfrak{g}$. However, even if σ is a quadratic, this extension need not be.

Example 4.2 The group Spin(5) has 4 positive roots: α, β, γ and δ with γ the highest root. For a root v let S_v be reflection in the hyperplane (in this case a line) of v. Consider the representation with highest weight ρ and denote the representation space by E. Then $\dim E = 16$ and there are 8 weight spaces, each of dimension 2, with weights $w\rho$ for w in the Weyl group.

Define $q\colon \mathfrak{t} \to \operatorname{End} \mathbb{R}^2$ by $q(x) = \| X \|^2 \operatorname{rot}(2\theta)$, where θ is the angle X makes with the initial line in $\mathfrak{t} \simeq \mathbb{R}^2$. Then define q_w by the following table.

ω	$1\&S_\alpha S_\gamma$	$S_\alpha\&S_\gamma$	$S_\beta\&S_\delta$	$S_\beta S_\alpha\&S_\gamma S_\delta$
$q_\omega(X)$	$q(X)$	$q(X)^t$	$-q(X)^t$	$-q(X)$

Now let $\sigma\colon t \to \operatorname{End} E$ by $\sigma(X) \mid E_{wp} = q_w(X)$. Then σ is a Weyl group invariant quadratic map on it, but its extension to Spin(5) is not quadratic. If the extension were quadratic it would factor through the second symmetric power: $\mathcal{S}^2(\mathfrak{g})$. An analysis of the weights of $\mathcal{S}^2(\mathfrak{g})$ as a representation space of Spin(5) shows this is not possible.

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If we restrict π to the maximal torus it can be decomposed into isotypic spaces E_i and irreducible spaces V_i:

$$E = \oplus_i E_i = \oplus_i m_i V_i, \text{ where } E_i = m_i V_i. \tag{4.2}$$

The spaces E_i are called weight spaces and the m_i are integers: the multiplicities.

Definition 4.3 The representation π has *s*imple multiplicities if $m_i = 1$ for all i.

If π has simple multiplicities we shall also say that the bundle **E** has simple multiplicities.

Theorem 4.4 *If E has simple multiplicities and D is elliptic, then there is a constant c and an invariant quadratic map $q\colon g \to \operatorname{End} E$ such that $\sigma(X) = c\|X\|^2 I + q(X)$, where q is not invertible for some $X \neq 0$. Further, if $\sigma(X) = c\|X\|^2 + tq(X)$ then $\sigma_t(X)$ is invertible for all $X \neq 0$ and $0 \leq t \leq 1$.*

Proof Since σ is G-invariant, it is also T-invariant, and so preserves the T-isotypic subspaces. These are irreducible, by simple multiplicities, and so consider each component $\sigma_i\colon \mathfrak{t} \to \operatorname{End} V_i$.

If rank $G \geq 3$, apply Corollary 3.2 to obtain $\sigma_i(X) = \|X\|_i^2 I$. If rank $G = 2$ consider the action of the Weyl group. When the Weyl group acts trivially on the weight space apply Corollary 3.3 to obtain $\sigma_i(X) = \|X\|_i^2 I$. When the Weyl group acts nontrivially it is generated by reflections in root lines (the root hyperplanes in the rank 2 case) and it permutes the weight spaces. Suppose the root line of root ν makes an angle ϕ with the initial line. If $\sigma_i(X) = \|X\|_i^2 \operatorname{rot}(2\theta)$ then, by some elementary plane geometry, $\sigma_i(S_\nu X) = \|X\|_i^2 \operatorname{rot}(4\phi - 2\theta)$. Now $\sigma_i(S_\nu X) = \sigma_j(X)$ for some j, thus $\operatorname{rot}(4\phi - 2\theta) = \operatorname{rot}(2\theta)$ or $\operatorname{rot}(-2\theta)$. Hence, θ is a multiple of $\pi/4$. This does not hold for the groups $SU(3)$ and G_2, so for these groups $\sigma_i(X) = \|X\|_i^2 I$. If the weight space E_i corresponds to a singular weight, there is an element of the Weyl group which leaves E_i fixed. This yields $\sigma_i(X) = \|X\|_i^2 I$ for singular weights. The symbol factors $\sigma\colon \mathfrak{g} \to \mathcal{S}^2(\mathfrak{g}) \to \operatorname{End} E$. Now for Spin(5) we decompose $S^2(g) = E_{2\gamma} \oplus E_\beta \oplus E_0$. Since each of these weights is singular, $\sigma_i(X) = \|X\|_i^2 I$ for the group Spin(5). ■

