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On the construction of soluble groups satisfying the minimal condition for normal subgroups

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A general method is described for constructing examples of soluble groups whose normal subgroups form a well-ordered chain under the ordering of inclusion. This method is a variant of one introduced in a recent paper by Heineken and Wilson. Each of the resulting groups is obtained by an embedding procedure from a pair of iterated wreath products $A_1 \text{ wr } A_2 \text{ wr } \ldots \text{ wr } A_n$, $B_1 \text{ wr } B_2 \text{ wr } \ldots \text{ wr } B_n$, where the constituent groups A_i, B_i are each either cyclic of prime power order or quasicyclic. Here n may be chosen arbitrarily, and the choice of constituent groups is subject only to a condition on the sequences of prime numbers that may occur as orders of elements in the groups

$$A_1, B_1, A_2, B_2, \ldots, A_n, B_n$$

respectively. The construction is applied to give certain examples which illustrate the limitations of results on particular classes of soluble groups satisfying the minimal condition for normal subgroups obtained in recent papers by Hartley, McDougall, and the present author.

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1.

In their paper [3] Hartley and McDougall have given a detailed classification of the metabelian groups that satisfy min-n (the minimal condition for normal subgroups) and have no proper subgroups of finite index. As every group satisfying min-n is a finite extension of a group that satisfies min-n and has no proper subgroups of finite index (see [13]), this classification gives a reasonably complete survey of the metabelian groups satisfying min-n . Moreover, it is shown in [12] that most of the results of McDougall's paper [5], which form the basis for the work of [3], can be extended, with suitable modifications, to the class of metanilpotent groups satisfying min-n . However, progress with the study of more general classes of soluble groups satisfying min-n is likely to be slow until we have more examples to illustrate the complexities that can arise. The aim of the present paper is to describe a method of constructing examples of this type, and to use the examples to illustrate some of the limitations of the methods of [5] and [12].

One simple method of constructing examples of soluble groups is by means of wreath products, and in the theory of soluble groups satisfying max-n (the maximal condition for normal subgroups) wreath products provide a convenient source of examples, as Hall showed in his well-known paper [1]. One of Hall's results shows that if $A_1, A_2, \ldots, A_n$ is a sequence of polycyclic groups, then the iterated wreath product

$$A_1 \text{ wr } A_2 \text{ wr } \ldots \text{ wr } A_n$$

is a soluble group satisfying max-n . (Here, and throughout the paper, X wr Y denotes the restricted standard wreath product of the groups X and Y , and unbracketed wreath products with several factors are to be interpreted as "left-normed", so that, for example, X wr Y wr Z stands for $(X \text{wr} Y)$ wr Z .) Unfortunately the analogous procedure of forming iterated wreath products of soluble groups satisfying min (the minimal condition for subgroups) does not lead directly to any interesting examples of soluble groups satisfying min-n . In fact it is not hard to show that the wreath product A_1 wr A_2 of two non-trivial soluble groups can only satisfy min-n if A_2 is finite. Nevertheless our main aim here is to show that iterated wreath products of soluble groups satisfying min do occur as subgroups of soluble groups satisfying min-n, and to describe a technique for embedding iterated wreath products of this kind in soluble groups satisfying min-n.

Before stating the theorem that underlies this construction, we introduce some terminology. If a and b are relatively prime positive integers then $\operatorname{ord}(a, b)$ will denote the *order* of a modulo b; that is, the least positive integer m such that $b \mid a^m - 1$. By an *admissible sequence* of prime numbers we shall mean a finite sequence of prime numbers $p_1, p_2, \ldots, p_n$, where $n \geq 2$, such that

(i) $p_i \neq p_{i+1}$ for each $i \leq n - 1$, and

(ii) if $n \geq 3$, then $p_{i+2} \nmid \operatorname{ord}(p_i, p_{i+1})$ for each $i \leq n - 2$.

It follows from results of elementary number theory (see, for example, Theorem 88 of [2]) that $\operatorname{ord}(p_i, p_{i+1})$ always divides $p_{i+1} - 1$; hence condition (ii) is satisfied, in particular, when $p_{j+1} \nmid p_j - 1$ for each j with $2 \leq j \leq n - 1$.

Our main result will be:

THEOREM A. *Let n be an integer with $n \geq 2$, and let $p_1, p_2, \ldots, p_n$ be an admissible sequence of primes. If A_i is a non-trivial locally cyclic p_i-group, for $i = 1, 2, \ldots, n$, then there is a soluble group G of derived length n, generated by isomorphic copies of the wreath products*

$$A_1 \operatorname{wr} A_3 \operatorname{wr} A_5 \operatorname{wr} \ldots \operatorname{wr} A_{n-1+\varepsilon}$$

and

$$A_2 \operatorname{wr} A_4 \operatorname{wr} A_6 \operatorname{wr} \ldots \operatorname{wr} A_{n-\varepsilon} ,$$

where $\varepsilon = 0$ if n is even and $\varepsilon = 1$ if n is odd, such that

(i) the normal subgroups of G form a well-ordered chain under the ordering of set theoretic inclusion (and, a fortiori, *G satisfies* min-n *), and*

(ii) the factor $G^{(i)}/G^{(i+1)}$ of the derived series of G is

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isomorphic to a direct power of A_{n-i} , *for* $i = 0, 1, \ldots, n-1$.

As in [12] we shall call a group $\underline{\underline{F}}$-*perfect* if it has no non-trivial finite homomorphic images, and hence no proper subgroups of finite index. A soluble group G has this property if and only if G/G' is a radicable group - that is, a group in which extraction of n-th roots is possible for every positive integer n . (This is a consequence of Lemma 4.1 of [5] and Lemma 9.22 of [10].) A related, but much stronger, condition that we shall consider is that G should have a finite series all of whose factors are radicable abelian groups. Periodic groups with this latter property were called $\mathrm{P}\underline{\underline{Q}}$-*groups* in [6] and we shall adopt this terminology here. For other unexplained notation and terminology we refer the reader to [9, 10].

One of the main results of [5] was that the p-subgroups of an $\underline{\underline{F}}$-perfect metabelian group satisfying min-n are abelian, for each prime p . This was generalized in [12], where we showed that if G is an $\underline{\underline{F}}$-perfect metanilpotent group satisfying min-n , then G' is nilpotent and, for each prime p , the p-subgroups of G are nilpotent with class not exceeding that of G' . It follows from Theorem A that no such restrictions apply to the p-subgroups of $\underline{\underline{F}}$-perfect soluble groups satisfying min-n in general, or even to the p-subgroups of $\mathrm{P}\underline{\underline{Q}}$-groups satisfying min-$n$. In fact we shall use Theorem A to prove

THEOREM B. *For every prime* p *and every integer* $n > 1$, *there is a* $\mathrm{P}\underline{\underline{Q}}$-*group satisfying* min-$n$ *that has a* p-*subgroup with trivial centre whose nilpotent length is equal to* n .

We shall see that it is also possible to deduce from Theorem A the known fact that the class of $\underline{\underline{F}}$-perfect groups satisfying min-n contains soluble groups of arbitrarily large derived lengths. This fact was proved independently by McDougall [5] and Roseblade and Wilson [11], and subsequent generalizations may be found in [3] (Lemma 3.4) and [12] (Theorem D).

The methods of this paper are similar to those of Heineken and Wilson [4].

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2.

We consider first the special case of Theorem A where $n = 2$. In this case the existence of a group G with the stated properties is already known, and the relevant facts may be found in [3]. However it will be convenient to summarize the information we need concerning this case in a rather different manner from that of [3]. We indicate briefly below how to derive the results in the form stated here from the results of [3].

In [4] a locally soluble group was called an L_ρ-group, for an ordinal number ρ, if its normal subgroups were linearly ordered by inclusion, with order type $\rho + 1$. Here we shall say that a locally soluble group is an *L^*-group* if its normal subgroups are well-ordered by inclusion; that is, if it is an L_ρ-group for some ordinal ρ.

LEMMA 1. *Let p and q be distinct primes, and let A be a non-trivial locally cyclic p-group and B a non-trivial locally cyclic q-group. Then there is a direct power $\bar{A}$ of A and a homomorphism $\theta : B \to \operatorname{aut}\bar{A}$ such that the associated semidirect product $G = \bar{A}B$ is a metabelian L^*-group, and*

(i) $G' = \bar{A}$,

(ii) the only normal subgroups of G are the subgroups $\bar{A}[p^i]$, for $i = 0, 1, 2, \ldots$, and the subgroups $\bar{A}\cdot B[q^j]$, for $j = 0, 1, 2, \ldots$,

(iii) if R/S is a chief factor of G with $R \le G'$ then $C_G(R/S) = G'$.

Proof. Since B is a locally cyclic p'-group, it has a faithful irreducible representation over the field Z_p with p elements. (See, for example, Lemma 2.5 of [3] and the remarks preceding that lemma.) Let V be a Z_pB-module affording such a representation. Then we may view V also as a module for the integral group ring ZB and Lemma 2.3 of [3] shows that the ZB-injective hull $\bar{V}$ of V has for its underlying group a minimal divisible group containing the additive group of V. Moreover, the remarks following the proof of Lemma 2.3 show that the only proper submodules of $\bar{V}$ are the submodules

$$\overline{V}[p^i] = \{v \in \overline{V} \mid p^i v = 0\}$$

for $i = 0, 1, 2, \ldots$; and that each of the factor-modules $\overline{V}[p^{i+1}]/\overline{V}[p^i]$ is isomorphic to V, and therefore affords a faithful irreducible representation of B over Z_p.

As the additive group of $\overline{V}$ is a divisible abelian p-group, it is a direct product of quasicyclic p-groups (groups of type p^∞). Now A is either a quasicyclic p-group or a cyclic p-group, so either $\overline{V}$ or one of its submodules $\overline{V}[p^i]$ has additive group isomorphic to a direct power $\overline{A}$ of A. We can use this isomorphism and the module action of B on $\overline{V}$ to define an action $\theta : B \to \operatorname{aut}\overline{A}$. Then, by what has been said about $\overline{V}$, the only proper B-invariant subgroups of $\overline{A}$ will be the subgroups $M_i = A[p^i]$, for $i = 0, 1, 2, \ldots$; and each non-trivial factor M_{i+1}/M_i will therefore be a chief factor in the semi-direct product $G = \overline{A}B$. Furthermore, for each such chief factor M_{i+1}/M_i we have $C_G(M_{i+1}/M_i) = \overline{A}$, as the factor arises from a faithful representation of B. Now B is either cyclic or quasicyclic, and in either case its only proper subgroups are those of the form $B[q^j]$, for $j = 0, 1, 2, \ldots$. Thus the subgroups M_i, for $i = 0, 1, 2, \ldots$, and the subgroups $\overline{A}\cdot B[q^j]$, for $j = 0, 1, 2, \ldots$, together form an ascending chief series of G, after repetitions have been suppressed.

Now let N be a normal subgroup of G with $\overline{A} \not\leq N$. Then for some i we have $\overline{A} \cap N = M_i < M_{i+1} \leq \overline{A}$. Consequently

$$[M_{i+1}, N] \leq [\overline{A}, N] \leq \overline{A} \cap N = M_i ,$$

so that $N \leq C_G(M_{i+1}/M_i) = \overline{A}$. This shows that every normal subgroup of G either contains or is contained in $\overline{A}$. Hence every normal subgroup of G occurs as a term of the chief series just described. Thus *(ii)* is established, and G is an L^*-group.

Since $G/\overline{A}$ is abelian, we have $G' \leq \overline{A}$. Also we have shown that $\overline{A}$ is the centralizer of every chief factor R/S of G with $R \leq \overline{A}$. Since the chief factors of G/G' are central, it follows that $G' = \overline{A}$. Thus

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both *(i)* and *(iii)* are proved, and the proof of the lemma is complete.

The case $n = 2$ of Theorem A now follows immediately. For if A_1 and A_2 are locally cyclic groups satisfying the hypotheses of Theorem A for $n = 2$, then taking $A = A_1$ and $B = A_2$ in Lemma 1 we obtain a metabelian L^*-group G, which is a semi-direct product of a direct power $\bar{A}_1$ of A_1 and the group A_2. Now identify A_1 with a direct factor of $\bar{A}_1$. Then A_1^G must coincide with $\bar{A}_1$ since these groups have the same exponent. Thus G is generated by A_1 and A_2. Moreover, $G' = \bar{A}_1$, which is a direct power of A_1, and $G/G' \cong A_2$.

We next record another property of the group G of Lemma 1 that we shall need later.