Some of the norms $\| \ \|_i$ may be positive-definite and some negative-definite. Let $E^+ = \oplus E_i$ over those i such that $\| \ \|_i$ is positive definite. Suppose that T' is another maximal torus with Lie algebra $\mathfrak{t}'$. We say that T' touches T if $\dim(\mathfrak{t} \cap \mathfrak{t}') \geq 1$. Notice T touches itself in all cases and only itself in the rank 1 case. Let $J = \bigcup T'$ and $\mathfrak{j} = \bigcup \mathfrak{t}'$, with the union over all tori T' that touch T.

Lemma 4.5 *If rank $G \geq 2$ then the span of $\mathfrak{j}$ is $\mathfrak{g}$.*

Proof Let $\mathfrak{g} = \mathfrak{t} \oplus \Sigma \mathfrak{g}_\alpha$ be the root space decomposition and let $X \in \mathfrak{g}_\alpha$ for some α. Then there is $Y \in \mathfrak{t}$ such that $\alpha(y) = 0$ and $[X, Y] = 0$ with $Y \neq 0$. Thus there is a maximal torus T_α with Lie algebra $\mathfrak{t}_\alpha$ such that $X, Y \in \mathfrak{t}_\alpha$. Hence T_X touches T, so $\mathfrak{g}_\alpha \subset \mathfrak{j}$ which proves the lemma. ■

Thus if $H = \{g \in G : \dim(\operatorname{Ad} g \cdot \mathfrak{j} \cap \mathfrak{j}) \geq 1\}$ then $J \subset H$ and H generates G since $\mathfrak{j}$ spans $\mathfrak{g}$.

Let $E^+(X)$ be the span of the eigenspaces of $\sigma(X)$ corresponding to positive eigenvalues. Then $E^+(X) = E^+(Y)$ for $X, Y \in \mathfrak{t}$ and hence also for $X, Y \in j$. Let $E^+ = E^+(X)$ for a non-zero X. Then E^+ is invariant under the action of any element of H and hence invariant under the action of G. As E is irreducible, either $E^+ = 0$ or $E^+ = E$. Replacing σ by $-\sigma$ if necessary, we can suppose all $\| \, \|_i$ are positive definite.

Now $\| \, \|_i$ is invariant under the torus, but not necessarily under the whole group, nor even under $N(T)$. Thus $\| \, \|_i$ need not to be the Killing from inner product, see Example 5.7. However, since T is compact and $\| \, \|_i$ is positive definite we have that there exist $c_i > 0$ and $d_i > 0$ such that for all $X \in \mathfrak{t}$:

$$c_i \|X\|^2 \leq \|X\|_i^2 \leq d_i \|X\|^2, \tag{4.3}$$

with equality achieved for some values of X. Let $c = \min(c_i)$ and set $q(X) = \sigma(X) - c\|X\|^2 I$. Then immediately

$$\sigma(X) = c\|X\|^2 I + q(X). \tag{4.4}$$

The non-invertibility of $q(X)$ for some X follows from the achievement of equality for $c_i\|X\|^2 \leq \|X\|_i^2$ in (4.3) for $c_i = c$ minimal. Just as immediately the invertibility of $\sigma_t(x) = c\|X\|^2 + tq(x)$ for $t \geq 0$ also follows from the inequality (4.3).

The space of elliptic symbols is the union of two subspaces: those with positive eigenvalues and those with negative. We shall restrict attention to the positive case; the negative one is similar and can be obtained from the positive one by changing signs.

Corollary 4.6 *The space of positive elliptic symbols is convex.*

Proof This follows immediately from (4.3). ■

The space of second order symbols is

$$\mathcal{S}^2(\mathfrak{g})^* \otimes \operatorname{End} E \subset \operatorname{End}(\mathfrak{g} \otimes E), \tag{4.5}$$

and so has an invariant inner product. The trace is a map

$$\operatorname{tr}\colon \operatorname{End}(\mathfrak{g} \otimes E) \to \mathbb{R}, \tag{4.6}$$

which is a positive constant times the projection to the trivial part representing Δ, the Laplacian. Denote the restriction of the trace to symbols on the torus by tr_T. The symbol of Δ is $\|X\|^2 I$ and $\operatorname{tr}(\|X\|^2 I) = l \dim E$. For a positive elliptic symbol σ, from (4.3), we have

$$\operatorname{tr}_T \sigma \geq cl \dim E, \tag{4.7}$$

where $c = \min(c_i)$ as before. Set $k = \operatorname{tr}_T \sigma / l \dim E$ then $k \geq c > 0$. Let $q(X) = \sigma(X) - k\|X\|^2 I$ then, by construction, $\operatorname{tr}_T q' = 0$. Thus, from (4.7), q' is not elliptic.

Theorem 4.7 *The symbols $\sigma_t(X) = k\|X\|^2 I + tq'(X), 0 \le t < 1$, form a family of elliptic symbols with $\operatorname{tr}_T \sigma_t = kl \dim E$, independent of t joining σ and $k\|X\|^2 I$.*

Proof By construction $\sigma_t(X)$ clearly joins $\sigma(X)$ and $k\|X\|^2 I$. Convexity shows that σ_t is elliptic, and $\operatorname{tr}_T \sigma_t = kl \dim E$ follows from $\operatorname{tr}_T q' = 0$. ■

This result gives an orthogonal homotopy between σ and the symbol of $k\Delta$. However, the symbols σ_t, which are elliptic for $0 \le t \le 1$, need not be elliptic for all $t \le 0$ unlike the result of Theorem 4.4. Example 5.7 illustrates this point.

5 The Structure of the Space of Second Order Operators and A Rank 2 Example

Let $\pi \colon G \to \operatorname{Aut} E$ be an irreducible representation with simple multiplicities, and D be an elliptic invariant second order operator on E. The symbol of D is σ_D and by Theorems 4.4 and 4.7 we have

$$\sigma_D(X) = c\|X\|^2 I + q(X) = k\|X\|^2 I + q'(X). \tag{5.1}$$

Let $Q = D - c\Delta$ and $Q' = D - k\Delta$ then Q and Q' are invariant second order operators with symbols q and q' which are non elliptic. Then if $D_t = c\Delta + tQ$ and $D'_t = k\Delta + tQ'$ the following is immediate.

Theorem 5.1 (a) *The operator D_t is invariant and elliptic for all $t \ge 0$ and D'_t for $0 \le t \le 1$.*

(b) *The eigenvalues of $q(X)$ and $q'(X)$ are either all non-negative when $c > 0$ or all non-positive when $c < 0$.*

(c) *D_t (respectively, D'_t) for $0 \le t \le 1$ is a homotopy through invariant elliptic operators between D and $c\Delta$ (respectively, $k\Delta$). The family D'_t gives a homotopy orthogonal to Δ.*

We next observe that Δ is not homotopic to $-\Delta$. If $\dim E$ is even then there is no homotopy through invariant elliptic operators. To see this we restrict to the maximal torus T and a weight space E_α. Then the homotopy would give rise to a function $c \colon [0,1] \to \mathbb{R}$ such that $c(t)\Delta + Q_t$ is the path form Δ to $-\Delta$. That is $c(0) = 1, c(1) = -1$ and c is a continuous non-vanishing function. This is impossible and so there is no such homotopy between Δ and $-\Delta$.