LEMMA 2. *In the notation of Lemma* 1, *the automorphisms* $b\theta$ *induced on* $\bar{A}[p]$ *by the elements* $b \in B$ *generate a subfield of the ring of endomorphisms of* $\bar{A}[p]$, *and the additive group of this subfield is isomorphic to* $\bar{A}[p]$.

Proof. From the proof of Lemma 1 it is clear that we may regard $\bar{A}[p]$ as a Z_pB-module affording a faithful irreducible representation of B over Z_p. The structure of such modules is described in Lemma 2.5 of [3], and the result may be readily deduced from the proof of that lemma.

3.

A basic tool in the construction of the groups required for the proof of Theorem A will be the *treble product*, which was introduced by Heineken and Wilson in [4]. This is a special case of the twisted wreath product of Neumann [7]. The data for its construction are three groups A, B, C and two homomorphisms $\sigma : B \to \operatorname{aut}A$ and $\tau : C \to \operatorname{aut}B$. The treble product associates with these data a group

$$T = \operatorname{tr}(A, B, C; \sigma, \tau)$$

generated by isomorphic copies of A, B, C (not here distinguished notationally from the originals) with the following properties:

(i) the subgroup $\langle A, B\rangle$ is the semi-direct product of A and

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B associated with the homomorphism σ, and the subgroup $\langle B, C\rangle$ is the semi-direct product of B and C associated with the homomorphism τ;

(ii) the normal closure $A^T = A^C$ of A in T is the direct product of all the conjugates $c^{-1}Ac$, where c ranges over C;

(iii) T is a semi-direct product of A^C and BC.

Notice that it follows from (ii) and (iii) that $\langle A, C\rangle \cong A \text{ wr } C$.

To construct T we may either proceed as in [4] or take T to be the twisted wreath product of A and the semi-direct product BC, with B doing the "twisting" according to the homomorphism σ. For details we refer the reader to [4].

We shall make use of the following lemma on minimal normal subgroups of treble products, in which we combine the results of Lemmas 1 and 2 of [4]. For the proof we refer the reader to [4].

LEMMA 3. *Let* $T = \operatorname{tr}(A, B, C; \sigma, \tau)$. *Suppose that* N *is a minimal normal subgroup of* AB *contained in* A *and that either* $N \nleq \zeta(A)$ *or the following condition is satisfied: for every element* $c \neq 1$ *in* C *there is a two-variable word* $p_c(a, b)$ *and there is an element* x_c *in* B *such that*

(i) $p_c(1, b) = 1$ *for all* b *in* B,

(ii) $p_c(a, x_c) \neq 1$ *for all* $a \neq 1$ *in* N, *and*

(iii) $p_c\left(a, cx_cc^{-1}\right) = 1$ *for all* a *in* N.

Then N^C *is a minimal normal subgroup of* T.

We also need a result similar to Lemma 3 of [4]. However, the conditions of that lemma are unfortunately a little too restrictive for our purposes. We therefore indicate below how the proof may be modified to yield the same conclusion under slightly weaker hypotheses.

LEMMA 4. *Let* $T = \operatorname{tr}(A, B, C; \sigma, \tau)$, *and let* N *be a normal subgroup of* AB *contained in* A *such that* $N \cap X \neq 1$ *for every normal sub-*

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group X *of* AB *with* $1 \neq X \leq A$. *Suppose also that* $C_B(A) = \ker\sigma$ *contains no non-trivial normal subgroup of* BC . *Then* $N^C \cap M \neq 1$ *for every normal subgroup* $M \neq 1$ *of* T .

Proof. Let M be a non-trivial normal subgroup of T . By the same argument used in the proof of Lemma 3 of [4], we see that $M_1 = A^C B \cap M$ is a non-trivial normal subgroup of T . Next suppose, if possible, that $M_2 = A^C \cap M_1 = 1$. Then

$$M_1 \leq C_T\left(A^C\right) \leq A^C C_{BC}\left(A^C\right) .$$

Since $\langle A, C\rangle \cong A \text{ wr } C$, the centralizer of A^C in C is trivial; hence $C_{BC}\left(A^C\right) = C_B\left(A^C\right)$, and this is a normal subgroup of BC contained in $C_B(A)$. Therefore, by our assumptions, $C_{BC}\left(A^C\right) = 1$, and hence $M_1 \leq A^C$. Thus $M_1 = M_1 \cap A^C = 1$, contradicting the first part of the proof. This contradiction shows that $M_2 \neq 1$.

To complete the proof we now argue exactly as in [4], noting that for the final part of the proof it is only necessary to know that N has non-trivial intersection with those non-trivial normal subgroups of AB that lie inside A .

Combining the results of Lemmas 3 and 4, we have the following result.

LEMMA 5. *Let* $T = \mathrm{tr}(A, B, C; \sigma, \tau)$ *and let* N *be a normal subgroup of* AB *that is contained in every normal subgroup* X *of* AB *with* $1 \neq X \leq A$. *Suppose further that the conditions of Lemmas* 3 *and* 4 *are satisfied. Then* N^C *is contained in every non-trivial normal subgroup of* G .

4.

We now deal with the special case of Theorem A where $n = 3$. Suppose that A_1, A_2, A_3 are non-trivial locally cyclic groups satisfying the conditions of Theorem A. By Lemma 1 there are direct powers $\overline{A}_1, \overline{A}_2$ of

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A_1, A_2 respectively and homomorphisms $A_2 \to \operatorname{aut}\bar{A}_1$ and $A_3 \to \operatorname{aut}\bar{A}_2$ such that the associated semi-direct products $H = \bar{A}_1 A_2$ and $K = \bar{A}_2 A_3$ are metabelian L^*-groups. We identify A_1 with a direct factor of $\bar{A}_1$ and A_2 with a direct factor of $\bar{A}_2$. Then as in §2 we see that the normal closures A_1^H and A_2^K coincide with $\bar{A}_1$ and $\bar{A}_2$ respectively, and we have

$$H = \langle A_1, A_2 \rangle , \quad K = \langle A_2, A_3 \rangle .$$

To simplify our notation we now denote $\bar{A}_1, \bar{A}_2, A_3$ by A, B, C respectively, and we write B_1 in place of A_2, so that

$$H = AB_1 = \langle A_1, B_1 \rangle$$

and

$$K = BC = \langle B_1, C \rangle .$$

Also we now write p, q, r instead of p_1, p_2, p_3 for the sequence of primes associated with the groups A, B, C.