Proposition 5.2 *The operator D is homotopic to either Δ or $-\Delta$ but not both.*

Using Propositions 5.1 and 5.2 it is straight forward to describe the space of invariant elliptic second order differentials operators. Let $\mathbb{R}^+$ be the positive real numbers, then the space in question if $\mathbb{R}^+ \times U$ for a suitable U. Now the space U is not uniquely determined by this, but one choice can be given.

Theorem 5.3 $U = U^+ \cup U^-$, *where* $U^\pm = \{\pm\Delta + tQ : t \ge 0$ *and Q is as in Proposition* 5.1$\}$.

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Corollary 5.4 *The space of second order, elliptic, invariant operators is the disjoint union of two simply connected subspaces.*

Remark 5.5 The space U^+ is a half space in the sense that if both $\Delta + Q_1$ and $\Delta + Q_2$ are in U^+ so is $\Delta + (Q_1 + Q_2)$. A similar statement is true for U^-.

Remark 5.6 We can say that the representation $\pi\colon G \to E$ has $N(T)$ simple multiplicities if $m_i = 1$ for all i where $\pi|N(T) = \bigoplus m_i\pi_i$ is the decomposition into irreducible representations of the normalizer of the torus, $N(T)$. The only difference between a T-simple and an $N(T)$-simple representation is in the zero-weight space, as exemplified by the adjoint representation. The result does not extend to $N(T)$-simple representations as the following rank 2 example shows

Let G be a rank 2 Lie group. Fix a maximal torus T and let e_1, e_2 be a basis for $\mathfrak{t}$. Let

$$g = t \oplus \Sigma\mathfrak{g}_\alpha. \tag{5.2}$$

Let D be an invariant operator on vector fields with symbol

$$\sigma_D(v)\colon \mathfrak{g} \to \mathfrak{g}, \text{ for } v \in \mathfrak{g}. \tag{5.3}$$

Because every orbit under the adjoint map intersects $\mathfrak{t}$, if we know the eigenvalues of $\sigma_D(v)$ for $v \in \mathfrak{t}$ we know them everywhere. Let $D = d^*d + kdd^*$ where k is a constant; one of the operators considered in [2, 7]. We have that $\sigma_d(v) = \in(v)$, exterior multiplication by v and $\sigma_{d^*}(v) = i(v)$, interior multiplication by v. Thus, on a root space $\mathfrak{g}_\alpha$ we have $\sigma_{d^*}(v) = 0$ and $\sigma_{d^*d}(v) = \|v\|^2$, for $v \in t$.

Hence $\sigma_D(v)|\mathfrak{g}_\alpha = \|v\|^2 I$ for all k. On $\mathfrak{t}$ the situation is different and, writing $v = (x, y)$, the result is

$$\begin{aligned}
\sigma_d(v)(ae_1 + be_2) &= (xb - ya)e_1 \wedge e_2\\
\sigma_{d^*}(v)(ae_1 + be_2) &= xa + yb\\
\sigma_{d^*d}(v)(ae_1 + be_2) &= (y^2a - xyb)e_1 + (x^2b - xya)e_2\\
\sigma_{dd^*}(v)(ae_1 + be_2) &= (x^2a + xyb)e_1 + (xya + y^2b)e_2.
\end{aligned}$$

Consequently we have that, as a matrix,

$$\sigma_{d^*d+kdd^*}(v) = \begin{pmatrix} kx^2 + y^2 & (k-1)xy \\ (k-1)xy & x^2 + ky^2 \end{pmatrix},$$

and this has eigenvalues $k(x^2 + y^2)$ and $x^2 + y^2$. Thus the eigenvalues of $\sigma_D(v)$ are $k\|v\|^2$, with multiplicity 1, and $\|v\|^2$, with multiplicity $\dim G - 1$.