Let $\phi : B_1 \to \operatorname{aut}A$ be the action associated with the semi-direct product H. To define G we first extend ϕ to a homomorphism $\sigma : B \to \operatorname{aut}A$ by composing it with the natural projection of B onto its direct factor B_1. Then, writing τ for the homomorphism $C \to \operatorname{aut}B$ associated with the semi-direct product K, we set

$$G = \operatorname{tr}(A, B, C; \sigma, \tau) .$$

The restriction of σ to B_1 agrees with ϕ, so the group G has a subgroup AB_1 isomorphic to H and we shall identify this subgroup with H.

We now want to show that G has the properties claimed in Theorem A. This will follow from

LEMMA 6. *The group G is a soluble L^*-group of derived length 3 generated by A_1, B_1, C and has the following properties:*

(i) $\langle A_1, C \rangle \cong A_1 \operatorname{wr} C$;

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(ii) $G'' = A^C$ and $G' = A^C B = H^G$;

(iii) if $0 \le i < 3$ and R/S is a chief factor of G with $G^{(i+1)} \le S < R \le G^{(i)}$ then $C_G(R/S) = G^{(i)}$.

Proof. By Lemma 1 the only normal subgroups of H contained in A are the subgroups $M_i = A\left[p^i\right]$, for $i = 0, 1, 2, \ldots$, and $\bigcup_{i=0}^{\infty} M_i = A$. To prove that G is an L^*-group we first show that if the factor M_{i+1}/M_i of H is non-trivial then the corresponding factor M_{i+1}^G/M_i^G of G is contained in every non-trivial normal subgroup of G/M_i^G . We do this by proving that M_1^G is contained in every non-trivial normal subgroup of G and then dealing with the remaining factors M_{i+1}^G/M_i^G by passing to appropriate factor-groups of G .

The main step in the proof is to show that the conditions of Lemma 5 are satisfied. Let us write $M = M_1$ and $L = C_B(A)$. Then M is contained in every non-trivial normal subgroup of H , and therefore also in every non-trivial normal subgroup of AB that lies inside A . By the definition of σ , we have

$$L = \ker\sigma = \ker\pi\phi ,$$

where π is the projection of B onto B_1 . But ϕ must be a monomorphism, otherwise its kernel would be a non-trivial normal subgroup of H intersecting A trivially. Thus $L = \ker\pi$, and from this we see that

$$B = B_1 \times L .$$

Since the non-trivial normal subgroups of $K = BC$ all contain $B[q]$ it follows that none of them can lie inside L . Therefore the hypotheses of Lemma 5 will be satisfied if we can establish the existence of a two-variable word $p_c(a, b)$ and an element $x_c \in B$ with the properties stipulated in Lemma 3, for every $c \ne 1$ in C . We distinguish two cases: either the automorphism induced on B by c leaves L invariant, or it does not.

Suppose first that c is an element such that $L^c \nleq L$. Then for some $y \in L$ we have $y^c \notin L$. We take $x_c = y^c$ and $p_c(a, b) = [a, b]$ in this case, and note that conditions *(i)* and *(iii)* of Lemma 3 are satisfied. By our choice of y we have $1 \neq x_c^{-1}u \in B_1$ for some $u \in L$, and since the non-trivial elements of B_1 induce non-trivial automorphisms on M it follows that $C_M(x_c) < M$. But $C_M(x_c)$ is normal in AB : for A is abelian and contains M, and if $z \in C_M(x_c)$ and $b \in B$ then using the commutativity of B we have

$$\left[z^b, x_c\right] = [z, x_c]^b = 1 ,$$

so that $z^b \in C_M(x_c)$. As M is a minimal normal subgroup of AB, it follows that $C_M(x_c) = 1$, and therefore

$$p_c(a, x_c) \neq 1$$

for all $a \neq 1$ in M. Thus the conditions *(i)*, *(ii)*, *(iii)* of Lemma 3 are all satisfied for this c.

Next suppose that $1 \neq c \in C$ and $L^c \leq L$. Then c induces an automorphism on B/L. Now B/L is isomorphic to B_1, so it is a non-trivial locally cyclic q-group and hence has a characteristic subgroup $\langle b_1L \rangle$ of order q, which must be invariant under the automorphism induced by c. Let s be an integer such that

$$b_1^cL = b_1^sL .$$

Then since the order of c is a power of the prime r we have

$$s^{r^t} \equiv 1 \pmod{q} \tag{1}$$

for some integer t.

By Lemma 2 the automorphisms induced on M by elements of B generate a subfield Ω of characteristic p in the ring of endomorphisms of A, and the automorphisms $\beta = b_1\sigma$ and $\beta^s = b_1^c\sigma$ will be q-th roots of unity in Ω . Hence β and β^s are roots of irreducible factors of the polynomial $x^q - 1$ over $\mathrm{GF}(p)$.

We assume first that β and β^s are roots of the same irreducible factor of $x^q - 1$, and show that this assumption leads to a contradiction. If β and β^s satisfy the same irreducible polynomial over $\mathrm{GF}(q)$ then they are conjugate roots of unity in Ω and hence also in the finite subfield Δ of Ω that they generate. Thus there is a field automorphism of Δ mapping β onto β^s . As Δ is finite its group of automorphisms is generated by the automorphism that maps each element to its p-th power: hence, for some integer k , we have

$$\beta^s = \beta^{p^k} . \tag{2}$$

Therefore, using (1), we have

$$p^{kr^t} \equiv s^{r^t} \equiv 1 \pmod{q} .$$

Consequently $\mathrm{ord}(p, q)$ is a divisor of kr^t . But the primes p, q, r form an admissible sequence, so

$$r \nmid \mathrm{ord}(p, q)$$

and therefore

$$\mathrm{ord}(p, q) \mid k .$$

Hence $p^k \equiv 1 \pmod{q}$, and it now follows from (2) that $\beta^s = \beta$. Thus b_1 and b_1^c induce the same automorphism on A , and so $[b_1, c] \in L$. As $\langle b_1 \rangle^C = B[q]$, we have

$$[B[q], c] \le L < B[q] .$$

However $[B[q], c]$ is normal in BC and $B[q]$ is a minimal normal subgroup: therefore $[B[q], c] = 1$. But this implies that $\langle c \rangle$ is a non-trivial normal subgroup of $B[q]C$ that intersects $B[q]$ trivially. Since $B[q]C$ is an L^*-group, this is a contradiction. Therefore β and β^s cannot be roots of the same irreducible factor of $x^q - 1$.