There is indeed an invariant elliptic operator orthogonal to the Laplacian.

Example 5.7 Let R be the invariant first order operator defined by

$$R(f \otimes s) = \sum_i X_i(f) \otimes \pi(X_i)s, \tag{5.4}$$

where $\{X_i\}$ is an orthonormal basis for $\mathfrak{g}$. Then R is an invariant first order operator with symbol $\sigma_R(X) = \pi(X)$. Thus, except for even dimensional representations over $SU(2)$, R is not elliptic. Set $D = \Delta + R^2$, take π to be the adjoint representation and let $n = \dim G$. Then the T-isotypic spaces are the root spaces and $\|X\|_i^2 = \|X\|^2 + \alpha(X)^2$, where α is the ith root and $\| \ \|^2$ is the Killing form norm. Then the c_i of (4.3) are all 1.

Thus $c = 1$. On the other hand

$$\operatorname{tr}_T \sigma_D = \sum_X \sum_\alpha (1 + \alpha(X)^2) = (n+1)1, \tag{5.5}$$

where the first sum is over X in an orthonormal basis of $\mathfrak{t}$ and the second over all the weights of the adjoint representation, including zero. Thus $k = (n+1)/n$. Hence, in Theorem 5.1

$$Q = R^2 \text{ and } Q' = R^2 - (1/n)\Delta. \tag{5.6}$$

So $D_t = \Delta + tR^2$ and is elliptic for all $t \geq 0$, and $D_t' = (n+1-t)\Delta/n + tR^2$. While D_t' is elliptic for $0 \leq t \leq 1$, it is not elliptic for $t = n+1$. However, D_t' is orthogonal to $(n+1)\Delta/n$.

6 Representation with Simple Multiplicities.

We shall restrict ourselves to the group $SU(\ell+1)$ for $\ell \geq 2$. Following [1, Planche 1] we have $\mathfrak{t} \subset \mathbb{R}^{\ell+1}$ where $\mathfrak{t} = \{(x_1, \dots, x_{\ell+1}) : \Sigma_i x_i = 0\}$. Let $\{e_1, \dots, e_{\ell+1}\}$ be the standard basis for $\mathbb{R}^{\ell+1}$ then the Weyl group is $W = S_{\ell+1}$, the symmetric group on $\ell+1$ letters, which acts by permuting $\{e_i\}$. The positive roots are $e_i - e_j$, $i < j$, and a root basis is

$$\alpha_i = e_i - e_{i+1} \quad 1 \leq i \leq \ell. \tag{6.1}$$

The fundamental weights are

$$w_i = \Big((\ell+1-i) \textstyle\sum_{j=1}^{i} e_j - i \displaystyle\sum_{j=i+1}^{\ell+1} e_j \Big) / (\ell+1). \tag{6.2}$$

Lemma 6.1 *The representations whose highest weight is a fundamental weight have simple multiplicities.*

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Proof Let W_i be the subgroup of W which stabilizes w_i. Then $W_i = S_i \times S_{\ell+1-i}$ and so the orbit of w_i under W has

$$|Ww_i| = \frac{(\ell+1)!}{i!(\ell+1-i)!} = \binom{\ell+1}{i} \tag{6.3}$$

elements.

Using the Weyl dimension formula we calculate that

$$\begin{aligned}\dim \pi_{wi} &= \prod_{\alpha>0} \frac{\langle \rho + w_i, \alpha\rangle}{\langle \rho, \alpha\rangle} = \prod_{1\le r\le i\le s\le \ell+1} \Bigl(1 + \frac{1}{s-r}\Bigr) \\ &= \prod_{r=1}^{i} \prod_{k=i+1-r}^{l+1-r} \Bigl(1+\frac{1}{k}\Bigr) = \frac{(\ell+1)!}{i!(\ell+1-i)!}.\end{aligned} \tag{6.4}$$

Now $\dim \pi_{w_i}$ is the number (counting multiplicities) of weights of π_{w_i}. On the other hand each element of W_{w_i} is a weight of π_{w_i} with multiplicity 1. Thus since $|W_{w_i}| = \dim_{w_i}$ the weights of π_{w_i} are precisely the elements of W_{w_i} and each occurs with multiplicity 1. ■

Lemma 6.2 *The tensor product $\pi_{w_1} \otimes \pi_{nw_1}$ decomposes into irreducible representations as $\pi_{w_i} \otimes \pi_{nw_1} = \pi_{(n-1)w_1} \oplus \pi_{(n-1)w_1+w_2}$.*

Proof From general results and Kostant's multiplicity formula in particular we have

$$\pi_\mu \otimes \pi_\lambda = \Sigma m_\nu \pi_\nu, \tag{6.5}$$

where $\nu = \mu + \lambda$ occurs with multiplicity 1, $m_{\mu+\lambda} = 1$, and each other ν occuring has the form $\nu = \alpha + \lambda$, where α is a weight of π_μ. Of course not all $\alpha + \lambda$ which are dominant necessarily need occur. By inspection the dominant weights of the form $\alpha + nw_1$ for α a weight of π_{wi} are $(n+1)w_1$ and $(n-1)w_1 + w_2$. Thus we have a decomposition

$$n_{w_i} \otimes \pi_{nw_1} = \pi_{(n+1)w_1} \oplus m\pi_{(n-1)w_1+w_2} \tag{6.6}$$

for some multiplicity m. Using the Weyl dimension formula to compute dimensions gives:

$$\begin{aligned}\dim(\pi_{w_i} \otimes \pi_{nw_1}) &= (\ell+1)\binom{\ell+n}{\ell-1}, \\ \dim(\pi_{(n+1)w_1}) &= \binom{\ell+n+1}{\ell}, \\ \dim(\pi_{(n-1)w_1+w_2}) &= n\binom{\ell+n}{\ell-1}.\end{aligned} \tag{6.7}$$

This yield $m = 1$. ■

For a representation π we denote the nth symmetric product by $\mathcal{S}^n(\pi)$.

Lemma 6.3 $\mathcal{S}^n(\pi_{w1}) = \pi_{nw_1}$.

Proof We proceed by induction on n. For $n = 1$ the result is immediate. For $n = 2$ we see that $2w_1$ is a highest weight for both $\mathcal{S}^2(\pi_{w_1})$ and π_{2w_1} furthermore these both have dimension $\frac{1}{2}(\ell+1)(\ell+2)$.

For the inductive step observe $\mathcal{S}^{n+1}(\pi_{w_1}) \subset \pi w_1 \otimes \mathcal{S}^n(\pi_{w_1})$. Thus $\mathcal{S}^{n+1}(\pi_{w_1}) \subset \pi_{(n+1)w_1} \oplus \pi_{(n-1)_{w_1+w_2}}$. Now since $\mathcal{S}^{n+1}(\pi_{w_1})$ is a representation space there are four possibilities.

$$\text{(6.8)} \qquad \begin{aligned} &\text{(a)} \quad S^{n+1}(\pi_{w_1}) = \pi_{(n+1)w_1} \oplus \pi_{(n-1)w_1+w_2},\\ &\text{(b)} \quad S^{n+1}(\pi_{w_1}) = \pi_{(n+1)w_1},\\ &\text{(c)} \quad S^{n+1}(\pi_{w_1}) = \pi_{(n-1)w_1+w_2},\\ &\text{(d)} \quad S^{n+1}(\pi_{w_1}) = \{0\}.\end{aligned}$$

Elementary linear algebra eliminates a and d. We see b is true rather than c by considering highest weights. ∎

Corollary 6.4 $\dim \mathcal{S}^n(\mathbb{C}^{\ell+1}) = \binom{l+n}{\ell}$.