This means that $x^q - 1$ has an irreducible factor

$$f(x) = a_0 + a_1 x + \ldots + a_m x^m$$

over $\mathrm{GF}(p)$ such that $f(\beta) = 0$ and $f(\beta^s) \neq 0$. We now define our two-variable word $p_c(a, b)$ by

$$p_c(a, b) = a^{\alpha_0} a^{\alpha_1 b} \ldots a^{\alpha_m b^m} = a^{f(b)} .$$

Then, substituting b_1 for b, we find that

$$p_c(a, b_1) = a^{f(\beta)} = a^0 = 1 ,$$

for all $a \in M$. On the other hand, as $f(\beta^s)$ is non-zero and hence invertible in Ω we have

$$p_c\left(a, b_1^c\right) = a^{f(\beta^s)} \neq 1 ,$$

for all $a \neq 1$ in M. Therefore the conditions *(i)*, *(ii)*, *(iii)* of Lemma 3 are satisfied if we take $x_c = b_1^c$.

We have now shown how to define x_c and $p_c(a, b)$ for all $c \neq 1$ in C, and it follows from Lemma 5 that M^G is contained in every non-trivial normal subgroup of G.

For each integer $i \geq 1$ the factor-group A/M_i is isomorphic to a direct power of a suitable factor-group of A_1, and the action of B on A induces an action of B on A/M_i. We can therefore repeat the construction used to define G, but taking A/M_i, B, C in place of A, B, C respectively. The resulting group

$$\overline{G} = \mathrm{tr}\left(A/M_i, B, C\right)$$

is easily seen to be isomorphic to G/M_i^G, since

$$(A \mathrm{wr} C)/M_i^G \cong \left(A/M_i\right) \mathrm{wr}\ C .$$

In $\overline{G}$ the role of M is taken by the subgroup M_{i+1}/M_i ; and the normal closure in $\overline{G}$ of this subgroup is mapped onto M_{i+1}^G/M_i^G under the isomorphism between $\overline{G}$ and G/M_i^G . Therefore, applying the above argument to $\overline{G}$, we find that M_{i+1}^G/M_i^G is contained in every non-trivial normal subgroup of G/M_i^G .

Since A is the union of the subgroups M_i , it follows that every normal subgroup of G either contains A^C or coincides with M_i^G for some $i \geq 0$. However, G/A^C is isomorphic to $BC = K$, which is an L^*-group, so the normal subgroups of G containing A^C form a well-ordered chain. As the subgroups M_i are also well-ordered, G is an L^*-group.

We can now identify the terms of the derived series of G . For by Lemma 1 *(i)* we have $K' = B$, so using the isomorphism between G/A^C and K we have $A^CG' = A^CB$. But G is an L^*-group and $G' \nleq A^C$, so $A^C \leq G'$, and therefore $G' = A^CB$. Also G/A^C is metabelian, so we have $G'' \leq A^C$. If this inclusion were strict then for some j we should have

$$G'' = M_j^G < M_{j+1}^G \leq A^C$$

and consequently

$$\left[G', M_{j+1}^G\right] \leq M_j^G . \tag{3}$$

Now M_j^G is a direct product of the conjugates M_j^c , as c runs over C , and similarly M_{j+1}^G is a direct product of the conjugates M_{j+1}^c . Thus

$$M_{j+1}^G/M_j^G \cong \underset{c\in C}{\mathrm{dr}}\ M_{j+1}^c/M_j^c \ ;$$

and each factor M_{j+1}^c/M_j^c in this direct product is invariant under the action of H . Hence if (3) is to hold then H must centralize each of these direct factors. However by Lemma 1 *(iii)* we have

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$$C_H(M_{j+1}/M_j) = H' < H ;$$

so this leads to a contradiction. Therefore $G'' = A^C$ and *(ii)* is proved.

Finally, let R/S be a chief factor of G and suppose that

$$G^{(i+1)} \leq S < R < G^{(i)} ,$$

where $0 \leq i < 3$. If $i = 0$, then obviously $C_G(R/S) = G$. If $i = 1$, then R/S corresponds to a chief factor of G/G'', and as this is a group isomorphic to K it follows from Lemma 1 *(iii)* that $C_G(R/S) = G'$. Suppose next that $i = 2$. Then there is an integer $j \geq 0$ such that $R = M_{j+1}^G$ and $S = M_j^G$. Now M_{j+1}^G/M_j^G is isomorphic to a direct product of groups M_{j+1}^c/M_j^c, as c ranges over C, and each of these groups is invariant under the action of B. Hence

$$C_B(R/S) \leq C_B(M_{j+1}/M_j) .$$

But $C_B(R/S)$ is a normal subgroup of BC, so either it is trivial or else it contains $B[q]$ and hence has non-trivial intersection with B_1. However,

$$\begin{aligned} B_1 \cap C_B(R/S) &\leq B_1 \cap C_H(M_{j+1}/M_j) \\ &= B_1 \cap H' = 1 , \end{aligned}$$

and so we conclude that $C_B(R/S) = 1$. As $G' = A^C B$, it follows that

$$C_{G'}(R/S) = A^C .$$

Also conjugation by a non-trivial element of C permutes the factors M_{j+1}^c/M_j^c in a non-trivial fashion, so the centralizer of R/S in C is trivial. As C complements G' in G, this shows that

$$C_G(R/S) = A^C = G'' .$$

Thus *(iii)* is established in all cases, and the proof of Lemma 6 is complete.

5.

We now describe the construction that will be used for the proof of Theorem A in the general case. The reader familiar with [4] will recognize a variant of Heineken and Wilson's "treble product tower".

Let n be a positive integer and let $p_1, p_2, \ldots, p_n$ be a sequence of primes such that $p_i \neq p_{i+1}$, for $i = 1, 2, \ldots, n-1$. Given a sequence of non-trivial locally cyclic groups $A_1, A_2, \ldots, A_n$, with A_i a p_i-group for each i, we shall define a group $L(A_1, A_2, \ldots, A_n)$, which is generated by isomorphic copies of $A_1, A_2, \ldots, A_n$ (as usual, these copies will be identified with the originals) and which has the following properties:

(1) if $1 \leq i < i + 2j \leq n$, then the subgroup of $L(A_1, A_2, \ldots, A_n)$ generated by $A_i, A_{i+2}, A_{i+4}, \ldots, A_{i+2j}$ is isomorphic to

$$A_i \text{ wr } A_{i+2} \text{ wr } \ldots \text{ wr } A_{i+2j} ,$$

(2) if $1 \leq m < n$, then the subgroup of $L(A_1, A_2, \ldots, A_n)$ generated by $A_1, A_2, \ldots, A_m$ is equal to $L(A_1, A_2, \ldots, A_m)$, and its normal closure is complemented by $\langle A_{m+1}, A_{m+2}, \ldots, A_n \rangle$,

(3) the subgroups $\langle A_i, A_{i+1} \rangle$ of $L(A_1, A_2, \ldots, A_n)$, for $i = 1, 2, \ldots, n-1$, are metabelian L^*-groups of the type described in Lemma 1.

The groups $L(A_1, A_2, \ldots, A_n)$ are defined inductively. We use a generalization of the method of §4, replacing the three groups A_1, A_2, A_3 used in §4 by the three groups $L(A_1, A_2, \ldots, A_{n-2}), A_{n-1}, A_n$. To start the induction we set $L(A_1) = A_1$ and take $L(A_1, A_2)$ to be one of the metabelian L^*-groups described in Lemma 1. Suppose now that $n > 2$ and assume inductively that the groups $L(A_1, A_2, \ldots, A_m)$ have been defined for all $m < n$. We write $H = L(A_1, A_2, \ldots, A_{n-1})$ and denote the normal closure of $L(A_1, A_2, \ldots, A_{n-2})$ in H by K . By our inductive assumptions, (2) above is satisfied when n is replaced by $n - 1$ and m by $n - 2$. Hence H is a semi-direct product of K and A_{n-1} . Let $\phi : A_{n-1} \to \operatorname{aut}K$ be the homomorphism associated with this semi-direct product. By Lemma 1, there is a direct power $\bar{A}_{n-1}$ of A_{n-1} and a homomorphism $\tau : A_n \to \operatorname{aut}\bar{A}_{n-1}$ such that the semi-direct product $\bar{A}_{n-1}A_n$ is a metabelian L^*-group. Composing the natural projection of $\bar{A}_{n-1}$ onto A_{n-1} with ϕ , we obtain a homomorphism $\sigma : \bar{A}_{n-1} \to \operatorname{aut}K$. With the actions σ and τ defined in this way, we now set

$$G = \operatorname{tr}(K, \bar{A}_{n-1}, A_n; \sigma, \tau)$$

and verify that G has the properties listed above. Observe that when $n = 3$ this construction coincides with that used in §4.

If we identify A_{n-1} with a direct factor of $\bar{A}_{n-1}$ not contained in $\ker\sigma$ then G contains a subgroup KA_{n-1} isomorphic to H . We shall suppose that this subgroup has been identified with H . Then, writing

$$G_m = L(A_1, A_2, \ldots, A_m) ,$$

for $m = 1, 2, \ldots, n$, we see by induction that the groups G_m form a chain

$$G_1 < G_2 < \ldots < G_{n-1} < G_n = G .$$

Suppose i and j are positive integers with $i + 2j \le n$. To establish property (1), we note first that $\langle K, A_n \rangle = K \operatorname{wr} A_n$, by the definition of the treble product. If $i + 2j = n$, then

$$\langle A_i, A_{i+2}, \ldots, A_{i+2j-2} \rangle \le K$$

so that

$$\langle A_i, A_{i+2}, \ldots, A_{i+2j} \rangle = \langle A_i, A_{i+2}, \ldots, A_{i+2j-2} \rangle \operatorname{wr} A_{i+2j} .$$

By induction we may assume that the left-hand factor of this wreath product is itself an iterated wreath product of the required type, so (1) holds in this case. On the other hand, if $i + 2j < n$, then

$$\langle A_i, A_{i+2}, \ldots, A_{i+2j}\rangle \leq H ,$$

and (1) is a consequence of our inductive assumptions about H. Thus (1) holds in all cases.

Let us now write $D_m = \langle A_{m+1}, A_{m+2}, \ldots, A_n\rangle$, for $m = 1, 2, \ldots, n-1$. To establish property (2) it will be sufficient to verify that

$$(*) \qquad G_m^G \cap D_m = 1$$

for all $m < n$. As $G_{n-2} = K$ we have

$$G_{n-2}^G \cap D_{n-2} = K^G \cap \bar{A}_{n-1}A_n = 1 ;$$

so (*) holds for $m = n - 2$. Also, as $\bar{A}_{n-1}$ is the normal closure of A_{n-1} in $\langle A_{n-1}, A_n\rangle$, we have

$$G_{n-1}^G = K^G\bar{A}_{n-1}$$

and therefore

$$G_{n-1}^G \cap D_{n-1} = K^G\bar{A}_{n-1} \cap A_n = 1 ,$$

showing that (*) also holds for $m = n - 1$. Next suppose that $m < n - 2$. Then

$$D_m = \langle A_{m+1}, \ldots, A_{n-2}\rangle^{D_m}\bar{A}_{n-1}A_n ,$$

and since

$$\langle A_{m+1}, \ldots, A_{n-2}\rangle^{D_m} \leq K^G ,$$

the modular law shows that

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$$G_{n-2}^{G} \cap D_m = \langle A_{m+1}, \ldots, A_{n-2} \rangle^{D_m \left(\bar{A}_{n-1} A_n \cap K^G\right)}$$
$$= \langle A_{m+1}, \ldots, A_{n-2} \rangle^{D_m} .$$

Hence $G_m^G \cap D_m = G_m^G \cap \langle A_{m+1}, \ldots, A_{n-2} \rangle^{D_m}$. This last group can be expressed as a direct product of conjugates

$$\left(G_m^H \cap \langle A_{m+1}, \ldots, A_{n-2} \rangle^{A_{n-1}}\right)^a ,$$

where a ranges over A, and by induction

$$G_m^H \cap \langle A_{m+1}, \ldots, A_{n-2} \rangle^{A_{n-1}} \leq G_m^H \cap \langle A_{m+1}, \ldots, A_{n-1} \rangle = 1 .$$

Therefore

$$G_m^G \cap D_m = 1 ,$$

and (2) is established in all cases.