Remark 6.5 It is possible to give an independent proof of this corollary by using induction and the formula:

$$\text{(6.9)} \qquad \mathcal{S}(C^{\ell+1}) = \Sigma_{k=0}^{n} \mathcal{S}^k(\mathbb{C}^\ell).$$

Remark 6.6 The results here are only true for the group $SU(\ell+1)$.

Theorem 6.7 *The weights of π_{nw_1} have simple multiplicities.*

Proof First define

$$\text{(6.10)} \qquad \beta_k = \Big((\ell+1)c_k - \sum_{i=1}^{\ell+1} c_i\Big)/(\ell+1),$$

so $\beta_1 = w_1$. Then $\beta_1, \dots, \beta_{\ell+1}$ are the weights of π_{w_1}. Thus, since $\pi_{nw_1} = \mathcal{S}^n(\pi_{w_1})$ the weights of π_n are $\sum_{i=1}^{\ell-1} a_i\beta_i$ with $a_i \in \mathbb{Z}, a_i > 0$ and $\sum_{i=1}^{\ell+1} a_i = n$.

If $\Sigma a_i\beta_i = \Sigma b_i\beta_i$ then to show π_{nw_1} has simple multiplicities we need to show $a_i = b_i$. Set $c_i = a_i - b_i$ then we have

$$\text{(6.11)} \qquad \sum_{i=1}^{\ell+1} c_i\beta_i = 0, \quad \sum_{i=1}^{\ell+1} c_i = 0$$

Now $\beta_{\ell+1} = -\beta_1 \cdots - \beta_\ell$ and $\{\beta_1, \dots, \beta_\ell\}$ is a linearly independent set. Thus $\sum_{i=1}^{\ell}(c_i - c_{\ell+1})\beta_i = 0$ and so $c_i = c_{\ell+1}$ for all i. Hence $\sum_{i=1}^{\ell+1} c_i = (\ell+1)c_{\ell+1} = 0$ so $c_i = 0$ for all i. ∎

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Corollary 6.8 *For the group $SU(\ell+1)$, the invariant elliptic second order differential operators on the homogenous bundle associated to the representation π_{nw_1} are homotopic to Δ, where c is a constant and Δ is the Laplacian.*

Remark 6.9 Since π_{w_ℓ} is the contragradiant representation to π_{w_1} all the results of this section apply equally to representation with w_ℓ replacing w_1 (and for Lemma 6.2 $\pi_{w_\ell} \otimes \pi_{nw\ell} = \pi_{(n+1)w\ell} \oplus \pi_{(n-1)}w_\ell + w_{\ell-1}$).

7 Some Examples

In any particular case the multiplicities of a representation can be explicitly computed. This is made easier by using a suitable program on a computer.

The following table was calculated by using the program LiE, written by Marc. A. A. vanLeeuwen, Arjeh M. Cohen and Bert Lisser. It was available at the web site http://wallis.uni-potiers.fr/ maavll/LIE/ The results are stated using the notation of [1, Planches].

group	highest weight with simple multiplicity	highest weights not with simple multiplicity
A_4	w_1 w_2 w_3 w_4	$2w_2$ $2w_3$
B_4	w_1 w_4	$2w_1$ w_2 w_3 $2w_4$
C_4	w_1	$2w_1$ w_2 w_3 w_4
D_4	w_1 w_3 w_4	$2w_1$ w_2 $2w_3$ $2w_4$
E_6	w_1 w_6	$2w_1$ w_2 w_3 w_4 w_5 $2w_6$
E_7	w_7	w_1 w_2 w_3 w_4 w_5 w_6 $2w_7$
E_8		w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8
F_4		w_1 w_2 w_3 w_4
G_2	w_1	$2w_1$ w_2

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