Finally, property (3) is an immediate consequence of the definition of G, since $\langle A_{n-1}, A_n \rangle = \bar{A}_{n-1} A_n$, and this is a metabelian L^*-group of the required type.

Thus by induction the groups $L(A_1, A_2, \ldots, A_n)$ are defined and have properties (1), (2), (3) for all $n \geq 1$. We now show that by choosing our prime sequences appropriately we obtain groups satisfying the conclusions of Theorem A.

LEMMA 7. *Let $n \geq 2$ and suppose the sequence of primes $p_1, p_2, \ldots, p_n$ is admissible. Then $G = L(A_1, A_2, \ldots, A_n)$ is an L^*-group and, for each integer i with $0 \leq i < n$, we have*

(i) $G^{(i)} = L(A_1, A_2, \ldots, A_{n-i})^G$,

(ii) $G^{(i)}/G^{(i+1)}$ *is isomorphic to a direct power of* A_{n-i},

(iii) if R/S is a chief factor of G with $G^{(i+1)} \leq S < R \leq G^{(i)}$, then $C_G(R/S) = G^{(i)}$.

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Proof. If $n = 2$ then the result follows from Lemma 1.

If $n = 3$ then the construction described above is essentially a repetition of that used in §4, and in this case the result is a consequence of Lemma 6.

Suppose now that $n \geq 4$. We define H and K as above and set

$$L = \langle A_1, A_2, \ldots, A_{n-3} \rangle^H .$$

Since property (2) holds with $n - 1$ in place of n, we see that L is complemented in H by $\langle A_{n-2}, A_{n-1} \rangle = \bar{A}_{n-2}A_{n-1}$. Now $L\bar{A}_{n-2} \leq K$, so we have

$$\begin{aligned} K &= K \cap L\bar{A}_{n-2}A_{n-1} \\ &= L\bar{A}_{n-2}\left(K \cap A_{n-1}\right) \\ &= L\bar{A}_{n-2} . \end{aligned}$$

Therefore $\bar{A}_{n-2}$ is a complement to L in K which admits the action of A_{n-1}, and hence also that of $\bar{A}_{n-1}$. Consequently K/L and $\bar{A}_{n-2}$ are isomorphic as operator groups under the action of $\bar{A}_{n-1}$. Now G/L^G is isomorphic to the treble product

$$\operatorname{tr}\left(K/L, \bar{A}_{n-1}, A_n\right)$$

formed using this action of $\bar{A}_{n-1}$, and therefore also to

$$\operatorname{tr}\left(\bar{A}_{n-2}, \bar{A}_{n-1}, A_n\right) .$$

But this last treble product is precisely the group that results from applying the construction of §4 to the sequence of groups A_{n-2}, A_{n-1}, A_n. Since the sequence p_{n-2}, p_{n-1}, p_n is admissible, Lemma 6 shows that G/L^G is an L^*-group. Hence the normal subgroups of G containing L^G form a well-ordered chain.

Now suppose that M is a minimal normal subgroup of H. By induction we may assume that H is an L^*-group and that the assertions of the Lemma are true with H in place of G and $n - 1$ in place of n.

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Thus M is contained in every non-trivial normal subgroup of H, and hence also in every non-trivial normal subgroup of $K\bar{A}_{n-1}$ that lies inside K. In particular $M \leq L = H'' < K$, so our inductive assumptions show that $C_H(M) < K$; in other words, M is not central in K. Furthermore $\ker\sigma$ contains no normal subgroup of $\bar{A}_{n-1}A_n$, by the argument used in the proof of Lemma 6. Therefore the conditions of Lemma 5 are satisfied, and we conclude from that lemma that M^G is contained in every non-trivial normal subgroup of G.

As H is an L^*-group there is a unique ascending chief series

$$1 = M_0 < M_1 < \ldots < M_\rho = L$$

of H between 1 and L, where ρ is some ordinal number. If $\alpha < \rho$ then, by passing to the factor-group

$$G/M_\alpha^G \cong \operatorname{tr}\left(K/M_\alpha, \bar{A}_{n-1}, A_n\right)$$

and using the above argument, we see that $M_{\alpha+1}^G/M_\alpha^G$ is contained in every non-trivial normal subgroup of G/M_α^G. Therefore the series

$$1 = M_0^G < M_1^G < \ldots < M_\rho^G = L^G$$

is a unique ascending chief series of G between 1 and L^G, and every normal subgroup of G either contains L^G or coincides with M_α^G for some $\alpha < \rho$. As we have already shown that the normal subgroups containing L^G form a well-ordered chain, it now follows that G is an L^*-group.

To identify the terms of the derived series of G we first use the isomorphism

$$G/L^G \cong \operatorname{tr}\left(\bar{A}_{n-2}, \bar{A}_{n-1}, A_n\right)$$

and Lemma 6 *(ii)* to deduce that

$$G'L^G = H^G, \quad G''L^G = K^G.$$

Now G is an L^*-group and G/L^G is not metabelian, so we must have

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$L^G \leq G''$. Hence the above equations show that $G' = H^G$ and $G'' = K^G$. Thus assertion *(i)* of the lemma is true if $i \leq 2$. Suppose next that $i > 2$. In this case we can write

$$G^{(i)} = \left(K^G\right)^{(i-2)} .$$

Now K^G is a direct product of conjugates of K , so

$$\left(K^G\right)^{(i-2)} = \left(K^{(i-2)}\right)^G .$$

Moreover, by induction we have

$$K^{(i-2)} = H^{(i-1)} = L\left(A_1, \ldots, A_{n-i}\right)^H ,$$

and it follows that

$$G^{(i)} = L\left(A_1, A_2, \ldots, A_{n-i}\right)^G ,$$

as claimed. Hence *(i)* is true in all cases.

As $G' = H^G = K^G\overline{A}_{n-1}$ and $G'' = K^G$, we see from the definition of G as a treble product of K , $\overline{A}_{n-1}$, and A_n that $G/G' \cong A_n$ and $G'/G'' \cong \overline{A}_{n-1}$. Thus $G^{(i)}/G^{(i+1)}$ is isomorphic to a direct power of A_{n-i} when $i = 0$ or $i = 1$. Suppose now that $i \geq 2$: then, using again the fact that $G'' = K^G$ is a direct product of conjugates of K , we find that $G^{(i)}/G^{(i+1)}$ is isomorphic to a direct power of $K^{(i-2)}/K^{(i-1)}$. However, $K^{(i-2)}/K^{(i-1)} = H^{(i-1)}/H^{(i)}$, and by induction $H^{(i-1)}/H^{(i)}$ is isomorphic to a direct power of A_{n-i} . Hence $G^{(i)}/G^{(i+1)}$ is isomorphic to a direct power of A_{n-i} , and *(ii)* is proved.

Finally, let R/S be a chief factor of G , and suppose that

$$G^{(i+1)} \leq S < R \leq G^{(i)} ,$$

where $0 \leq i < n$. If $i \leq 2$ then R/S is effectively a chief factor of $G/G^{(3)} = G/L^G$, and, using again the isomorphism

$$G/L^G \cong \operatorname{tr}\left(\overline{A}_{n-2}, \overline{A}_{n-1}, A_n\right)$$

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and Lemma 6 *(iii)*, we have

$$C_G(R/S) = G^{(i)} .$$

On the other hand, if $i > 2$ then $R \leq G^{(3)} = L^G$, and so for some $\alpha < \rho$ we have $R = M_{\alpha+1}^G$ and $S = M_\alpha^G$. Now $M_{\alpha+1}^G/M_\alpha^G$ is isomorphic to a direct product of the groups $M_{\alpha+1}^a/M_\alpha^a$, as a ranges over A_n, and the direct factors are all invariant under the action of H. Furthermore $M_{\alpha+1}/M_\alpha$ is a chief factor of H and

$$H^{(i)} \leq M_\alpha < M_{\alpha+1} \leq H^{(i-1)} ,$$

so by induction

$$C_H(M_{\alpha+1}/M_\alpha) = H^{(i-1)} .$$

Using an argument like that given in the last paragraph of the proof of Lemma 6, we deduce from this that

$$C_G(R/S) = \left(H^{(i-1)}\right)^G = G^{(i)} .$$

Thus *(iii)* is established and the proof of Lemma 7 is complete.

Theorem A now follows immediately from Lemma 7. For $G = L(A_1, A_2, \ldots, A_n)$ is generated by $A_1, A_2, \ldots, A_n$, and by the properties established in the course of construction of $L(A_1, A_2, \ldots, A_n)$ we see that the subgroups

$$\langle A_1, A_3, A_5, \ldots, A_{n-1+\varepsilon} \rangle$$

and

$$\langle A_2, A_4, A_6, \ldots, A_{n-\varepsilon} \rangle$$

are expressible as wreath products of the required type. Hence G is generated by these wreath products, and the other properties are established in Lemma 7.

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6.

Proof of Theorem B. Given a prime p and an integer $n > 1$, we want to construct a $p\underline{\underline{Q}}$-group G satisfying min-n with a p-subgroup P of nilpotent length n, such that $\zeta(P) = 1$.

We choose a prime $q > p$ such that $p \nmid \operatorname{ord}(p, q)$: this is possible, for if $p \neq 2$ we can use any prime $q > p$ with $q \equiv 1 \pmod{p}$, and if $p = 2$ we can take $q = 7$, for example. Let $p_1, p_2, \ldots, p_{2n}$ be the sequence of primes defined by

$$p_{2i-1} = p , \quad p_{2i} = q ,$$

for $i = 1, 2, \ldots, n$. Our choice of q ensures that this sequence is admissible. Now take A_i to be a quasicyclic p_i-group for each $i \leq 2n$. Then by Theorem A there is a soluble L^*-group G of derived length $2n$ generated by

$$P = A_1 \operatorname{wr} A_3 \operatorname{wr} \ldots \operatorname{wr} A_{2n-1}$$

and

$$Q = A_2 \operatorname{wr} A_4 \operatorname{wr} \ldots \operatorname{wr} A_{2n}$$

such that $G^{(i)}/G^{(i+1)}$ is isomorphic to a direct power of A_{2n-i}, for $i = 0, 1, 2, \ldots, n-1$. As the groups A_i are radicable, so are all the factors $G^{(i)}/G^{(i+1)}$: hence G is a $p\underline{\underline{Q}}$-group.

Now P is a p-subgroup of G, and we have $\zeta(P) = 1$, for it is known (see, for example, Corollary 3.4 of [8]) that the wreath product of a non-trivial group and an infinite group always has trivial centre. Further P is evidently a $p\underline{\underline{Q}}$-group of derived length n, so Theorem 4.5 of [6] shows that the nilpotent length of P is also equal to n. Finally, G is an L^*-group, and *a fortiori* satisfies min-n, so the proof of Theorem C is complete.

The group G constructed above shows also that there is no upper bound on the derived lengths of $p\underline{\underline{Q}}$-groups satisfying min-$n$, for G was a $p\underline{\underline{Q}}$-group of derived length $2n$, and n was an arbitrary integer with $n > 1$. Thus we have a new proof of Lemma 3.4 of [3].

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We conclude by remarking that if we drop the requirement that our sequences of primes be admissible, then the constructions of §4 and §5 do not in general produce L^*-groups. This may be shown by the following example. Let $\langle a \rangle$ and $\langle c \rangle$ be cyclic groups of order 2 and let $\langle b \rangle$ be a cyclic group of order 3. The construction of §4, applied to the three groups $\langle a \rangle, \langle b \rangle, \langle c \rangle$, gives us a group

$$G = \mathrm{tr}(A, B, C; \sigma, \tau) ,$$

where A and B are direct powers of $\langle a \rangle$ and $\langle b \rangle$ respectively, and $C = \langle c \rangle$. The direct powers A and B are chosen so that the semi-direct products $H = A\langle b \rangle$ and $K = B\langle c \rangle$ are metabelian L^*-groups, and it is not hard to verify that the only possibilities for H and K are groups isomorphic to the alternating group A_4 and the symmetric group S_3 respectively. Thus we may assume that $A = \langle a, a^b \rangle$ and $B = \langle b \rangle$, and that the following relations hold in G :

$$[a, a^b] = 1 , \quad a^{b^2} = aa^b , \quad b^c \quad b^{-1} .$$

Now $A^G = A^C = A \times A^c$, and we can express this subgroup also as a direct product

$$A^G = \langle aa^{bc}, a^c a^b \rangle \times \langle aa^c, aa^b a^{bc} \rangle$$

in which each of the direct factors is a normal subgroup of G. Thus the normal subgroups of G are not well-ordered, and consequently G is not an L^*-group.

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